Infinite Electrical Networks: Forward and Inverse Problems

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Abstract

We consider infinite electrical networks. We work to generalize many of the notions from the finite case. Using Hilbert Space techniques we develop a framework for discussing infinite networks and finite power voltage and current functions on them. In a similar fashion to the finite case, we show how to recover a large class of infinite networks called "supercritical half-planar networks". We then proceed to show that two major ideas from the finite case persist in the infinite case, namely the idea of using determinants to find connections, and using geodesics to count connections via the cutpoint lemma.

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Chapter 1

Forward Problems

1.1 Introduction

In this chapter we begin our development of infinite electrical networks. After initial preliminaries about infinite graphs, we present some several results about naive approaches to infinite graphs, such as considering L^p spaces of voltage functions and bounded voltage functions. Though we are able to prove several existence and uniqueness results about these graphs, it becomes clear that these are not the right spaces to consider. Instead, we introduce the vector spaces of finite power voltage and current functions. These vector spaces turn out to be Hilbert Spaces, which allows us to use linear algebra in the infinite setting. We introduce the notion of a minimal function, which will become a common theme for the rest of the paper. The property of being γ-harmonic turns out to be worse behaved than minimality, and we prove various facts about boundary value maps which generalize the notion of the Neumann to Dirichlet and Dirichlet to Neumann maps.

1.2 Preliminaries and Definitions

Firstly, we need some basic information about infinite graphs.

Definition 1.2.1. We call a graph **topologically connected** if there are no two subgraphs A and B such that $A \cup B = G$ and $A \cap B = \varnothing$.

Definition 1.2.2. We call a graph **finitely connected** if every two vertices in the graph can be connected by a finite path.

Lemma 1.2.3. A graph G is topologically connected iff it is finitely connected.

Proof. If G is finitely connected then obviously it is topologically connected. Now suppose that G is topologically connected but not finitely connected and let v and w be vertices that can't be connected with a finite path. We will define a subgraph $G_n(v)$. A vertex a is in $G_n(v)$ iff there is a path from v to a in G

such that is of length n or less. An edge $ab \in G_n$ iff $a, b \in G_n$ and $ab \in G$. We will now define the graph

$$C(v) = \bigcup_{n \in \mathbb{N}} G_n(v)$$

which we will call the maximal component containing $C(v)$. Now notice that $\{C(v) : v \in G\}$ is just the collection of maximal finitely connected components of G, so if $C(v) \neq C(w)$ (which occurs iff there is not a finite path from v to w) then $C(v) \cap C(w) = \varnothing$. Define the graphs

$$A = C(v) \quad \text{and} \quad B = \bigcup_{u \in \{G : C(u) \neq C(v)\}} C(u)$$

and note that $A \cap B = \varnothing$ but $A \cup B = G$ so G is not topologically connected. □

Lemma 1.2.4. If G is a finitely connected infinite graph with finite valence at each vertex, then G has countably many vertices and (at most) countably many edges.

Proof. Using the above notation, we have just seen that if $v \in G$ is any vertex, then $G = C(v) = \bigcup_{n \in \mathbb{N}} G_n(v)$. Thus if we can show that each $G_n(v)$ is finite we will be done. The rest of the proof will be by induction. Since every vertex has finite valence, if w is any vertex in G, we know that $G_1(w)$ is finite. Now suppose that $G_n(v)$ is finite for some $n \geq 1$. Then we have that

$$G_{n+1}(v) = \bigcup_{w \in G_n(v)} G_1(w)$$

which is a finite union of finite sets and is thus finite. By induction each $G_n(v)$ is finite so the union is countable. □

Now we need a definition of an infinite electrical network. We do this much the same as the finite case.

Definition 1.2.5. Let $G = (\partial G, \operatorname{int} G, E)$ be an infinite graph divided into boundary and interior vertices ∂G and $\operatorname{int} G$. We will call the pair $\Gamma = (G, \gamma)$ an infinite electrical network if the following conditions are satisfied

1. G is finitely connected,
2. each vertex of G has finite valence (i.e. finitely many edges connected to it),
3. γ is a function from E (the edges of the graph) to $\mathbb{R}^+$ (strictly positive reals).

1.3 Naive Approaches to Infinite Graphs

In this section we develop some of the theory for L^p voltage functions, which is sort of the most naive route to approach this problem. We have some useful results, which we present for completeness, but the theory of L^p spaces in this context turns out to be not as robust as the theory of finite power voltages, which we will discuss later.

Definition 1.3.1. The spaces $L^p(G)$ and $L^p(E)$ are as follows

1. For $1 \leq p < \infty$ $L^p(G)$ is the space of real valued vertex functions u such that $\sum_{v \in V(G)} |u(v)|^p < \infty$
2. $L^\infty(G)$ is the space of bounded real valued vertex functions.
3. $L^p(E)$ (resp. $L^\infty(E)$) is the space of positive valued conductivity functions γ such that $\sum_{e \in E} \gamma^p < \infty$ (resp. is bounded).

We recall that from Minkowski's inequality that these spaces are actually $\mathbb{R}$-vector spaces.

We note that since we assumed that vertices have finite valence, given an electrical network $\Gamma = (G, \gamma)$ and a real valued function $u : G \to \mathbb{R}$ and a, we can make sense of the current at each vertex. In fact, this lets us define will define the operator $K : \mathbb{R}^G \to \mathbb{R}^G$ as

$$(Ku)(v) = \sum_{v' \sim v} (v - v')\gamma_{vv'}.$$

Thus a function u is called γ harmonic if $(Ku)(v) = 0$ for $v \in \operatorname{int} G$.

1.3.1 Results about $L^\infty(G)$

We can now state a result about existence.

Theorem 1.3.2. Let Γ be an infinite electrical network. Let $\phi \in L^\infty(\partial G)$. Then there exists a (not necessarily unique) function $u \in L^\infty(G)$ such that $u|_{\partial G} = \phi$ and u is γ-harmonic on $\operatorname{int} G$. Furthermore, we have that $\|u\|_{L^\infty(G)} = \|\phi\|_{L^\infty(\partial G)}$.

Proof. Arbitrarily pick a boundary vertex v_0. We can define the distance between two vertices v and v' as the minimum path length between v and v'. Let G_n denote the subgraph of G consisting of all vertices v with distance less than or equal to n from v_0 and let an edge be between two vertices in G_n there is a corresponding edge in G. We can make G_n into an electrical network by setting $\partial G_n = (\partial G \cap G_n) \cup (G_n \setminus G_{n-1})$ and defining a conductivity function on G_n to be just the restriction of γ onto G_n. We note that G_n is a finite graph. Define the function $\phi_n : \partial G_n \to \mathbb{R}$ as $\phi_n(v) = \phi(v)$ if $v \in \partial G \cap G_n$ and set $\phi_n(v) = 0$ if $v \in \partial G_n \setminus \partial G$. Using basic theory, the Dirichlet problem has a unique solution on finite graphs, so there is a function $u_n : G_n \to \mathbb{R}$ such that $u_n|_{\partial G_n} = \phi_n$ and

u_n is γ harmonic on $\operatorname{int} G_n$. We can extend u_n to all of G by setting u to be zero outside of G_n. We note that $||u_n||_{L^\infty(G)} \leq ||\phi||_{L^\infty(G)}$ and hence $\{u_n\}$ is a pointwise bounded sequence. Now order the vertices of G arbitrarily as the sequence $\{v_i\}$. Since u_n is bounded, we can find a subsequence $u_{k_n^1}$ such that $u_{k_n^1}(v_1)$ converges. Now find a subsequence $\{u_{k_n^2}\}$ of this last sequence such that $u_{k_n^2}(v_2)$ converges. Repeat this process to get a chain of subsequences such that $u_{k_n^j}(v_j)$ converges. Now consider the diagonal subsequence $u_{k_j^j}$. This converges pointwise at every vertex, say to a function u. Now, since $u_{k_j^j}$ is γ-harmonic on $\operatorname{int} G_\ell$ for all $\ell \leq k_j$ and every interior vertex of G is contained in G_n for all n sufficiently large, we know that u will be γ-harmonic on G, and furthermore will obviously take the right boundary values. Finally, we note that since u has the right boundary values $\|u\|_{L^\infty(G)} \geq \|\phi\|_{L^\infty(\partial G)}$. On the other hand, since $||u_n||_{L^\infty(G)} \leq ||\phi||_{L^\infty(G)}$, in the limit we will have $||u||_{L^\infty(G)} \leq ||\phi||_{L^\infty(G)}$ and hence combining the two inequalities yields $||u||_{L^\infty(G)} \leq ||\phi||_{L^\infty(G)}$. □

1.3.2 Lack of Uniqueness

Unfortunately, uniqueness is in general not true, even if we only consider bounded functions. We might hope it would be true, since for instance in the half plane, harmonic functions that are zero on the real axis are zero on the upper half plane if we assume that they are bounded. The reader is advised to consider the case of an infinite string of conductors with a single conductor as a boundary vertex. By making the conductors have conductance 2^n say, we can get γ-harmonic voltages of $0, 1/2, 3/4, 7/8, \ldots$ which are bounded (the 0 voltage corresponds to the boundary vertex). Thus there's no hope of having uniqueness in this case without restrictions on γ. At the present time there are no known restrictions on γ to give uniqueness in the case of $u \in L^\infty$.

1.3.3 More on $L^p(G)$ spaces

It turns out that uniqueness is pretty easy easy in if we assume all of our functions are in L^p so that things go to zero as we move "far" into the graph, but the existence is much harder and we don't have any useful results. We will assume that the reader is somewhat familiar with measure spaces, and all measures will be assumed to be positive.

1.3.4 Some Real Analysis

Here we develop a bit of machinery and terminology. We note that by Lemma 1.2.4, we can make G into a σ-finite measure space by simply putting the counting measure with weight 1 on each vertex. The set of measurable sets is just $\mathcal{P}(V(G))$, the set of all subsets of $V(G)$ (the vertices of G). We begin with some useful but extremely basic remarks.

Lemma 1.3.3. Let (X, M, μ) be a measure space and let $E_1 \subseteq E_2 \subseteq \cdots$ be an increasing sequence of measurable sets.. If $f \in L^p(X)$, then $\int_{E_j \setminus E_{j-1}} |f|^p \to 0$.

Proof. This is super trivial. Define $F_1 = E_1$ and let $F_n = E_n \setminus E_{n-1}$ for $n \geq 2$. Note that the collection $\{F_n\}$ is a pairwise disjoint collection of sets and hence, by the definition of a measure we have that

$$\int_E |f|^p = \sum_j \int_{F_j} |f|^p \leq \int |f|^p < \infty$$

and since each $\int_{F_j} |f|^p$ is nonnegative, we have absolute convergence and hence the summands must tend to zero as $j \to \infty$. □

Lemma 1.3.4. In the case of G, which we treat as a measure space with the counting measure, we have that $\|f\|_{L^\infty} \leq \|f\|_{L^p}$.

This is obvious since the weights on each vertex are 1.

1.3.5 First Results for $L^p(G)$

Theorem 1.3.5 (Maximum Principle for Infinite Graphs)**.** If u is γ-harmonic on Γ and $u \in L^p(G)$, then $\|u\|_{L^\infty(G)} = \|u|_{\partial G}\|_{L^\infty(\partial G)}$.

Proof. This is basically just an application of the maximum principle for finite graphs. Let $u \in L^p(G)$ for $1 \leq p < \infty$ and let ϕ denote $u|_{\partial G}$. Pick a vertex $v \in G$ arbitrarily and let $G_n = G_n(v)$ be the finite electrical network as defined in our discussion of L^∞. Let ∂G_n and $\operatorname{int} G_n$ also be as defined above. Note that $G_1 \subseteq G_2 \subseteq \cdots$ is an increasing sequence of sets. Hence by Lemma 1.3.3 we know that

$$\int_{G_{n+1} \setminus G_n} |u|^p \to 0.$$

By Lemma 1.3.4 this implies that $\|u\|_{L^\infty(G_{n+1} \setminus G_n)} \to 0$. Since u is γ-harmonic on G_n, by the maximum principle for finite graphs, we know that

$$\|u\|_{L^\infty(G_n)} \leq \|u\|_{L^\infty(\partial G_n)}. \tag{1.1}$$

We recall that by the definition of ∂G_n, we have that

$$\partial G_{n+1} = (\partial G \cap G_{n+1}) \cup (G_{n+1} \setminus G_n) \subseteq \partial G \cup (G_{n+1} \setminus G_n).$$

Hence

$$\|u\|_{L^\infty(\partial G_n)} \leq \|u\|_{L^\infty(\partial G)} + \|u\|_{L^\infty(G_{n+1} \setminus G_n)}. \tag{1.2}$$

Clearly $\|u\|_{L^\infty(G_n)} \to \|u\|_{L^\infty(G)}$. Combining equations (1.1) and (1.2) we get that

$$\|u\|_{L^\infty(G_n)} \leq \|u\|_{L^\infty(\partial G)} + \|u\|_{L^\infty(G_{n+1} \setminus G_n)}.$$

Letting $n \to \infty$ and using the various results about convergence of various terms shows that

$$\|u\|_{L^\infty(G)} \leq \|u\|_{L^\infty(\partial G)}.$$

Since the reverse inequality is trivial, we have equality in the above expression. □

Theorem 1.3.6. Let $u_1, u_2 \in L^p(\partial G)$ for $1 \le p < \infty$ and $u_1|_{\partial G} = u_2|_{\partial G}$. Then $u_1 = u_2$.

Proof. Consider the function $h = u_1 - u_2$. Note that h is γ-harmonic, $h \in L^2$ and $h = 0$ on ∂G. By the maximum principle for infinite graphs, we know that $\|h\|_{L^\infty(G)} \le \|h\|_{L^\infty(\partial G)} = 0$ and hence $h = 0$ on G so $u_1 = u_2$. □

1.3.6 Existence Theorems for $L^p(G)$

In general, we do not have existence in $L^p(G)$, for example take an infinite series of conductors with unit conductivity on each edge and a single boundary vertex (the exact configuration of these conductors is not that important, they can extend in both directions from the boundary vertex or they can extend in only one direction). Set voltage 1 on the boundary. Clearly this is in $L^p(G)$ for all p. But for $p \neq \infty$, we readily see that no L^p solution will exist, since every nonconstant γ-harmonic function will be unbounded while the constant voltage $\phi = 1$ will not be in L^p.

1.4 The Spaces of Finite Power Functions

It turns out the most natural condition to consider for functions on an infinite graph is the space of voltages that satisfy finite power. This turns out to be much more natural than L^p spaces since it takes into account the conductivities better than it seems is possible for L^p spaces. It turns out that there are sort of two dual finite power spaces. There is the space of finite power voltage functions, and there is the space of finite power current functions, which are basically be functionally dual to each other.

1.4.1 Finite Power Voltage Functions

Definition 1.4.1. If Γ is an infinite resistor network and ϕ is a real valued vertex function, we define the power of ϕ to be

$$P(\phi) \stackrel{\text{def}}{=} \sum_{v \in G} \sum_{v' \sim v} \gamma_{vv'}(\phi(v) - \phi(v'))^2.$$

We note that by convention $\gamma_{vv'} = 0$ iff $v \not\sim v'$ and hence we can write the above sum as

$$\sum_{(v,v') \in V \times V} \gamma_{vv'}(\phi(v) - \phi(v'))^2$$

or just

$$\sum_{V \times V} \gamma_{vv'}(\phi(v) - \phi(v'))^2$$

for brevity.

We note that there is no reason to assume that $P(\phi)$ is finite for a particular ϕ or even a γ harmonic ϕ.

Definition 1.4.2. We define $F(\Gamma)$ to be the set of real valued vertex functions on Γ of finite power.

Lemma 1.4.3. *$F(\Gamma)$ is a vector space over $\mathbb{R}$.*

Proof. Clearly $F(\Gamma)$ is closed under multiplication by scalars. That $F(\Gamma)$ is closed under addition is just the triangle inequality for L^2 since

$$\sqrt{P(f+g)} = \left(\sum_{V \times V} \gamma_{vv'} (f(v) + g(v) - f(v') - g(v')^2 \right)^{1/2}$$

$$= \left(\sum_{V \times V} \left(\sqrt{\gamma_{vv'}} (f(v) - f(v')) + \sqrt{\gamma_{vv'}} (g(v) - g(v') \right)^2 \right)^{1/2}$$

$$\leq \left(\sum_{V \times V} (\sqrt{\gamma_{vv'}} (f(v) - f(v'))^2 \right)^{1/2} + \left(\sum_{V \times V} (\sqrt{\gamma_{vv'}} (g(v) - g(v'))^2 \right)^{1/2}$$

$$= \sqrt{P(f)} + \sqrt{P(g)}.$$

□

It turns out that we can in some sense solve the Dirichlet problem in $F(\Gamma)$, but that $F(\Gamma)$ isn't quite the right space to look at. We note that adding a constant to every vertex does not change the power, and that the constant function has 0 power, this leads us to make the following definition:

Definition 1.4.4. We define the space $Z(\Gamma)$ to be $F(\Gamma)/\text{span}\{1\}$ where 1 denotes the constant function 1.

We note that the power function is well defined on $Z(\Gamma)$. We recall that power was almost a norm on $F(\Gamma)$ but on $Z(\Gamma)$ it turns out to be a norm:

Theorem 1.4.5. *$Z(\Gamma)$ is a Hilbert space with inner product given by*

$$(f, g) = \sum_{v,v'} \gamma_{vv'} (f(v) - f(v'))(g(v) - g(v')).$$

Proof. The only claim that requires justification is that $Z(\Gamma)$ with the inner product above is Cauchy complete. We use the standard theorem that says a normed vector space is complete iff $\sum_{\mathbb{N}} \|f_n\| < \infty$ implies $\sum_{\mathbb{N}} f_n$ exists as a limit in the norm topology (Theorem 5.1 of Folland). Thus suppose that $\sum_{\mathbb{N}} \sqrt{P(f_n)} < \infty$. We first claim that we get pointwise convergence in a certain sense. Let v_0 be an arbitrary vertex in G and pick representatives of f_n in $F(\Gamma)$ such that $f_n(v_0) = 0$. We first claim that $\sum_n |f_n(v)| < \infty$ for all $v \in G$. This part of the proof will be by induction. Suppose the claim holds for all vertices of distance k or less from v_0 and consider a vertex v_{k+1} of distance exactly $k+1$. Let $v_0 v_1 \dots v_k v_{k+1}$ be a path from v_0 to v_{k+1}. So by assumption

$$\sum_{n \in \mathbb{N}} |f_n(v_{k+1})| < \infty. \tag{1.3}$$

Since $\sum_{n\in\mathbb{N}}\sqrt{P(f_n)}<\infty$, we in particular have that

$$\sqrt{\gamma_{v_kv_{k+1}}}|f_n(v_k)-f_n(v_{k+1})|\leq\sqrt{P(f_n)}$$

and hence

$$\sum_{n\in\mathbb{N}}|f_n(v_k)-f_n(v_{k+1})|<\infty.$$

Applying the triangle inequality shows that

$$\sum_{n=1}^{N}|f_n(v_{k+1})|-\sum_{n=1}|f_n(v_k)|\leq\sum_{n=1}^{\infty}|f_n(v_k)-f_n(v_{k+1})|.$$

Letting $N\to\infty$ and using equation (1.3), we see that $\sum_{n\in\mathbb{N}}|f_n(v_{k+1})|<\infty$. By induction on k we thus know that $\sum_n f_n(v)$ converges absolutely for each v. Denote the limiting function by f. Firstly, we note that $f\in F(\Gamma)$, since an application of Fatou's lemma shows that

$$\sqrt{P(f)}=\left(\sum_{V\times V}\gamma_{vv'}(f(v)-f(v'))^2\right)^{1/2}$$

$$\leq\lim_{n\to\infty}\left(P\left(\sum_{i=1}^{n}f_i\right)\right)^{1/2}\leq\sum_{i=1}^{\infty}\sqrt{P(f_i)}.$$

To see that $\sum_{n=1}^{N}f_n$ converges to f in the power norm, we will apply the dominated convergence theorem. Firstly, let $G_N(v,v')=\sum_{i=1}^{N}|f_n(v)-f_n(v')|$ and let $G(v,v')=\sum_{i=1}^{\infty}|f_n(v)-f_n(v')|$. We note that G is finite everywhere by the triangle inequality and the fact that $\sum_{n\in\mathbb{N}}|f_n(v)|<\infty$ for all $v\in G$. We first claim that $\sum_{v,v'}\gamma_{vv'}(G(v,v'))^2$ is finite. To see this, note that G_n is pointwise nondecreasing in n and

$$\left(\sum_{V\times V}\gamma_{vv'}\left(\sum_{i=1}^{N}|f_n(v)-f_n(v')|\right)^2\right)^{1/2}\leq\sum_{i=1}^{N}\left(\sum_{V\times V}\gamma_{vv'}(f_n(v)-f_n(v')^2\right)^{1/2}$$

by the triangle inequality. By assumption the latter sum is finite (since it is just $\sum_{\mathbb{N}}\sqrt{P(f_n)}$) and hence by the Monotone Convergence Theorem, we know that

$$\sum_{v,v'}\gamma_{vv'}(G(v,v'))^2=\lim_{n\to\infty}\sum_{v,v'}\gamma_{vv'}(G_n(v,v'))^2<\infty. \tag{1.4}$$

We use the estimate $(a+b)^2\leq 4(a^2+b^2)$ to get that

$$\gamma_{vv'}\left|(\sum_{i=1}^{N}f_n(v)-f_n(v'))-\sum_{i=1}^{\infty}(f_n(v)-f_n(v'))\right|^2$$

$$\leq 4\gamma_{vv'} \left|\sum_{i=1}^{N}(f_n(v) - f_n(v'))\right|^2 + \left|\sum_{i=1}^{\infty}(f_n(v) - f_n(v'))\right|^2$$

$$\leq 4\gamma_{vv'} \left(\sum_{i=1}^{N}|f_n(v) - f_n(v')|\right)^2 + \left(\sum_{i=1}^{\infty}|f_n(v) - f_n(v')|\right)^2$$

$$\leq 8\gamma_{vv'}G(v,v')^2$$

which is in $L^1(V \times V)$ by the estimate on in (1.4). Thus by the dominated convergence theorem, we know that

$$P\left(\sum_{i=1}^{N} f_i - \sum_{i=1}^{\infty} f\right) = \sum_{V\times V}\gamma_{vv'}\left|(\sum_{i=1}^{N} f_n(v) - f_n(v')) - \sum_{i=1}^{\infty}(f_n(v) - f_n(v'))\right|^2 \to 0$$

since the integrand goes to zero pointwise and is pointwise bounded by $8\gamma_{vv'}G(v,v')^2 \in L^1(V \times V)$. Hence $Z(\Gamma)$ is Cauchy complete. □

1.4.2 Interlude about Hilbert Spaces

We now need to present some basic machinery about Hilbert spaces in order to discuss the space of finite power functions. Many of the existence and uniqueness theorems that we will state and prove can be proven (and indeed were initially proven) without using the language of Hilbert spaces, but it turns out that by introducing some basic machinery, we can simplify many of the proofs significantly. We begin with a basic result, the proof of which is left as a reference.

Lemma 1.4.6 (from pg 175 of [3])**.** If M is a closed subspace of a Hilbert space H then $H = M \oplus M^{\perp}$, that is each $x \in H$ can be uniquely expressed as $x = y + z$ where $y \in M$ and $z \in M^{\perp}$. Moreover, y and z are the unique elements of M and $M^{\perp}$ whose distance to x is minimal.

See Folland's book on Real Analysis for the proof, though the theorem is extremely standard in the subject of Hilbert spaces

Lemma 1.4.7. If H is a Hilbert space and M is a closed affine subspace, then there is a unique element x of M such that $\|x\|$ is minimal in M.

Proof. The proof follows essentially from the last lemma, in fact only from the last part of the last lemma. By an affine subspace, we mean that $M - y$ is a linear subspace for some $y \in H$. By the Lemma, there is a unique $x \in M - y$ such that the distance from x to $-y$ is minimal in $M - y$. Hence $\|x + y\|$ is minimal over $M - y$. But $x = z - y$ where $z \in M$, but $\|x + y\| = \|z\|$ so clearly $\|z\|$ is minimal over M, as we wanted. Also z is clearly the unique minimizer. □

1.4.3 More on Finite Power Voltage Functions

We are now in a position to state several theorems about existence and uniqueness of the Dirichlet problem.

Definition 1.4.8. We will denote the set of functions in $Z(\Gamma)$ which are constant on ∂G by W.

Lemma 1.4.9. The set W is a closed subspace of $Z(\Gamma)$.

Proof. Obviously W is a subspace, so it is only necessary to show that W is closed. We note that convergence of a sequence in $Z(\Gamma)$ implies pointwise convergence of that sequence (in the sense that we can pick representatives of the sequence which converge pointwise) as was shown in the proof that $Z(\Gamma)$ was Cauchy complete. Hence the property of being constant on the boundary will be preserved under limits. Hence W is a closed subspace. $\square$

Lemma 1.4.10. Let $\phi \in Z(\Gamma)$. Then there is a function $u \in Z(\Gamma)$ such that $P(u)$ is minimum over all functions $f \in Z(\Gamma)$ such that $f|_{\partial G} = \phi|_{\partial G}$.

Proof. Since $\phi + W$ is a closed affine subspace, we apply Lemma 1.4.7 to see that there is a unique element u of $\phi + W$ of minimal norm and furthermore, we have that $u|_{\partial G} = \phi|_{\partial G}$. Lastly we note that every function $f \in Z(\Gamma)$ that agrees with ϕ on the boundary will be in $\phi + W$, so the statement of the theorem follows. $\square$

We now show that these minimal solutions are γ-harmonic.

Lemma 1.4.11. Suppose that $\phi \in Z(\Gamma)$ and that u is of minimal norm in $\phi + W$. Then u is γ-harmonic on $\operatorname{int} G$.

Proof. If ϕ is of minimal norm in $\phi + W$ then by basic facts about Hilbert Spaces, since W is a closed subspace of $Z(\Gamma)$ we have that $\phi \in W^{\perp}$. In particular, the set indicator function χ_v is in W for all $v \in \operatorname{int} G$. Hence $(\phi, \chi_v) = 0$, but a trivial computation shows that

$$(\phi, \chi_v) = 2 \sum_{v' \sim v} \gamma_{vv'}(\phi(v) - \phi(v')),$$

and since this is zero for all $v \in \operatorname{int} G$ we know that ϕ is γ-harmonic. $\square$

Theorem 1.4.12. If $\phi \in Z(\Gamma)$, then there is a unique element $u \in Z(\Gamma)$ such that u is minimal in power in $\phi + W$. Furthermore, u is γ-harmonic on $\operatorname{int} G$.

This is literally just a restatement of the previous two theorems. We note that this is in some ways an existence and uniqueness result for the Dirichlet problem. But we should note that it is not a solution as we may want. This gives us a map from valid boundary voltages to a currents coming from a minimal power solution, but in general, there may be other finite power γ-harmonic functions with the same boundary voltages but different currents leaving the boundary.

Remark 1.4.13. The Dirichlet problem as it is often formulated in the finite case is ill posed, even in $Z(\Gamma)$. In particular, functions in $Z(\Gamma)$ with the same boundary voltages may have different current boundary currents.

We should note that we only have a unique *energy minimizing* solution and not a unique γ-harmonic solution. In fact, even the assumption of finite power does not imply that there is a unique finite power γ-harmonic function with given boundary voltages. An example is an infinite string of conductors in series, with conductance such that $\sum_{\mathbb{Z}} 1/\gamma^2 < \infty$, and a single boundary vertex in the middle.

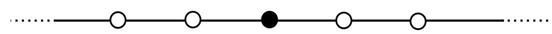


Figure 1.1: The infinite series of conductors discussed in Remark 1.4.13.

Consider the voltage function with constant current 1 flowing constantly the right on the entire graph. This will give us a finite power voltage function on the entire graph since the power can be rewritten as $\sum I/\gamma^2 = 1 \sum 1/\gamma^2$. There is no current flowing out of the network with this voltage. Note that we can alter the above example to have constant 1 current flowing to the right, on the left side of the graph, and then have constant 1 current leaving the boundary vertex, and no current flowing on the right side of the graph. These are two γ-harmonic finite power solutions with different boundary currents. This fact will turn out to be important later: *there is not in general a Dirichlet map from boundary voltages to boundary currents.*

1.4.4 Characterizing Minimal Power Solutions

We continue with our discussion of the the minimal power solutions of $Z(\Gamma)$. We note that if $\phi \in Z(\Gamma)$, by the above theorem, we can find a $u \in Z(\Gamma)$ such that u has minimal energy and $u|_{\partial G} = \phi|_{\partial G}$. Thus given a $u \in Z(\Gamma)$ we can determine whether it has the property of having minimal power with respect to its boundary conditions, i.e. whether u is the element of least norm in the space $u + W$.

Definition 1.4.14. The set of $u \in Z(\Gamma)$ which have minimal power will be denoted by $M(\Gamma)$.

Lemma 1.4.15. $M(\Gamma)$ is a subspace of $Z(\Gamma)$.

Proof. The proof is remarkably straightforward, and extremely enlightening. Obviously scaling by constants doesn't change the property of being minimal. We note that the minimal power functions correspond exactly to minimal norm elements in closed affine subspaces of $Z(\Gamma)$. To see this, note that if $f \in M(\Gamma)$ then f has minimal norm over all elements u in $Z(\Gamma)$ such that $f-u$ is supported only in the interior of G (or to be more exact is constant on ∂G). Recall that the space of functions that are constant on the boundary is denoted by W. Then

f is minimal iff f has minimal norm in $f + W$, but this happens iff $f \in W^\perp$. We note that $f, g \in M(\Gamma)$ iff $f, g \in W^\perp$ which implies that $f + g \in W^\perp$ which occurs iff $f + g \in M(\Gamma)$ so $M(\Gamma)$ is a subspace. $\square$

As a corollary of the proof of the previous theorem, we have that

Corollary 1.4.16. If W denotes the subspace of $Z(\Gamma)$ consisting of elements which are constant on ∂G, then $M(\Gamma) = W^\perp$. Since W is closed we know that $W^\perp = M(G)$ is a closed subspace.

1.4.5 Finite Power Current Functions

We will discuss the Neumann problem in much the same way that we discussed the Dirichlet problem above. We will define a space of finite power current functions, which will turn out to be a Hilbert space,

Definition 1.4.17. Let $\phi \in F(\Gamma)$. Then we define the function $I(\phi)$ to be the real valued directed edge function defined by

$$I(\phi)(vv') = \gamma_{vv'}(\phi(v) - \phi(v')).$$

Definition 1.4.18. We define a current function I on the directed edges of a graph to be a function such that $I(vv') = -I(v'v)$ and such that $\sum_{v' \sim v} I(vv') = 0$ for all $v \in \operatorname{int} G$.

We note that $I(\phi)$ is a current function iff ϕ is γ-harmonic on $\operatorname{int} G$.

Definition 1.4.19. We recall the definition of the map $K : Z(\Gamma) \to \mathbb{R}^G$. We define analogously the map $\widetilde{K} : F(E) \to \mathbb{R}^G$ define by

$$\widetilde{K}(I)(v) = \sum_{v' \sim v} I(vv').$$

Definition 1.4.20. We define the power of a current function I to be

$$\sum_{vv' : \gamma_{vv'} \neq 0} \frac{I(vv')^2}{\gamma_{vv'}}.$$

Definition 1.4.21. We define the space of finite power current $F(E)$ to the be set of current functions with finite power.

Lemma 1.4.22. $F(E)$ is a Hilbert space with inner product

$$(I_1, I_2) = \sum_{v, v' : \gamma_{v,v'} \neq 0} \frac{I_1(vv') I_2(vv')}{\gamma_{v,v'}}.$$

Proof. As with the case of voltages, the only point worth mentioning is Cauchy completeness. The proof of this fact is similar to the proof of Cauchy completeness of L^2 for a general measure space. We observe that if we let $d\#$ be the counting measure on $V \times V$, then the above inner product is exactly the inner product on $L^2(V \times V, \mu)$ where $d\mu = \gamma(v, v')d\#$. Hence to show that $F(E)$ is a Cauchy complete, it is sufficient to show that $F(E)$ is closed in $L^2(V \times V, \mu)$. If I_n is Cauchy, it will converge to a function $\widetilde{I} \in L^2(V \times V, \mu)$. To show that the limiting function is a current function, we note that in general, convergence in L^p for $1 \leq p < \infty$ for any counting measure implies pointwise convergence on sets of positive measure. Hence we will have that $I_n \to \widetilde{I}$ pointwise, which will imply that $\widetilde{I}(vv') = \widetilde{I}(v'v)$ and that the sum of currents coming into a vertex will be zero since it is zero for each I_n. Hence $F(E)$ is a closed subset of a Cauchy complete space and is hence Cauchy complete. □

Theorem 1.4.23. Let $i_0 \in F(E)$. Then there is a unique current function i such that $P(i)$ is minimal over all functions $i' \in F(E)$ such that $\widetilde{K}i' = \widetilde{K}i_0$. Furthermore, there is a unique $u \in Z(\Gamma)$ such that $I(u) = i$.

Proof. Let Y denote the space of finite power current functions i such that $\widetilde{K}i = 0$. We note that i is a closed subspace since convergence in $F(E)$ implies pointwise convergence since the topology on $F(E)$ is the same as the subspace topology given from the topology on $L^2(V \times V)$ under the appropriate counting measure and convergence in L^p under a counting measure (for $1 \leq p < \infty$) implies pointwise convergence since each point has positive measure. But as in the case of $Z(\Gamma)$, by 1.4.7 we know that there is always a unique element of $i_0 + Y$ of minimal norm. Call this element i.

Now we just need to establish that there is a unique $u \in Z(\Gamma)$ such that $I(u) = I$. Uniqueness is fairly straightforward, since if u_1 and u_2 both satisfy $I(u_1) = I(u_2)$ then $I(u_1 - u_2) = 0$ and hence there is no current flowing anywhere for the voltage function $u_1 - u_2$ and hence $u_1 - u_2$ is constant and hence equal to zero in $Z(\Gamma)$. The existence of such a u is harder, but not overly difficult. It relies on an argument made by Will Johnson. It is sufficient to show that if the sum of $I(ij)^2/\gamma_{ij}$ around any loop in G is zero, then such a u will exist. This is clear since if the sum around every loop is zero, then we can just define u by arbitrarily setting u to be zero at a single vertex, and then defining u by extending u to neighboring vertices by defining it to so that the voltage difference yields the desired current. This is well defined if the sum around a loop of the necessary voltage differences is zero.

Let $C = v_0 v_1 v_2 \cdots v_n v_0$ be a loop in G. Let i_C have current have current one along each edge $v_k v_{k+1}$ and have -1 along each edge $v_k v_{k-1}$ (where say $v_{-1} = v_n$ and $v_{n+1} = v_0$). We note that $\widetilde{K}i_C = 0$ and hence $i_C \in Y$. But by 1.4.7 we know that $i \in Y^\perp$ and hence $(i, i_C) = 0$. But, we just observe by explicit computation that

$$(i, i_C) = 2 \sum_{j=0}^{n} \frac{i(v_i v_{j+1})}{\gamma_{v_j v_{j+1}}}$$

which must be zero, exactly as we needed. Hence there is a ϕ such that $I(\phi) = i$. We should also note that clearly $\phi \in Z(\Gamma)$ since $P(\phi) = P(i)$ by definition. $\square$

This formulation of the Neumann problem is somewhat different than in the finite graph case. We should note that something happens which is unexpected:

Remark 1.4.24. There is not a well defined Neumann-to-Dirichlet map in general, even if we restrict ourselves to functions in $Z(\Gamma)$.

In particular, given boundary currents on a network G, even if we know that they came from a voltage function in $Z(\Gamma)$, we cannot say that they came from a unique function. For example, take a countably infinitely many conductors in series as in Figure 1.1 and suppose that the conductivities satsify $\sum_i \frac{1}{\gamma} < \infty$ (for instance take $\gamma_n = n^2$). Take a vertex and arbitrarily set the voltage u to zero. Assume that the current is constant on the entire graph, say 1 to the right. Clearly with this information we can extend u γ-harmonically to the rest of the graph, but we see that the power on the graph is $\sum_{\mathbb{Z}} \frac{I^2}{\gamma} = 1^2 \sum \frac{1}{\gamma} < \infty$! If we set any subset of the graph to be boundary vertices, we note that the current leaving the network is 0 at each boundary vertex, even though the solution is not constant.

1.4.6 Minimal Currents

Just like in the case of minimal power voltage functions in $Z(\Gamma)$, we can consider minimal current functions, and we will analogous results.

Definition 1.4.25. Let $M(E)$ denote the set of minimal current functions in $F(E)$.

Lemma 1.4.26. $M(E)$ is a closed subspace of $F(E)$. Moreover, if Y denotes the set of all current functions i in $F(E)$ with $\widetilde{K}i = 0$, then $M(E)$ is exactly $Y^{\perp}$.

Proof. As in the case of voltage functions, we note that i is minimal iff i has minimal norm over the set of current functions i' such that $\widetilde{K}i' = \widetilde{K}I$. The set of such functions is exactly $i + Y$. Hence i has minimal norm over $i + Y$ iff $i \in Y^{\perp}$. $\square$

1.4.7 Duality of Current and Voltages

We now have two interesting spaces, namely the space of minimal voltage functions $M(\Gamma)$, and the space of minimal current functions $M(E)$. Since all countably infinite dimensional Hilbert spaces are isometrically isomorphic, it is too easy to say that $M(\Gamma) \cong M(E)$ since that is trivially true. They turn out to not be naturally isomorphic under the obvious maps, so instead, we try to see how close these spaces are isomorphic in the *natural* way. We need some definitions.

Definition 1.4.27. Let $H(\Gamma)$ denote the subspace of $Z(\Gamma)$ of functions which are γ-harmonic on $\operatorname{int} G$. Similarly let $L(E)$ denote the subspace of $F(E)$ consisting of current functions such that the sum of the necessary potential differences in any loop is zero.

We note that $M(\Gamma) \subseteq H(\Gamma)$ and $M(E) \subseteq L(E)$. We have natural functions between $L(E)$ and $H(\Gamma)$.

Definition 1.4.28. Let $I : H(\Gamma) \to L(E)$ denote the function which takes a γ-harmonic function in $Z(\Gamma)$ and maps it to the natural current function induced. Similarly let $\Phi : L(E) \to H(\Gamma)$ denote the function which takes a current function with necessary loop sums that are zero and maps it to the unique voltage function that satisfies those currents.

Lemma 1.4.29. Both I and Φ are bijective linear functions. Furthermore both are isometries. Lastly $I = \Phi^{-1}$.

Proof. The fact that they are isometries is trivial. To see that I and Φ are bijective, we just observe that given a function in $I \in L(E)$, we have already shown that there is a unique function $u \in Z(\Gamma)$ such that $I(u) = I$, and hence we clearly have that $I\Phi = id|_{L(E)}$ and $\Phi I = id|_{H(\Gamma)}$ so the claim follows. □

Hence we have the following diagram illustrating the relationship between the respective sets concerning voltages and currents:

$$\begin{array}{ccc} Z(\Gamma) & & F(E) \\ \cup & & \cup \\ H(\Gamma) & \overset{I=\Phi^{-1}}{\longleftrightarrow} & L(E) \\ \cup & & \cup \\ M(\Gamma) & & M(E) \end{array}$$

A natural conjecture would be that I and Φ map minimal elements to minimal elements. This is unfortunately not true. A counterexample to Φ mapping minimal currents to minimal voltages is again offered by an infinite series of conductors with a single boundary vertex in the middle. There will be a minimal current such that the current leaving the system is nonzero, while any minimal voltage will just be constant, and hence there are strictly more minimal currents in this example than minimal voltages. We will later see that I cannot in general map minimal voltages to minimal currents.

We should note that in general, we can imagine evaluating I at voltages which are not in $H(\Gamma)$, but it wouldn't immediately be obvious how to evaluate Φ on edge functions which are not in $L(E)$. We will use Hilbert space duality to get around this problem. We remark that there is a sort of symmetry between current and voltage, which leads us to define the following function.

Definition 1.4.30. We will define the pseudoinner product $G : Z(\Gamma) \times F(E) \to \mathbb{R}$ defined by

$$G(u, i) = \sum_{V \times V} (u(v) - u(v'))i(vv').$$

We note that $G(u,i)$ is always finite since by Cauchy Schwarz we know that

$$\sum_{V\times V} |(u(v)-u(v'))i(vv')| = \sum_{V\times V} \sqrt{\gamma_{vv'}}|(u(v)-u(v'))|\frac{|i(vv')|}{\sqrt{\gamma_{vv'}}}$$
$$\leq \left(\sum_{V\times V} \gamma_{vv'}(u(v)-u(v'))^2\right)^{1/2} \left(\sum_{V\times V} \frac{|i(vv')|^2}{\gamma_{vv'}}\right)^{1/2}$$

which the understanding that the integrand is 0 whenever $v' \not\sim v$. The above computation shows immediately that

$$|G(u,i)| \leq \|u\|_{Z(\Gamma)}\|i\|_{L(E)}. \tag{1.5}$$

We now recall the Riesz Representation theorem:

Theorem 1.4.31. Let H be a Hilbert space and let $f : H \to \mathbb{R}$ be a bounded linear functional. Then there is a unique $y_f \in H$ such that $f(x) = (y_f, x)$ for all $x \in H$. Furthermore, $\|f\|_{\sup} = \|y_f\|$.

Thus we have the following result:

Lemma 1.4.32. The function G defined above yields well defined, bounded linear functions $\widetilde{I}$ and $\widetilde{\Phi}$ defined on $Z(\Gamma)$ and $F(E)$ respectively such that $\widetilde{I}|_{H(\Gamma)} = I$ and similarly $\widetilde{\Phi}_{L(E)} = \Phi$.

Proof. For a fixed $u \in Z(\Gamma)$, set $f_u(i) = G(u,i)$. By (1.5) we know that f_u is a bounded linear functional and hence by the Riesz Representation theorem we know that there is a unique $i_u \in F(E)$ such that $f_u(i) = (i_u, i)$. Define $\widetilde{I}(u) = i_u$. This is a well defined function by the uniqueness of the Riesz Representation theorem. Also, from the uniqueness of the Riesz representation theorem we see that $\widetilde{I}$ is linear. By the bound given by the Riesz representation theorem we know that $\|\widetilde{I}\| \leq 1$. If $u \in H(\Gamma)$, then we just observe that $G(u,i) = (Iu,i)_{F(E)}$ and hence by uniqueness of the Riesz Representation theorem we know that $Iu = i_u = \widetilde{I}(u)$. The exact same analysis works for $\widetilde{\Phi}$. $\square$

Lemma 1.4.33. We have that $\ker \widetilde{I} \subseteq W$. Similarly we have that $\ker \widetilde{\Phi} \subseteq Y$.

Proof. We can see this fairly easily just by looking at appropriate test functions. Suppose that $u \in \ker \widetilde{I}$, i.e. $G(u,i) = 0$ for all $i \in F(E)$. Let v_1 and v_n be boundary vertices such that $C = v_1 v_2 \cdots v_n$ is a path from v_1 to v_n. We define the current function I_C to be just 1 along each edge $v_j v_{j+1}$ and to be -1 along each edge $v_i v_{i-1}$ and we observe that I_C is clearly in $F(E)$ since the sum of the currents at any interior vertex is 0 and I_C is finitely supported. Hence $G(u, I_C) = 0$. But we observe that

$$0 = G(u, I_C) = 2(u(v_0) - u(v_n))$$

and hence u is constant on all boundary vertices and hence in W.

Now for the other claim, suppose that $i \in \ker \widetilde{\Phi}$, i.e. that $G(u,i) = 0$ for all $u \in Z(\Gamma)$. The claim is also not that difficult. Just consider the set indicator function $u = \chi_{\{v\}}$ where v is any vertex. We simply note that

$$0 = G(u,i) = 2 \sum_{v \sim v'} u(v) - u(v')$$

and hence i has zero current sums. This holds on the boundary, and hence $i \in Y$.

$\square$

We have a stronger result for $\ker \widetilde{I}$ and $\ker \widetilde{\Phi}$.

Lemma 1.4.34. We have that $\ker \widetilde{I} \subseteq H(\Gamma)^{\perp}$ and similarly $\ker \widetilde{\Phi} \subseteq L(E)^{\perp}$.

Proof. We will show only the first claim since the argument for the second is identical. Let $\phi \in \ker \widetilde{I}$. Then $G(\phi, i) = 0$ for all $i \in F(E)$. *A fortiori* we know that $G(\phi, i) = 0$ for $i \in L(E)$. Using the fact that I is a bijection from $L(E)$ to $H(\Gamma)$, we have that if $u \in H(\Gamma)$ then $0 = G(\phi, Iu) = (\phi, u)_{Z(\Gamma)}$. Since this holds for all $u \in H(\Gamma)$, we have that $\phi \in H(\Gamma)^{\perp}$. $\square$

Lemma 1.4.35. The map $\widetilde{I}$ maps $H(\Gamma)^{\perp}$ into $L(E)^{\perp}$ and similarly $\widetilde{\Phi}$ maps $L(E)^{\perp}$ in $H(\Gamma)^{\perp}$.

Proof. If $\phi \in H(\Gamma)^{\perp}$ then $(\phi, u)_{Z(\Gamma)} = 0$ for all $u \in H(\Gamma)^{\perp}$. Hence $G(\phi, Iu) = 0$. Since I maps $H(\Gamma)$ onto $L(E)$, we know that $(\widetilde{\Phi}, \phi i) = G(\phi, i) = 0$ for all $i \in L(E)$ so $\widetilde{\Phi}\phi \in L(E)^{\perp}$. The other claim follows from an identical argument. $\square$

Remark 1.4.36. The spaces $H(\Gamma)$ and $L(E)$ are *naturally* functionally dual to each other. We get this by noting that

$$(u_1, u_2)_{Z(\Gamma)} = G(I(u_1), u_2) = G(u_1, I(u_2)) = (I(u_1), I(u_2))_{F(E)}$$

and similarly

$$(i_1, i_2)_{F(E)} = G(\Phi(i_1), i_2) = G(i_1, \Phi(i_2)) = (\Phi(i_1), \Phi(i_2))_{Z(\Gamma)}.$$

1.4.8 Dirichlet to Neumann and Neumann to Dirichlet maps

In a certain sense we have both Dirichlet-to-Neumann and Neumann-to-Dirichlet maps. We have maps which send vaild boundary data to the data from unique minimal functions. We have shown that these unique minimal functions are not in general the unique finite power γ-harmonic (resp. loop sum zero) voltage or current functions. Thus we don't in general have a Dirichlet-to-Neumann map or a Neumann to Dirichlet map in the traditional sense. We will create several definitions to make discussing these issues easier:

Definition 1.4.37. We define the map which sends valid boundary voltages to the boundary currents of the unique power minimizing function with those boundary values to be Λ_M and similarly we call H_M the function which sends valid boundary currents to the boundary voltages of the unique minimal power function with those currents. We call these maps the **minimal boundary data maps**. We call Λ_M the minimal Dirichlet-to-Neumann map and similarly for H_M.

Definition 1.4.38. If there exist maps which take valid boundary data to unique valid dual boundary data, we call the appropriate maps the **harmonic boundary data maps**.

The harmonic boundary data maps are the closest analogue traditional Neumann-to-Dirichlet maps as we have in the finite case. We now prove a theorem that gives conditions for when we could have such maps.

Theorem 1.4.39. If the function I maps $M(\Gamma)$ into $M(E)$, then there exists a well defined harmonic Neumann-to-Dirichlet map on finite power functions. If the function Φ maps $M(E)$ into $M(\Gamma)$ then there is a well defined harmonic Dirichlet-to-Neumann map on finite power functions.

Proof. First suppose that I maps $M(\Gamma)$ into $M(E)$ and let $i_0 \in L(E)$ be a current function that has zero boundary values, i.e. $i_0 \in Y$. Then $(i, i_0) = 0$ for all $i \in M(E)$ since $M(E)^\perp = Y$. But then, since I maps $M(\Gamma)$ into $M(E)$ and I is a bijection from $H(\Gamma)$ to $L(E)$, we know that for all $u \in M(\Gamma)$, there is an $i \in M(E)$ such that $\Phi(i) = u$, and hence since Φ is an isometry, we know that $(i, i_0) = (u, \Phi(i)) = 0$. Since this holds for all $u \in M(\Gamma)$ we know that $\Phi(i) \in W = M(\Gamma)^\perp$. But, this means that if a finite power function has zero current on the boundary, then it has constant voltage, so that the harmonic Neumann-to-Dirichlet map is well defined. The other direction, i.e. showing that if Φ maps $M(E)$ into $M(\Gamma)$ then there is a well defined harmonic Dirichlet-to-Neumann map on power functions, is proved just by switching the appropriate symbols. □

Corollary 1.4.40. In general, we don't have that $M(\Gamma)$ is mapped into $M(E)$ by I or that $M(E)$ is mapped into $M(\Gamma)$ by Φ since in general we don't have well defined harmonic boundary data maps.

Remark 1.4.41. If G is finite, then the spaces $H(\Gamma)$ and $M(\Gamma)$ correspond because there is a unique solution for given boundary data. Similarly $L(E)$ and $M(E)$ correspond.

We now analyze Λ_M and H_M a bit further. Perhaps our discussion is a bit out of order, but we now define the spaces of relevant boundary information.

Definition 1.4.42. Let $\Omega(V)$ be the vector space of valid boundary voltages and let $\Omega(E)$ denote the vector space of valid boundary currents (as sets we can view both as subspaces of the rather large space $\mathbb{R}^V$).

The maps Λ_M and H_M are defined between these spaces.

1.4.9 Miscellanea

Here we present some basic facts which are analogues of results we have in finite resistor networks. Let $\Lambda : Z(\Gamma) \to M(\Gamma)$ denote the map which sends a finite power voltage function ϕ to the unique minimal function u which shares the same boundary values as ϕ.

Theorem 1.4.43. The map Λ is self adjoint.

Proof. Suppose that $f, g \in Z(\Gamma)$. Note that $f - \Lambda f, g - \Lambda g \in W = M(\Gamma)^{\perp}$. Hence

$$(f, \Lambda g) = (f + (\Lambda f - f), \Lambda g) = (\Lambda f, \Lambda g) = (\Lambda f, \Lambda g + (g - \Lambda g)) = (\Lambda f, g)$$

and hence by definition Λ_M is self adjoint. □

Now we have an easy computation that follows from the same reasoning as above.

Proposition 1.4.44. If $f \in Z(\Gamma)$, then $P(\Lambda f) = (f, \Lambda f)$.

Proof. Just note

$$P(\Lambda f) = (\Lambda f, \Lambda f) = (\Lambda f + (f - \Lambda f), \Lambda f) = (f, \Lambda f).$$

□

1.5 The Space of Finitely Supported Functions

Here we analyze the space of finitely supported functions in both $H(\Gamma)$ and $L(E)$.

Definition 1.5.1. We define the space of finitely supported γ-harmonic functions to be $H_f(\Gamma)$ and the space of finitely supported minimal functions to be $M_f(\Gamma)$. Similarly we define the space of finitely supported current loop sum zero current functions to be $L_f(E)$ and we define the space of finitely supported minimal current functions to be $M_f(E)$.

Lemma 1.5.2. We have that $H_f(\Gamma) = M_f(\Gamma)$ and $L_f(E) = M_f(E)$.

Proof. Obviously $M_f(\Gamma) \subseteq H_f(\Gamma)$ and $M_f(E) \subseteq L_f(E)$ by previous theorems. We now show that $H_f(\Gamma) \subseteq M_f(\Gamma)$ Suppose v_0 is an arbitrary vertex in G and suppose that ϕ is constant outside of $G_n(v_0)$. In particular, we know ϕ is constant across any edge that is not in $G_n(v_0)$. We need to show that $\phi \in W^{\perp}$, so let $w \in W$. Considering the network Γ_{n+2} with graph defined by $G_{n+2}(v_0)$ where the boundary vertices are $(G_{n+2} \setminus G_{n+1}) \cup (\partial G \cap G_{n+2})$. We observe that ϕ is γ-harmonic on $G_{n+2}(v_0)$ and hence is in $W(\Gamma_{n+2})^{\perp}$. There is a canonical map $r_{n+2} : Z(\Gamma) \to Z(\Gamma_{n+2})$ defined by restricting a function on $V(G)$ to a function on $V(G_{n+2})$ (if $u \in Z(\Gamma)$ denote its restriction to $Z(\Gamma_{n+2})$ by u_{n+2}).

Noting that ϕ has zero voltage difference across any edge that is not in G_{n+1}, so in particular for all the edges in $G_{n+2} \setminus G_{n+1}$, we note that

$$(\phi, w)_{Z(\Gamma)} = \sum_{V \times V} \gamma_{vv'}(\phi(v) - \phi(v'))(w(v) - w(v'))$$

$$= \sum_{G_{n+1} \times G_{n+1}} \gamma_{vv'}(\phi(v) - \phi(v'))(w(v) - w(v'))$$

$$= (\phi_{n+1}, w_{n+1})_{Z(\Gamma_{n+1})}.$$

But we notice that w_{n+1} is some constant c on $\partial G \cap G_{n+1}$. So we can extend w_{n+1} to be the function w' on G_{n+2} by setting w' to be c on all $G_{n+2} \setminus G_{n+1}$. We note then that $w' \in W(G_{n+2})$ and furthermore

$$(\phi, w)_{Z(\Gamma)} = (\phi_{n+2}, w')_{G_{n+2}} = 0$$

since ϕ_{n+2} is γ-harmonic and γ-harmonic functions correspond to minimal functions on finite graphs and $w' \in W(G_{n+1})$. Hence $\phi \in W^{\perp}$ and hence ϕ is minimal.

The proof that $L_f(E) \subseteq M_f(E)$ is identical. □

Lemma 1.5.3. In general we have that $M_f(\Gamma)$ is mapped in $M_f(E)$ by I but that $M_f(E)$ is only mapped into $M(\Gamma)$ by Φ.

Proof. Clearly $M_f(\Gamma)$ is mapped into $L(E)$ by I. Furthermore, if $u \in M_f(\Gamma)$ is finitely supported, then Iu will be finitely supported so $Iu \in L_f(E)$. By the previous theorem we know that $L_f(E) = M_f(E)$.

Now if $i \in M_f(E)$, suppose that i is supported on some $G_n(v)$. Then in particular, we know that Φi will be constant on each connected component of $G_{n+1}(v) \setminus G_n(v)$. We note that Φi is γ-harmonic on $\operatorname{int} G$. Pick a representative of w such that w is zero on ∂G. If $w \in W$ then we observe that

$$(\Phi i, w) = \sum_{V \times V} \gamma_{vv'} \left[(\Phi i)(v) - (\Phi i)(v')\right](w(v) - w(v'))$$

$$= \sum_{G_{n+1} \times G_{n+1}} \gamma_{vv'} \left[(\Phi i)(v) - (\Phi i)(v')\right](w(v) - w(v'))$$

$$= \sum_{v \in G_{n+1}} \sum_{v' \sim v} \gamma_{vv'} \left[(\Phi i)(v) - (\Phi i)(v')\right](w(v) - w(v')). \tag{1.6}$$

This expression is a finite sum and is linear in w. Let δ_v denote the function which is 1 at v and 0 at all other vertices. We note that equation (1.6) can thus be rewritten as

$$\sum_{\rho \in (\operatorname{int} G) \cap G_n} w(\rho) \sum_{v \in G_{n+1}} \sum_{v' \sim v} \gamma_{vv'} \left[(\Phi i)(v) - (\Phi i)(v')\right](\delta_\rho(v) - \delta_\rho(v')),$$

but we notice that for a fixed ρ we have that

$$\sum_{v \in G_{n+1}} \sum_{v' \sim v} \gamma_{vv'} \left[(\Phi i)(v) - (\Phi i)(v') \right] (\delta_\rho(v) - \delta_\rho(v'))$$

$$2 \sum_{v \sim \rho} \gamma_{v\rho} \left[(\Phi i)(v) - (\Phi i)(v') \right]$$

which is zero since Φi is γ-harmonic on $\operatorname{int} G$ and $\rho \in \operatorname{int} G$. Hence by linearity and the finiteness of the sums involved, we know that

$$(\Phi i, w) = \sum_{\rho \in (\operatorname{int} G) \cap G_n} w(\rho) \sum_{v \in G_{n+1}} \sum_{v' \sim v} \gamma_{vv'} \left[(\Phi i)(v) - (\Phi i)(v') \right] (\delta_\rho(v) - \delta_\rho(v')) = 0$$

so $\Phi i \in W^\perp = M(\Gamma)$, so we are done.

□

1.6 Half Planar and Dual Graphs

The natural analogue of circular planar graphs for infinite graphs turns out to be half planar:

Definition 1.6.1. We say an electrical network Γ is **half-planar** if Γ is embeddable in the upper half plane $\overline{\mathbb{H}} \subseteq \mathbb{C}$ such that the boundary vertices are all on $\mathbb{R}$ and the interior vertices are in $\mathbb{H}$ (the strict upper half plane).

1.6.1 Dual Networks

Just as in circular planar graphs we can construct a dual graph in the case of half planar networks. *A-priori* the dual graph doesn't need to have finite valence, so we will only consider primal graphs such that the dual graph has finite valence at every vertex. We can form an electrical network $\Gamma^\dagger$ using $G^\dagger$ by setting the conductance across any edge $a'b'$ in the dual graph to be $1/\gamma_{ab}$ where ab is the edge in G which crosses $a'b'$. Now given a $\phi \in H(\Gamma)$, then we wish to form the dual function $\phi^\dagger$ on the dual function by requiring that if $a'b'$ is the dual edge crossing ab and $a'b'$ is the counterclockwise rotation of ab then

$$\phi^\dagger(b') - \phi^\dagger(a') = \gamma_{ab}(\phi(b) - \phi(a)).$$

There is nothing super sophisticated about this, but what we will do instead is form a current function on $\Gamma^\dagger$ by setting $i_\phi(b'a') = \phi(b) - \phi(a)$. We have that i_ϕ is γ-harmonic on $\operatorname{int} G^\dagger$ since ϕ is a well defined voltage function and summing i_ϕ at a vertex corresponds to summing voltage differences around a loop, which obviously yields zero. Similarly i_ϕ has loop sum zero since it clearly has loop sum zero around any loop which bounds a single cell in the dual graph since summing around any such loop corresponds to checking that ϕ is γ-harmonic. We don't go into these details too explicitly since they are identical to what happens in the finite case. We note that if $\phi \in H(\Gamma)$ then $P(i_\phi)_{\Gamma^\dagger} = P(\phi)_\Gamma$ and hence $i_\phi \in L(E^\dagger)$. We define the map D_Γ by

$$D_\Gamma : H(\Gamma) \to L(E^\dagger), \quad \phi \mapsto i_\phi$$

and we note that D is a linear isometry. As is easily verified, the composition of all the functions below in the obvious order is the identity function from $H(\Gamma)$ to $H(\Gamma)$

$$H(\Gamma) \xrightarrow{D_\Gamma} L(E^\dagger) \xrightarrow{\Phi_{\Gamma^\dagger}} H(\Gamma^\dagger) \xrightarrow{D_{\Gamma^\dagger}} L(E) \xrightarrow{\Phi_\Gamma} H(\Gamma)$$

is the identity.

1.6.2 Voltage-Covoltage

Let $\Omega(V)$ denote the vector space of valid boundary voltages and let $\Omega(E)$ denote the space of valid boundary currents γ-harmonic functions. We observe that there is not in general well defined map from these space into any of the spaces of functions defined on all vertices, or all edges, but we do have a well defined map from $\Omega(E)$ into the set of valid boundary covoltages Θ, and this map is a bijection, as is easily verified. We will denote the map from $\Omega(E)$ by ∂. We leave the details to the reader since they are just as they are in the finite case (see [4]). Hence we have the following diagram

$$\Omega(V) \xrightarrow{\Lambda_M} \Omega(E) \xrightarrow[\text{iso}]{\partial} \Theta,$$

and similarly we have

$$\Theta \xrightarrow[\text{iso}]{\partial^{-1}} \Omega(E) \xrightarrow{H_M} \Omega(V)$$

Corollary 1.6.2. The minimal boundary data maps Λ_M and H_M contain equivalent information to the minimal voltage-covoltage and covoltage-voltage maps respectively (for half-planar graphs).

Chapter 2

Inverse Problems

2.1 Introduction

Here we present a proof that we can recover a conductivities of a class of infinite networks that we will can supercritical half planar. Much of the work we present here is a generalization of work by Will Johnson, and a lot of theory, especially of convex sets of the medial graph, originated from him. The general strategy of our recovery process mirrors his process for finite graphs, though many of the proofs for the infinite case differ significantly from the finite case.

2.2 Preliminaries

2.2.1 Half-Planar Graphs

The class of graphs that we will attempt to recover is a subset of what we will call the half-planar graphs:

Definition 2.2.1. Let $\Gamma = (\partial G, \operatorname{int} G, K)$ be a (possibly infinite) electrical network. We will say that Γ is **half planar** if there is an embedding of G into the closed upper half plane $\overline{\mathbb{H}} \subseteq \mathbb{C}$ such that the following conditions are satisfied:

1. all vertices in ∂G are in $\mathbb{R}$,
2. all vertices in $\operatorname{int} G$ are in $\mathbb{H} = \overline{\mathbb{H}} \setminus \mathbb{R}$,
3. $V(G)$ is a discrete set of $\mathbb{C}$,

We can define the dual and medial graphs exactly as we would in the finite case, but things don't have to be as well behaved. Unfortunately for the extension, we do need things to be well behaved, so the class of graphs that we wish to recover becomes somewhat restrictive. We use as the infinite analogue of critical circular planar graphs a class of graphs which we call critical half planar graphs. Many of the below conditions are probably redundant, but we list them as assumptions to simplify the below exposition as much as possible.

Definition 2.2.2. A half planar graph G will be called supercritical with respect to a particular embedding if, with respect to that embedding, the following conditions are satisfied:

1. every vertex in the dual graph his finite valence,
2. there are no loops or self intersection of geodesics,
3. two geodesics intersect at at most one point,
4. the geodesics intersect the real axis at exactly two points. This implies that the geodesics are compact.
5. if $K \subseteq \mathbb{C}$ is compact, then $\{g : g[0,1] \cap K \neq \varnothing\}$ is finite.
6. the geodesics can be parametrized by smooth curves with nonvanishing derivative.
7. each geodesic cell is compact and has as boundary only finitely many geodesics.

2.2.2 Dirichlet and Neumann Data

We now recall some foundational material about infinite electrical networks, as was proven in the previous chapter. We encourage the reader to first read that chapter, since much of the material and terminology will be assumed. Firstly, in that paper we showed that there were well defined maps (both Dirichlet-to-Neumann and Neumann-to-Dirichlet) which take valid boundary data to the dual boundary data of a unique minimal function assuming both boundary data. We call these maps the minimal boundary data maps. We denote these maps by Λ_M and H_M. We still need to be careful though, since we don't not have harmonic boundary data maps in general. I present several examples in that chapter. It might be a natural conjecture that for supercritical circular planar that we get well defined harmonic boundary data maps, but this turns out to be false, as we exhibit with the following example.

Example 2.2.3. We examine the "railroad track" graph in Figure 2.1.

We explicitly verify that this graph is supercritical half planar with the embedding below in Figure 2.2. Notice that technically this isn't embedded in a "half plan" but that doesn't matter since we can just conformally map the region onto the half plane.

Now we put conductivities on Γ in the way below and we immediately see that there is a γ-harmonic vertex function with finite power that is zero on the boundary and yields nonzero current flowing out of the boundary, so there is no Dirichlet-to-Neumann map in the traditional sense.

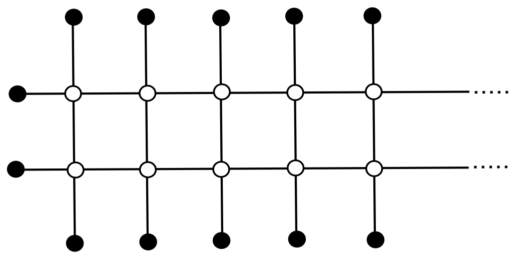


Figure 2.1: The “infinite railroad” graph examined in Example 2.2.3.

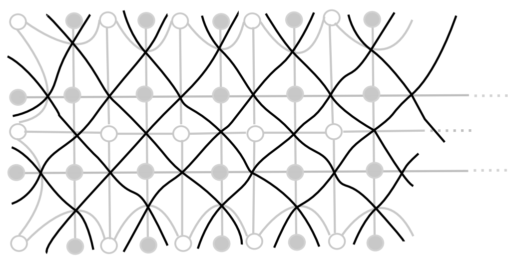


Figure 2.2: The medial and dual graphs of the “infinite railroad” 2.2.3. Note that primal vertices are notated with a solid circle and dual vertices are notated with an empty circle.

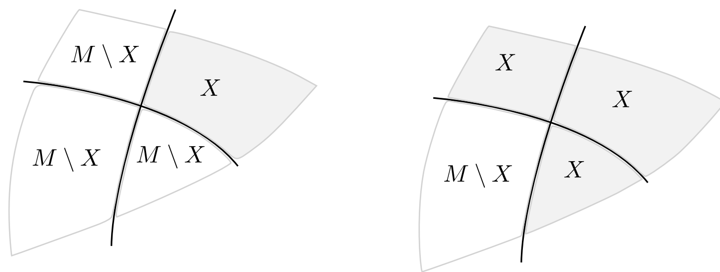
2.3 Convex and Closed Sets in the Medial Graph.

Here we develop the theory of convex and closed sets of the medial graph for infinite graphs, which was first developed by Will Johnson in [4] for finite critical circular planar. As in the finite case, this section will form the technical heart of the paper. Some results carry immediately over from the finite case, but most require substantially different proofs. Sometimes if c is a cell in the medial graph, we will regard c as a subset of $\mathbb{C}$, in which case we will always use c to refer to the Euclidean interior of the region c. Hopefully no confusion will be had.

2.3.1 Basic Definitions

Definition 2.3.1. Two cells in a medial graph M are adjacent if they share an edge. A connected set of cells X is one that is connected through adjacency.

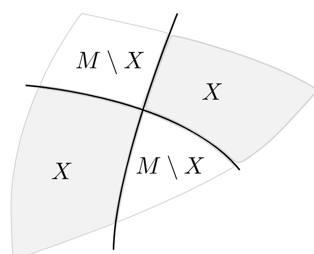
Definition 2.3.2. If $X \subseteq M$ we say that X has a corner at a vertex v in the medial graph if X contains exactly one of the cells that touch v. We see that X has an anticorner at v if X contains exactly three of the cells next to v (see Figure 2.3)



A corner. An anticorner.

Figure 2.3: Corners and anticorners.

Definition 2.3.3. We will say X has a degenerate corner at a vertex v in the medial graph if X contains two cells which are diagonally opposite each other across v, but neither of the two cells adjacent to these two cells. See Figure 2.4.



A degenerate corner.

Figure 2.4: Degenerate Corners.

2.3.2 Results from the Jordan Curve Theorem

We assume as part of the embedding that each geodesic g can be parametrized by a function $g : [0, 1] \to \overline{\mathbb{H}}$ such that $g(1)$ and $g(0)$ but the image of g doesn't intersect $\mathbb{R}$ at any other points. We form the function $\widetilde{g} : [0, 1] \to \overline{\mathbb{H}}$ such that on $[0, 1/2]$ we define $\widetilde{g}(t)$ to be $g(2t)$ and on $[1/2, 1]$ we define $\widetilde{g}$ to parametrize the straight line segment from $g(1)$ to $g(0)$ on $\mathbb{R}$. Hence $\widetilde{g}$ is a Jordan curve and hence by the Jordan curve theorem $\mathbb{C} \setminus \widetilde{g}[0, 1]$ consists of exactly two connected sets, one which is bounded and one which is unbounded. The bounded component is clearly a subset of $\mathbb{H}$. Furthermore, it's really easy to see that if U is the unbounded component then $U \cap \mathbb{H}$ is also unbounded and connected. If g is any such geodesic, we will define $B(g)$ to be the cells of the medial graph whose interiors lie in the bounded component of $\mathbb{C} \setminus \widetilde{g}[0, 1] = \mathbb{H} \setminus \widetilde{g}[0, 1]$ and we will define $U(g)$ to be the cells in the medial graph whose cells lie in the unbounded component of $\mathbb{H} \setminus \widetilde{g}[0,1]$. Clearly $U(g) \cup B(g) = M$.

We first need some facts given by the piecewise Jordan curve theorem.

Lemma 2.3.4. Let $\gamma : [0,1] \to \mathbb{C}$ be a piecewise continuously differentiable, closed, simple curve such with only finitely many nonsmooth points and suppose that γ'^{+} and γ'^{-} exist at every point where γ is not continuously differentiable. Suppose further that γ' is nonzero at the points where γ is differentiable. By possibly reversing the orientation of γ, we can ensure that $i\gamma'$ (rotation counterclockwise) points into the bounded region bounded by γ and $-i\gamma'$ (rotation clockwise) points into the unbounded component. Reversing the orientation of γ interchanges which vector points into which region.

Proof. I might fill in this proof later, but this is essentially taken out of Gamelin (page 250). ☐

We now state some alternate versions of the Jordan curve theorem.

Lemma 2.3.5. Let $\gamma : (0,1)$ be a simple curve in $\mathbb{H}$ which escapes every compact set $K \subseteq \mathbb{C}$. Then $\mathbb{H} \setminus \gamma(0,1)$ consists of exactly two components, at least one of which is unbounded.

Proof. Note that $\gamma \to \infty$ as $t \to 0$ and $t \to 1$. Hence we can extend γ to a simple closed curve in $\hat{\mathbb{C}}$ (which is $\mathbb{C}$ adjoined with a single point at ∞) by setting $\hat{\gamma}(0) = \hat{\gamma}(1) = \infty$. The claim follows from the regular Jordan curve theorem for $\hat{\mathbb{C}}$ along with the observation that one of the components that $\hat{\mathbb{C}} \setminus \hat{\gamma}[0,1]$ will contain the closed lower half plane, and neither component contains ∞ and hence if U_1 and U_2 are the components of $\hat{\mathbb{C}} \setminus \hat{\gamma}[0,1]$ then $U_1 \cap \mathbb{H}$ and $U_2 \cap \mathbb{H}$ will both be connected. ☐

Lemma 2.3.6. Let x be a cell in the medial graph such that all boundaries of x are geodesics. Then x is in $B(g)$ for some geodesic g which travels adjacent to x.

Proof. Suppose to the contrary that $x \in U(g)$ for all g which border x. We will use Lemma 2.3.4 in a strong way. Furthermore, we can assume our geodesics to be smooth except at the points where they intersect the real axis. Let the edges of x be (in clockwise order) $e_1, e_2, \ldots, e_n$. Let γ_0 denote the piecewise smooth curve parametrizing these edges clockwise (i.e. points in the bounded region have winding number 1). Let the edge e_i of x correspond to geodesic g_i. Let $x_{i,\ell}$ and $x_{i,r}$ denote the points where g_i intersects the real axis, ordered so that $x_{i,\ell} < x_{i,r}$. We now need to consider parametrizations of g_i. Let $\hat{g}_i : [0,1] \to \mathbb{C}$ denote the simple closed piecewise smooth curve defined by having $\hat{g}_i$ first parametrize the straight line from $x_{i,\ell}$ to $x_{i,r}$ and then parametrize the image of g_i starting at $x_{i,r}$ and travelling to $x_{i,\ell}$. Since there are ϵ_1, ϵ_2 such that $[x_{i,\ell} + \epsilon_1, x_{i,r} - \epsilon_1] \times (0, \epsilon_2)$ is nonempty and a subset of $B(g_i)$, and that furthermore, the points in $[x_{i,\ell} + \epsilon_1, x_{i,r} - \epsilon_1] \times (0, \epsilon_2)$ are clearly going to be to the left of g_i in the sense given in Lemma 2.3.4, we know that x is to the left of g_i (in sense of the previous lemma) iff $x \in B(g_i)$. Since x is not in $B(g)$, we know that the points in x are to the right of every geodesic $\hat{g}_i$. We oriented γ_0 so that the interior of x corresponded to being to the right of γ_0. Hence we know that the orientation on γ_0 agrees with the orientation of $\hat{g}_i$ on the intersection of their images (on the appropriate edges of x). We note that exactly one of $x_{i+1,\ell}, x_{i+1,r}$ is in $[x_{i,\ell}, x_{i,r}]$ since G has a supercritical embedding and so there are no lenses. If we start on a portion of g_{i+1} which is not in $B(g_1)$ (such a portion exists because of the assumptions on x) then as t increases, since the orientation of g_2 and γ_0 agree, we must have that g_2 intersects g_1 and enters into the interior of $B(g_1)$. Hence $x_{i+1,\ell} \in [x_{i,\ell}, x_{i,r}]$ and $x_{i+1,\ell} > x_{i,\ell}$. We repeat this to get that $x_{1,\ell} < x_{2,\ell} < x_{3,\ell} < \cdots < x_{n,\ell}$. But since our ordering was cyclic (it didn't which edge on x we started with), we get that $x_{n,\ell} < x_{1,\ell}$ and hence $x_{1,\ell} < x_{1,\ell}$, which is nonsense. Hence x must be in the bounded region of at least one of the geodesics which are adjacent to x. □

Lemma 2.3.7. Let x be a cell in the medial graph such that x does not border $\mathbb{R}$. Then x is in $U(g)$ for some g which neighbors x.

Proof. The proof is basically the same as the proof of Lemma 2.3.6. Suppose that x is in the bounded component bounded by every geodesic that borders x. Let γ_0 parametrize the boundary of x in a counterclockwise fashion, i.e. x is to the left of γ. Now let $\hat{g}_i$ be defined as in the proof of Lemma 2.3.6. We note that the orientation of $\hat{g}_i$ and γ_0 agrees on the intersection of their images for each i. Since there can be no lenses in the medial graph, and using the assumptions about orientation, we see that $x_{i+1,r} < x_{i,r}$ for all i. But because of the cyclic ordering of the edges of x, we see that $x_{1,r} < x_{n,r} < x_{n-1,r} < \cdots < x_{1,r}$, which is nonsense. □

Lemma 2.3.8. Let x be a cell which borders $\mathbb{R}$. Then $x \in B(g)$ for at least one of the geodesics which borders x.

Proof. The proof is almost identical to the proof of Lemma 2.3.6 with a slight modification. To do this, we will introduce *pseudogeodesics*, which are just closed intervals of $\mathbb{R}$, with the convention that if I is a closed interval, then $U(I) = \mathbb{H}$ and $B(I) = \varnothing$. We note that if x borders $\mathbb{R}$, then x has a boundary which consists of pseudogeodesics and geodesics. If g_i is a pseudogeodesic, then we will define $\hat{g}_i$ to be the linear parametrization of g_i from right to left. It is easily verified with this definition that the proof from Lemma 2.3.6 goes through essentially without change. We note that if g is a pseudogeodesic then $x \notin B(g) = \varnothing$ and hence $x \in B(g)$ for some real geodesic which borders x. We leave this verification to the reader.

□

Combining Lemma 2.3.6 and 2.3.8 we get

Corollary 2.3.9. Let x be a cell in the medial graph, then $x \in B(g)$ for some geodesic g which borders x. *A-fortiori* $x \in B(g)$ for some geodesic.

2.3.3 Theorems about Minimal Numbers of Corners

Definition 2.3.10. We say that X is simply connected if X is connected and every cell in $M \setminus X$ can be connected to a boundary cell in $M \setminus X$ by a path of pairwise adjacent cells in $M \setminus X$

Definition 2.3.11. A **geodesic path** is a path in the edges of the medial graph. A **simple geodesic path** is a path of geodesics which is a path of geodesics that is either never reaches a vertex twice, or a closed loop (i.e. doesn't reach a vertex twice except that the first and last vertices are the same).

Definition 2.3.12. A **pseudosimple** geodesic path is a geodesic path that either never reaches a single edge twice, or is a closed loop such that no edges are reached twice, except that the first and last edges are the same.

Definition 2.3.13. We define the boundary ∂X of a set $X \subseteq M$ as the set of medial edges which border both a cell in X and a cell in $M \setminus X$.

Definition 2.3.14. If γ is a geodesic path which traverses part of the boundary of X. Then we say γ is **left-inwardly-oriented** (with respect to X) if every cell that is to the left of the edges of γ (with respect to the direction of traversal of *gamma*) is in X and every cell to the right is in $M \setminus X$.

Definition 2.3.15. If γ traverses a portion of ∂X, we define γ to be **component-following** if at every degenerate corner of X that γ reaches, γ follows the edge as shown in Figure 2.5. More precisely, if at every degenerate corner γ will turn so as to continue along the boundary of the same cell along which it came to the degenerate corner.

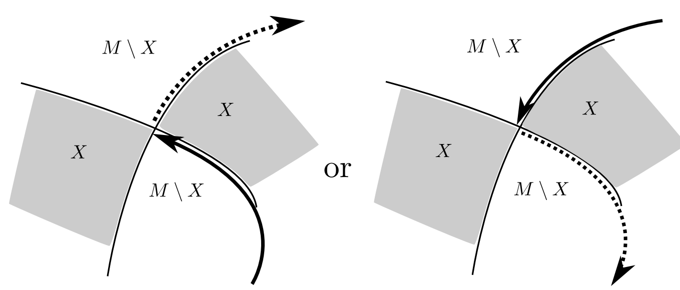


Figure 2.5: A curve γ is component-following if whenever it reaches a degenerate corner as above, it follows the dotted line.

Lemma 2.3.16. If $X \subseteq M$ then ∂X can be parametrized by an edge disjoint family of left-inwardly-oriented, complement-following, pseudosimple geodesic paths or loops.

Proof. We will do this by induction. We will create a family of curves such that "to the right" corresponds to being in X. Suppose such a (possibly empty) family $\mathcal{F}$ of pseudosimple, left-inwardly-oriented, and component-following geodesic curves has been defined. Let e_1 be geodesic edge which is in ∂X but not in any path. Now just extend e along ∂X in both directions. We now consider different possibilities of corners that we can reach. If we reach a node where two adjacent cells are in X but the other two are not, then we only have one choice. If we reach a corner then we also have no choice. Thus having started at e, we cannot reach an edge that's in another curve of $\mathcal{F}$ by only passing along geodesics and turning at corners and anticorners since there have been no choices for any other path to make. The only case that we need to discuss is if the path reaches a degenerate corner. In that case, we have up to two choices, as illustrated in Figure 2.6.

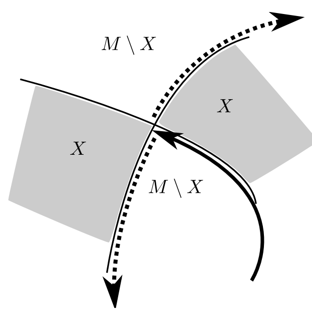


Figure 2.6: The dotted lines represent the possible choices we have to pick at a degenerate corner.

If any of the the dashed edges in Figure 2.6 is in a curve in $\mathcal{F}$, then we pick the direction that preserves the component-following property. If any of the edges entering the node are are already in a path in $\mathcal{F}$, then we know that exactly two of them must be (since if three were already traversed then the fourth would have to be as well, contradicting the fact that we got to this node without traversing edges that are in $\mathcal{F}$), and furthermore, since $\mathcal{F}$ is assumed to have the component-following property, by looking at 2.6, we immediately see that the edge allowing our curve to have the component-following property cannot be the one in another path.

The only possibility for running into an edge that has already been defined is if we meet an edge of the same path we are defining, which is allowable. Also, these curves are pseudosimple since we assume that we just stop if we are ever forced to take an edge that we've already encountered. They are left-inwardly-oriented and component-following by our choice of travel at anticorners. Thus such a family of curves exists as stated. □

Lemma 2.3.17. Let $X \subseteq M$ be an arbitrary subset which doesn't contain any boundary cells of the medial graph. Then X is simply connected iff ∂X can be parametrized as simple geodesic path. The set X is finite and simply connected iff ∂X can be parametrized by a single closed simple geodesic path.

Proof. Let $\mathcal{F}$ be an edge disjoint family of left-inwardly-oriented, complement following pseudosimple geodesics which traverse the entire boundary of X. Let γ be an arbitrary element of $\mathcal{F}$. We wish to show that γ is simple. Suppose γ is not simple. Let v be a medial node of self intersection of γ. Note that v must occur at a degenerate corner of X. We claim that $\mathbb{C} \setminus \gamma$ must consist of at least three components. Clearly $\mathbb{C} \setminus \gamma$ must have at least two components, since it is either a closed curve in $\mathbb{C}$ or can be extended to a closed curve on $\hat{\mathbb{C}}$ by setting the left and right endpoints to be $\infty \in \hat{\mathbb{C}}$, and hence by basic complex analysis, the winding number is constant in connected components, but the winding number must change as we cross γ since γ doesn't repeat any edges. Suppose that $\mathbb{C} \setminus \gamma$ consists of exactly two components. Since γ is component-following, we now that the set of cells in X which are adjacent to an edge in γ must be connected. Hence the cells in X which are diagonally opposite must be in the same component. Since each of the cells in $M \setminus X$ which touch v cannot be in the same component as the two cells in X since we cross γ to get to them (and hence the winding number with respect to γ would change), we know those two cells must be in the same connected component. Let x_1 and x_2 be arbitrary points in the two cells in $M \setminus X$ which are adjacent to v (with x_1 and x_2 be difference cells). Since x_1 and x_2 are in the same component, there is a path (a continuous 1-1 function from $[0, 1]$ to $\mathbb{C}$) from x_1 to x_2 which does not cross γ (and we can even assume that this path is linear in the two cells which contain x_1 and x_2). Concatenating this with a "nice" path that goes through v and starts at x_1 and ends at x_2 (what "nice" means is left to the reader) as in Figure 2.7, we get a simple closed curve, and hence we can apply the Jordan curve theorem to get that there are exactly two regions.

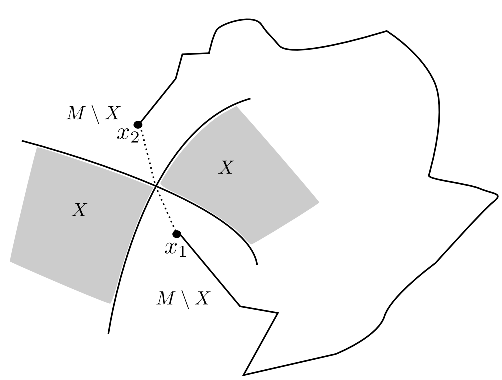


Figure 2.7: The path discussed above.

Clearly *exactly* one of the two cells in X which is adjacent to v must have an interior which is entirely in the bounded region and the other one must have an interior which is entirely contained in the unbounded region. But this contradicts the fact that the two cells in X which touch v are in the same component of $\mathbb{C} \setminus \gamma$.

Now we now that all curves in $\mathcal{F}$ must be simple. Now we claim that $\mathcal{F}$ must consist of at most one path. Let γ be any of the paths in $\mathcal{F}$. Note that by the Jordan curve theorem γ will divide $\mathbb{H}$ into exactly two connected regions. Note that exactly one of these regions can contain any cells in the boundary of the medial graph since X contains no cells of the medial graph and hence ∂M (the boundary cells of the medial graph) must be contained in the single component. Let Y consist of all of the cells whose interiors are in the other component. Obviously $\partial Y = \operatorname{Im} \gamma$. Suppose that X had some boundary edge e which was not contained in the image of γ. Then we note that either both cells must be in Y or both cells must be in $M \setminus Y$. If there is a cell from X in $M \setminus Y$ then X cannot be connected and if there is a cell from $M \setminus X$ in Y then that cell clearly cannot be connected to the boundary. We leave it to the reader to verify these last two assertions, but they are very clear.

For the last statement, if X is finite and simply connected and doesn't touch the boundary, then ∂X must be finite, and the only possibility for a simple geodesic curve to be finite is for it to be closed. If γ is a closed simple geodesic curve, then the fact that X is simply connected and finite follows from the Jordan curve theorem. The details are left to the reader.

□

Corollary 2.3.18. If X is simply connected, then X contains no degenerate corners.

Lemma 2.3.19 (Lemma 5.4 of [4])**.** Let $X \subseteq M$ be connected and finite and suppose X contains no boundary cells of the medial graph. Then X has at least three corners.

We leave it to the reader to verify that the proof from [4] carries over without change.

Lemma 2.3.20. Let X by a simply connected set and let g be a geodesic. Then $X \cap B(g)$ and $X \cap U(g)$ contain no degenerate corners.

Proof. If $X \cap B(g)$ contained a degenerate corner, then X must too since none of the other two adjacent cells can be added when we pass to X since none of the geodesic passing through the vertex at this degenerate corner cannot be g since otherwise those two cells would not be diagonally adjacent in $X \cap B(g)$. The same argument holds for $X \cap U(g)$. □

Lemma 2.3.21. Let X be simply connected and g a geodesic such that $B(g) \cap X$ is nonempty. Then every component of $B(g) \cap X$ has a corner which is also a corner of X.

Proof. Let C be a component of $B(g) \cap X$. If the segment of g that corresponds to the boundary of C is not connected, then we can fill in the components of $B(g) \setminus X$ that touch the intermediate segments of g as shown in Figure 2.8.

This clearly does not reduce the number of corners. Call the resulting region $\hat{C}$. Clearly $\hat{C}$ is finite and doesn't contain any of the boundary of M. Since $\hat{C}$

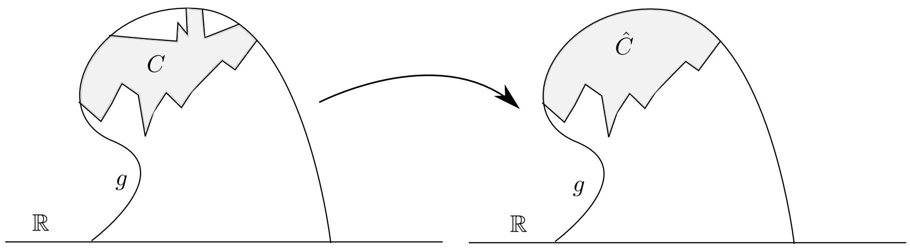


Figure 2.8: Filling components along g.

is connected we know by Lemma 2.3.19 that $\hat{C}$ has at least 3 corners. At most two of these cells can be adjacent to g since the segment of g that borders $\hat{C}$ is connected. Hence one of these corners must occur inside of $B(g)$ but not along g. Let c be such a corner. We note that clearly this corner is also a corner of C. Furthermore, by Lemma 2.3.20, we know that c is a corner of $B(g) \cap X$. Since is not adjacent to g, we know that c is not adjacent to any cells in $U(g)$ and hence c is a corner of X. This holds for every component of $B(g) \cap X$. □

We wish to show the infinite analogue of Lemma 2.3.19. It turns out this theorem is actually quite difficult to prove. It is relatively straightforward to show that an infinite simply connected subset which doesn't contain any boundary cells has at least one corner by just applying Lemma 2.3.19 to a subregion bounded by a geodesic. It is somewhat more complicated but still relatively straightforward to find a second corner by traveling in both directions from the corner we have already found. It turns out that finding the third corner is quite difficult, and the only proof that I could figure out actually implied that we have infinitely many corners.

Lemma 2.3.22. Let $X \subseteq M$ be simply connected and infinite and suppose that X contains no boundary cells. Then X has infinitely many corners.

Proof. Let γ be a geodesic path which traverses the boundary of X. Suppose that X has no more than k corners (the case $k = 0$ is allowed). We observe that γ must be an infinite curve, since otherwise X would be bounded, contradicting our assumptions. After a certain point in traversing the boundary, γ can only travel along geodesics or meet anticorners and there must be infinitely many anticorners, since the geodesics are compact. Let $c_1, \ldots, c_k$ be the corners of X. Since X is connected there are paths p_i of adjacent cells in X from c_i to c_1. By Lemma 2.3.9, we know that each cell c in the any of the paths p_i (which includes all c_i) is in a region $B(g_c)$ for some geodesic g_c. Define

$$R = \bigcup_{\{c:\exists p_j, c \in p_j\}} B(g_c)$$

and note that R is finite and contains all c_i for $i = 1, \ldots, k$ and also R contains a path from each c_i to each c_j. Since R is finite, there can only be finitely many geodesics which pass through any portion of the interior of R or pass along the boundary of R. Since γ must be an infinite path which doesn't intersect itself (since otherwise X would be finite), we know that eventually one of the geodesics which has an edge on γ must not pass through R. Let g be such a geodesic. We note any component of g which γ traverses must start and end with an anticorner, since otherwise there would have to be a corner along g, which isn't possible since all the corners are contained in R and g doesn't border any cells in R. Hence $X \cap B(g)$ and $X \cap U(g)$ are both nonempty. Either $B(g)$ or $U(g)$ must contain all of R since g doesn't intersect the boundary of R. If $R \subseteq U(g)$ then we apply Lemma 2.3.21 to show that $B(g) \cap X$ contains a corner, which can't be any of the corners in R since $R \subseteq U(g)$ and hence X has at least $k+1$ corners. Thus suppose that $R \subseteq B(g)$. Now we have to break into cases about which side of g corresponds to the bounded side with respect to the anticorners. Note that the anticorners on the segment of g that we are discussing must occur on the same side of g since otherwise one would be a corner.

Let the "sharp" side denote the side of g where the single cells of the anticorner are and let the "smooth" side of g refer to the side where the two adjacent cells of of the anticorners are. See Figure 2.9 for a picture of what this means.

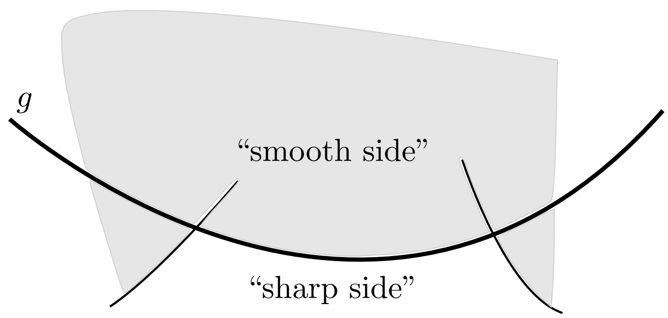


Figure 2.9: The smooth and sharp sides

If the "sharp" side of g is $B(g)$, then we first claim that $B(g) \cap X$ must consist of at least two components. To see this, let x and y be the two "sharp" cells in the anticorners adjacent to g as in Figure 2.10.

Suppose to the contrary that $B(g) \cap X$ is connected and hence there is a path from x to y in $B(g) \cap X$. Let z be any cell along g between x and y which is not in X (such a cell obviously exists since otherwise there wouldn't be anticorners. But then there is clearly a loop of cells in X which bound z, and hence X is not simply connected. This is illustrated in Figure 2.11. Since all of the corners are connected by paths in R and $R \subseteq B(g)$, we know that $R \cap B(g)$ all of the corners $c_1, \ldots, c_k$ will be contained in a single component of $X \cap B(g)$. But since there is at least one other component of $B(g) \cap X$, we just apply Lemma 2.3.21 to get an additional corner in X, which can't be one of $c_1, \ldots, c_k$ since it's in a different component of $X \cap B(g)$ and hence X has $k+1$ corners. Now suppose that the "smooth" side of g is bounded. If $B(g) \cap X$ is

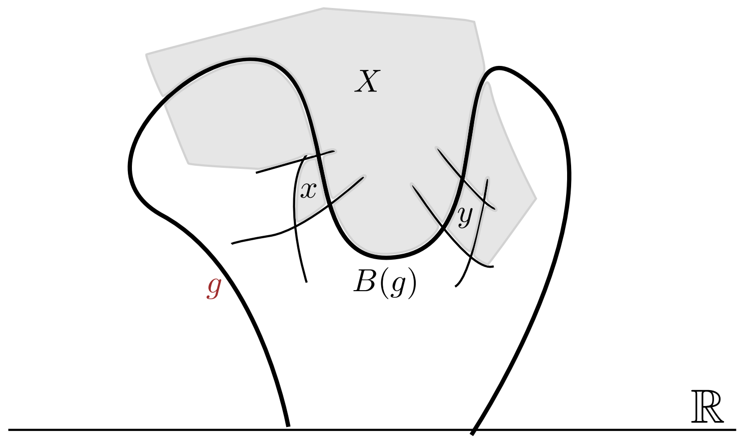


Figure 2.10: The cells x and y.

not connected, then one of the components must contain all of the corners in R as before and then another component will contain an extra corner of X. The last possible case is that $B(g) \cap X$ is connected. Let e_1 be any geodesic edge on the boundary of $B(g) \cap X$ which is not along g and let e_2 be any geodesic edge along g between the two anticorners x and y. We have the situation in Figure 2.12.

Since X is simply connected, by Lemma 2.3.17 there is a geodesic path α from e_1 to e_2. Since α starts at e_1 and the entire connected segment from x and y is part of the boundary of X, we know that α must first travel along g and then turn at one of the anticorners x and y. If we truncate α at the last edge before it either travels along g again or enters $B(g)$ and then concatenate this curve with g to get back to the anticorner that α turns at (see picture), then we form simple geodesic loop, which must bound a set of cells S in M which are not in $B(g)$. Furthermore $S \cap X$ is nonempty. We summarize with Figures 2.13 and 2.14.

By filling in holes along g if $X \cap S \neq S$, we can apply the same argument in Lemma 2.3.21 to see that S must have a corner that is also a corner of X. Since this corner is not in $B(g)$, we know that it must be distinct from $c_1, \ldots, c_k$. Hence X has $k+1$ corners. □

Corollary 2.3.23. If $X \subseteq M$ is connected and infinite and doesn't contain any boundary cells then X has infinitely many corners.

Proof. Just fill in all of the components of $M \setminus X$ except the one that contains the boundary ∂M, which leaves a simply connected region. □

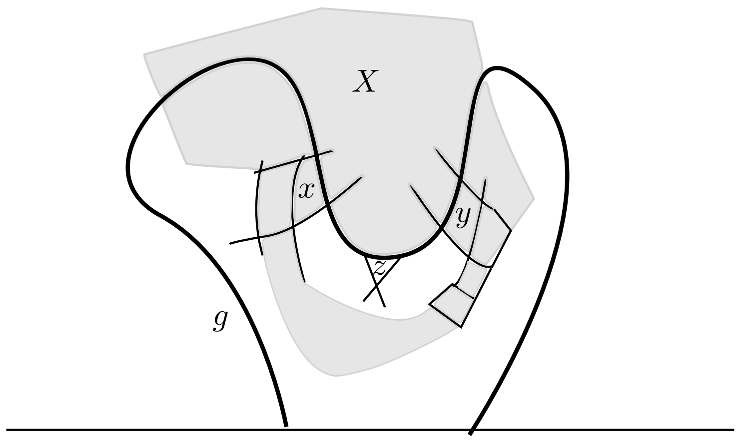


Figure 2.11: A path in X which surrounds a cell in $z \notin X$.

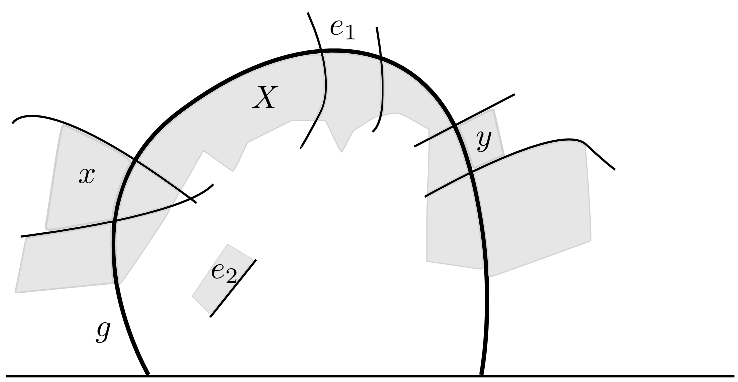


Figure 2.12: The edges e_1 and e_2.

2.3.4 Convex and Closed Sets

Definition 2.3.24. We define a half plane to be a set of the form $B(g)$ or $U(g)$ for some geodesic g.

Definition 2.3.25. A set $X \subseteq M$ is closed if it has no anticorners.

Lemma 2.3.26. The intersection of arbitrarily many closed sets is closed.

Proof. Consider an arbitrary collection $\{X_\alpha\}_{\alpha \in A}$ of closed sets. Let x_1, x_2, x_3, x_4 be cells around a vertex v in the medial graph. If $v_i \notin \bigcap_A X_\alpha$ then $v_i \notin X_\alpha$ for some α. Since X_α is closed, another $x_j \notin X_\alpha$ and hence $x_j \notin \bigcap X_\alpha$ so $\bigcap X_\alpha$ does not contain an anitcorner and hence is closed. □

Lemma 2.3.27. Half planes are closed.

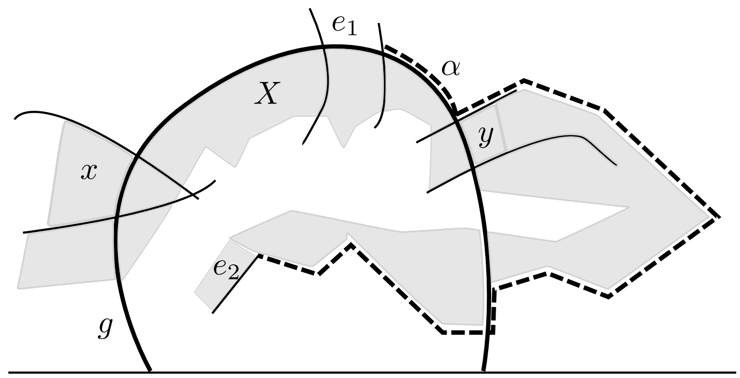


Figure 2.13: The curve α. The curve α is represented with a dashed line.

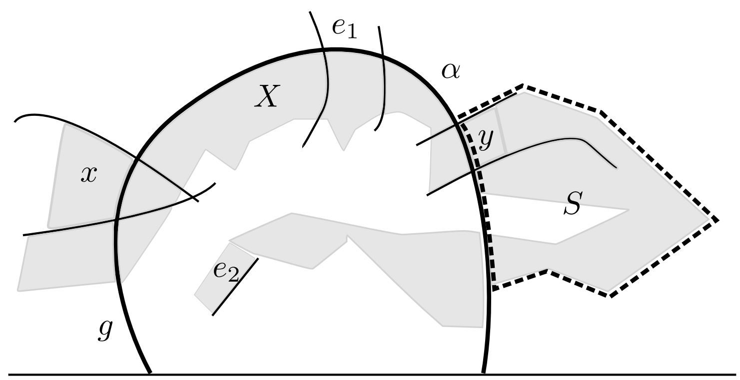


Figure 2.14: The region S. It is the region bounded by the dashed line.

Proof. Obvious. □

Corollary 2.3.28. Convex sets are closed.

Proof. This follows from Lemma 2.3.27 and Lemma 2.3.26. □

Definition 2.3.29. We define a set $X \subseteq M$ to be convex if it is an intersection halfplanes.

Theorem 2.3.30. Every convex set X of cells is connected.

Proof. The proof caries over without alteration. □

Definition 2.3.31. We define the closure of X to be the intersection of all closed sets which contain X.

Lemma 2.3.32. The closure of X can also be attained as a countable increasing chain of sets $X = X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ such that each $X_{i+1} \setminus X_i$ consists of at most one cell in the medial graph, and if $x \in X_{i+1} \setminus X_i$ then x fits into an anticorner of X_i.

Proof. Let $\mathcal{F}$ the collection of all supersets Y of X such that there is an at most countable chain of sets $X = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ such that $Y = \bigcup_i X_i$ and each X_i differs from X_{i-1} by filling in a single corner. Let $\mathcal{F}$ be ordered by set inclusion. We claim that $\mathcal{F}$ contains upper bounds. Let $Y_1 \subseteq Y_2 \subseteq \cdots$ all be sets in $\mathcal{F}$. Then we claim that $\bigcup_i Y_i$ is in $\mathcal{F}$. This follows essentially from the proof that $\mathbb{N} \times \mathbb{N}$ is countable. We'll use a picture to demonstrate what to do, but the argument is exactly the same as what one would canonically do to show that $\mathbb{N} \times \mathbb{N}$ was countable. Let $X = X_0^i \subseteq X_1^i \subseteq X_2^i \subseteq \cdots$ be an increasing chain of sets which increases at each step at most by filling in a single anticorner and whose union is Y_i. Take an increasing union of the sets X_j^i in the order presented in Figure 2.15.

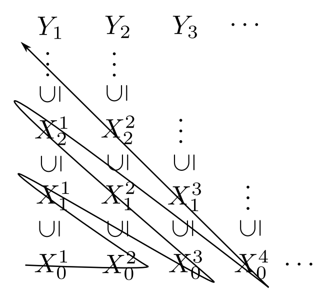


Figure 2.15: Take an increasing union in this order.

If Z_j is the j^{th} set that we hit, let $\widetilde{Z_j} = \bigcup_{i \leq j} Z_j$ and clearly $Z_0 = X$ and $Z_j \setminus Z_{j-1}$ consists of at most a single cell filled into an anticorner of Z_{j-1} and also $Z \stackrel{\text{def}}{=} \bigcup_{\mathbb{N}} \widetilde{Z}_j = \bigcup_{\mathbb{N}} Y_i$. So Z is an upper bound for all Y_i and $Z \in \mathcal{F}$. Finally we apply Zorn's lemma to get a maximal element $M \in \mathcal{F}$, which we claim is closed. Since M is written as a countable chain on increasing sets starting with X which differ by at most 1 cell added to a corner, we know that M must be in the closure of X. Hence $M = \overline{X}$. □

Corollary 2.3.33. The closure of a connected set is connected.

Proof. This is a consequence of the previous lemma. □

Theorem 2.3.34. If X is convex, then X is closed.

Proof. By Lemma 2.3.26, it is sufficient to show that half planes are closed, but this is obvious, since clearly you can't have an anticorner in a half plane. □

Definition 2.3.35. If $X \subseteq M$, then let $\widetilde{X}$ denote the intersection of all half planes containing X.

Lemma 2.3.36. If $X \subseteq M$, then $\widetilde{X}$ is closed.

Proof. By Lemma 2.3.27 and Lemma 2.3.26 we know that $\widetilde{X}$ is an intersection of closed sets and is hence closed. □

Corollary 2.3.37. If $X \subseteq M$ then $\overline{X} \subseteq \widetilde{X}$.

Lemma 2.3.38. If X is connected and closed then X is simply connected.

Proof. We will use a similar argument to that found in [4]. Suppose X is not simply connected Consider all of the components of $M \setminus X$. Let S be a component which does not include any boundary cells. By Lemmas 2.3.23 and 2.3.19 we know that S must contain at least three corners (v_i, x_i) (using Will's notation). The two cells adjacent to x_i must be in X since (v_i, x_i) are corners, but the cell across from x_i must be in $M \setminus X$ since if the cell were in X, then X would contain an anticorner, which would contradict the fact that X is closed. Thus we can form an adjacency multigraph A for connected components of $M \setminus X$ by having each component be a vertex and connecting to vertices iff they have corners which are diagonally opposite to each other at the same vertex. By Lemma 2.3.19 and Lemma 2.3.23, we know that each component of $M \setminus X$ which does not intersect the boundary has at least three edges connected to it. Now pick an interior component A_0 of $M \setminus X$ arbitrarily. We now we will define a path inductively in A which extends in both directions from A_0. Since each interior component has degree at least 2, we know that if our path ends in an interior vertex, we can always extend farther in that direction. If we get a cycle, then we know X is disconnected (by applying the Jordan Curve theorem). There are four possibilities:

1. the path forms a cycle,
2. the path extends indefinitely in both directions,
3. the path extends indefinitely in one direction, but reaches the boundary in the other,
4. the path reaches the boundary in both directions.

In all cases, we get that X is disconnected and hence there can be no connected components of $M \setminus X$ which don't intersect the boundary. □

We now diverge from the treatment of closed and compact sets that is found in [4] and introduce some new lemmas and results.

Definition 2.3.39. Let $x = x_0, x_1, \ldots, x_n = y$ be a path of adjacent cells in the medial graph such that no cell is repeated. Let $p_0, \ldots, p_n$ be arbitrary points such that $p_i \in x_i^\circ$ (the interior of x_i) for all i. Let $\gamma : [0,1] \to \mathbb{C}$ be a curve such that the following conditions are satisfied:

1. $\gamma(k/n) = p_k$ for $k = 0, \ldots, n$,

2. $\gamma[k/n, (k+1)/n] \subseteq \operatorname{int} \overline{x_k \cup x_{k+1}}$

3. γ is piecewise smooth.

4. γ does not intersect itself.

Then we will call γ a **continuous curve associated with the path $x_0, \ldots, x_n$.**

Lemma 2.3.40 (Filling Lemma)**.** Let $X \subseteq M$ be a closed connected set. Let x_1, x_2 be two cells on the same side of g and let $y_1 = x_1, y_2, \ldots, y_n = x_2$ be a path of adjacent cells from x_1 to x_2 which lies entirely on the same side of g as x_1 and x_2. Then there is a region R of medial cells which is bounded by g and the path $y_1, \ldots, y_n$ and R consists of a finite union of simple connected regions and furthermore $R \subseteq X$. In particular all of the cells along g between x_1 and x_2 are in X.

Proof. Let $\gamma : [0,1] \to \bigcup_i \overline{y_i}$ be a continuous curve associated with the medial path $y_1, \ldots, y_n$. Extend γ to get a function $\hat{\gamma} : [-1, 2]$ such that

1. $\hat{\gamma}(t) = \gamma(t)$ for $t \in [0,1]$;

2. $\hat{\gamma}(t) \in \overline{y_1}$ for $t \in [-1, 0]$;

3. $\hat{\gamma}(t) \in \overline{y_n}$ for $t \in [1, 2]$;

4. $\hat{\gamma}(-1) \in g \cap \overline{y_1}$ and $\hat{\gamma}(2) \in g \cap \overline{y_n}$

5. $\hat{\gamma}$ has no self intersections.

6. $\hat{\gamma}$ intersects the geodesic g at exactly two locations.

Under these assumptions we can extend and reparametrize $\hat{\gamma}$ to a function $\widetilde{\gamma}$ that is a Jordan curve by letting $\widetilde{\gamma}$ traverse the geodesic arc between $\hat{\gamma}(-1)$ and $\hat{\gamma}(2)$. We will assume that $\widetilde{\gamma}$ is parametrized on the interval $[0,1]$. We summarize in Figure 2.16.

Thus R exists as stated by noting that R is just the set of medial cells in $M \setminus X$ which are in the region bounded by $\widetilde{\gamma}$. It may not be true that R is connected, but it is sufficient to show that the connected component closest to x_1 (along g) is in X, so without loss of generality we may assume that x_2 is the first element of the path $y_1, y_2, \ldots, y_n$ other than x_1 which touches g. Let $c_1, c_2, \ldots, c_m$ denote the cells between x_1 and x_2 along g such that $x_1 = c_1$ and $x_2 = c_m$. Without loss of generality we may assume that $\{c_2, \ldots, c_{m-1}\} \cap \{y_1, \ldots, y_n\}$ is empty, i.e. that c_m is the closest cell to x_1 in $\{y_2, \ldots, y_n\}$ which is along g and in the same direction as x_2. Our goal is to show that R is empty. We draw a picture to summarize the situation in Figure 2.17.

We first show that R is connected. Let $\mathcal{R}$ denote the set of connected components of R. Define an adjacency graph A on $\mathcal{R}$ by setting two components adjacent if they share a degenerate corner. By Lemma 2.3.19 each component

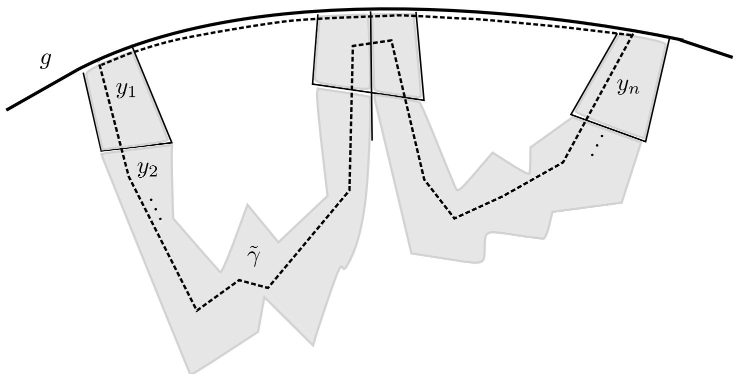


Figure 2.16: The curve $\widetilde{\gamma}$.

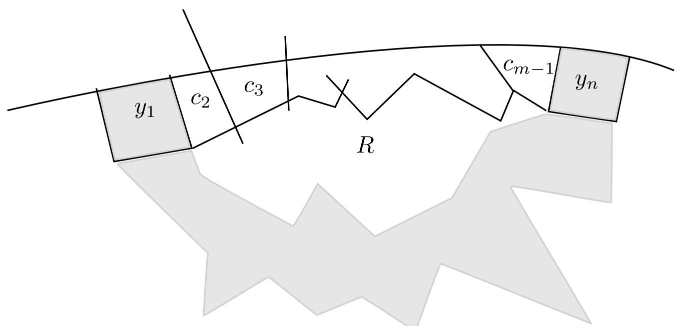


Figure 2.17: The cells $c_1, \cdots, c_m$ and R. Then entire white region bounded by the grey loop and g is R.

of R has at least 3 corners. Since by assumption there is at most one component which has cells on g, and the geodesic segment which borders R must be connected (since all of the cells in the path $y_1, \ldots, y_n$ are on one side of g), we know at most two cells from the component along g can correspond to corners along g, and that all other components have no corners along g. Hence at least one must correspond to an actual to either an anticorner of X or a degenerate corner of R. But X has no anticorners, and hence each connected component of S must have a degenerate corner with another connected component of S, and hence A is connected. But furthermore, we know that at most one connected component can have cells which are adjacent to g, and hence every vertex of A except for possibly a single vertex has valence 3 or greater. By forming a path in A by starting at the component along g and travelling in any manner such that we don't backtrack, we must eventually form a loop, thus forming a disconnection of X, a contradiction. Hence R must be empty.

□

Lemma 2.3.41. Suppose X is a closed connected subset of M, then $\widetilde{X} = X$.

Proof. We will show that there are no cells in $\widetilde{X} \setminus X$ which are adjacent to X. Since $\widetilde{X}$ is connected and contains X, this would imply that $\widetilde{X} = X$. We use the previous lemma. Suppose that $c \in \widetilde{X} \setminus X$ is adjacent to X. We will show that $c \in X$. Let c be adjacent to $x_0 \in X$ and let g denote the geodesic that travels between c and x_0. Since $c \in \widetilde{X}$, there must be some medial cell y in X that is on the same side of g as c. Since X is connected there is a path $z_1, \ldots, z_n$ of adjacent cells in X such that $z_1 = x$ and $z_n = y$. Let k denote the last index such that all z_i are on the same side of g as x for all $i = 1, \ldots, k$. Notice that z_{k+1} is on the same side of g as c. We note that the path $z_1, \ldots, z_k$ is a loop only on one side of a geodesic with z_1 and z_k along g. We summarize with the situation in Figure 2.18

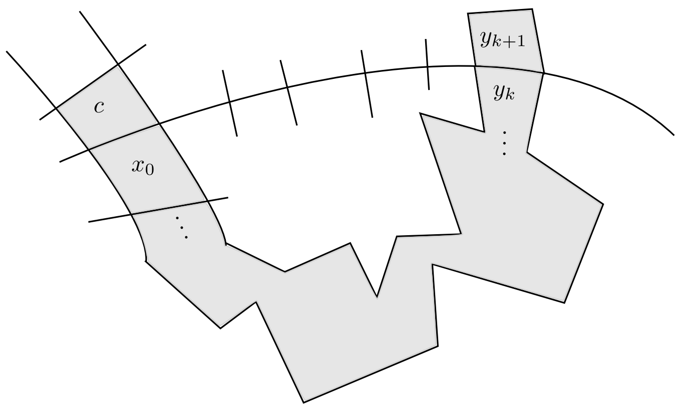


Figure 2.18: The situation: cells $y_1, \ldots, y_k, y_{k+1}$ and c.

Thus we are in a position to apply Lemma 2.3.40, and hence we know that all the cells along g between z_1 and z_k on the same side of g as the path z_i must be in $\overline{X}$. But then since z_{k+1} is on the opposite side of g, and is adjacent to z_k, which is adjacent to a cell x' in X which is diagonally across from z_{k+1} (which is guaranteed to be in X by the filling lemma), we know that the fourth cell adjacent to z_{k+1} and x' is in X. By induction, all of the cells along g on the same side as z_{k+1} which are between z_{k+1} and c must be in X. Hence c must be in X, so we are done. □

Corollary 2.3.42. If $X \subseteq M$ is connected, then $\overline{X} = \widetilde{X}$.

Proof. We have that $\widetilde{X} \supseteq \overline{X}$ since $\widetilde{X}$ is closed and contains X. On the other hand, we have by the previous lemma that $\widetilde{X} \subseteq \widetilde{\overline{X}} = \overline{X}$ and hence combining these two results yields the desired equality. □

In the case of infinite graphs, it turns out something far more powerful is true:

Theorem. Suppose that $X \subseteq M$ is a subset of the medial graph. If every component of X contains infinitely many cells, then $\overline{X} = \widetilde{X}$.

We will not have occasion to use this theorem in its full glory, so we prove this at the end of the paper. It is Theorem 3.7.2. Weaker versions of this theorem will pop up in various places.

2.4 An Important Computation

Lemma 2.4.1. If X and Y are subsets of M then we have $\overline{X} \cup \overline{Y} \subseteq \overline{X \cup Y}$ and $\widetilde{X} \cup \widetilde{Y} \subseteq \widetilde{X \cup Y}$.

Proof. Left to reader. □

Lemma 2.4.2. Let g be a geodesic and let X_1 consist of all of the boundary cells in the medial graph to the left of $B(g)$ and let X_2 consist of all of the boundary cells to the right of $B(g)$. Then $\overline{X_1 \cup X_2} = U(g)$.

Proof. Clearly $\widetilde{X_1 \cup X_2} \subseteq U(g)$ since $U(g)$ is a half plane which contains both X_1 and X_2. Note by Lemma 2.3.42 that since X_1 and X_2 are closed we have $\overline{X_1} = \widetilde{X_1}$ and $\overline{X_2} = \widetilde{X_2}$. Let $R = \overline{X_1 \cup X_2} \cup B(g)$ and consider $S = M \setminus R$. By Lemma 2.3.19 and Corollary 2.3.23 we know that each component of S has at least three corners. Now we observe that by the Filling lemma we know that the component of g which is not adjacent to any cells in $\overline{X_1 \cup X_2}$ is connected. Now if S_0 is the component of S which is adjacent to this connected arc of g, then by the previous lemmas we know that S_0 has at least three corners, and at most two of them can be adjacent to g. The third corner cannot be an anticorner of $\overline{X_1 \cup X_2}$ since $\overline{X_1 \cup X_2}$ is closed. Hence the diagonal cell must be another component of S, which must have at least three corners. Furthermore, every component of S other than S_0 must have at least three corners, all of which must be degenerate corners. Form an adjacency graph A on the components of S other than S_0 and note that any loop or infinite path in A will create portion of $\overline{X_1 \cup X_2}$ which cannot be connected to the boundary with a path of adjacent cells in $\overline{X_1 \cup X_2}$. Hence S must be empty, so $\overline{X_1 \cup X_2} = M \setminus B(g)$. □

2.5 Extending Consistent Functions

We will now reference some work done by Will Johnson in [4] that will carry over essentially without change to the infinite case. These initial theorems and definitions are essentially identical to what is presented in [4].

Definition 2.5.1. Let $X \subseteq M$. Let a be a cell in the medial graph. Then if $X' = X \cup \{a\}$ we say that X' is a **simple** extension of X if a and three corners in X meet at an anticorner of X. We say X' is a **nice** simple extension if a touches exactly one anticorner of X (i.e. is adjacent to exactly two cells in X). If X'' is obtained from X by a series of simple extensions then X'' is an extension of X. If X'' is obtained by a sequence of simple nice extensions, then X'' is a nice extension of X.

Theorem 2.5.2 ([4])**.** Let M be a finite critical medial graph and let $X \subseteq M$ be convex such that $X \cap \partial M \neq \varnothing$. Then there is some set of cells $S \subseteq \partial M \setminus X$ such that the entire medial graph is a nice extension of $X \cup S$.

Another result of Will Johnson's work that we get trivially is that

Remark 2.5.3. If c is a cell in $\partial M \setminus X$ which is adjacent to X, then we can pick S in Theorem 2.5.2 such that $c \in S$.

We leave it to the reader to verify the last statement, but it follows trivially. We wish to apply Theorem 2.5.2 to sets of the form $B(g)$ for geodesics in a medial graph. Though the theorem seems like it should obviously carry over in some form, we will provide justification, and then leave some of the details to the reader. The idea is that if $X = U(g)$ for some geodesic g, then we want to be able to a single cell a from ∂M which borders X so that $\overline{X \cup \{a\}}$ is a simple extension of X, but the theorem doesn't quite apply. To make it apply, we will construct a new medial graph, M', which has $B(g)$ embedded so that

1. all geodesics in $B(g)$ are geodesics in M';
2. the geodesic g is a geodesic in M'
3. $M' \setminus B(g)$ is a closed and connected set of cells
4. M' is a critical medial graph.

If the above four conditions are satisfied, then we can just apply 2.5.2 to see that all M is a simple extension of $X = U(g)$ since any simple extensions of a superset of X in M corresponds to a simple extension of a superset of $M' \setminus B(g)$ in the obvious fashion.

We now need to construct such a medial graph M'. Thus turns out to actually be really easy. First, conformally map $B(g)$ (as a closed subset of $\mathbb{C}$) onto the closed unit disc. This is possible via the Riemann Mapping Theorem (which ensures that such a conformal map exists between the interiors of those regions) and Catheodory's Theorem (which states that since the boundaries of both regions are Jordan curves, the conformal map shown to exist by the Riemann mapping theorem extends continuously to the boundary). Now smooth each of the geodesics near the boundary so that if g is a geodesic then (if we view g as a function from $[0, 1]$ to $B(g)$ (reparametrizing if necessary)) then $\lim_{t\to 1^-} g'(t)$ and $\lim_{t\to 0^+} g'(t)$ both exist. We can do this by simply replacing g with a linear function sufficiently close to the boundary such that we don't get any additional intersections of regions. Now let z_1 and z_2 be the two points on $\partial\mathbb{D}$ which correspond to the points of intersection of g and $\mathbb{R}$ in our original medial graph and let R_1 and R_2 denote the two arcs of of $\partial\mathbb{D}$ between z_1 and z_2. Without loss of generality let R_1 denote the arc corresponding to the geodesic g. Extend two rays segments from z_1 and z_2 radially. Call these line segments

ℓ_1 and ℓ_2. Now extend each geodesic which intersects R_1 along a straight line corresponding to the limit of its derivative as it approaches R_1. Now using a basic compactness argument, we can pick a $\rho > 1$ such that none of the extended geodesics or ℓ_1 or ℓ_2 intersect in the closed ball of radius ρ. Let x_1 and x_2 denote two points along ℓ_1 and ℓ_2 of norm ρ and let $\widetilde{R}$ denote the circular arc ρR_1. Let H be the region bounded by R_2, the two line segments $[z_1, x_1]$, $[z_2, x_2]$ and the arc $\widetilde{R}$. Clearly the extended geodesics will divide H into regions, which can be interpreted as medial graph cells for a medial graph M' in the obvious way. If so desired we can map the region H to the disc and then smooth the geodesics at the boundary as before. This procedure is summarized in Figure 2.19

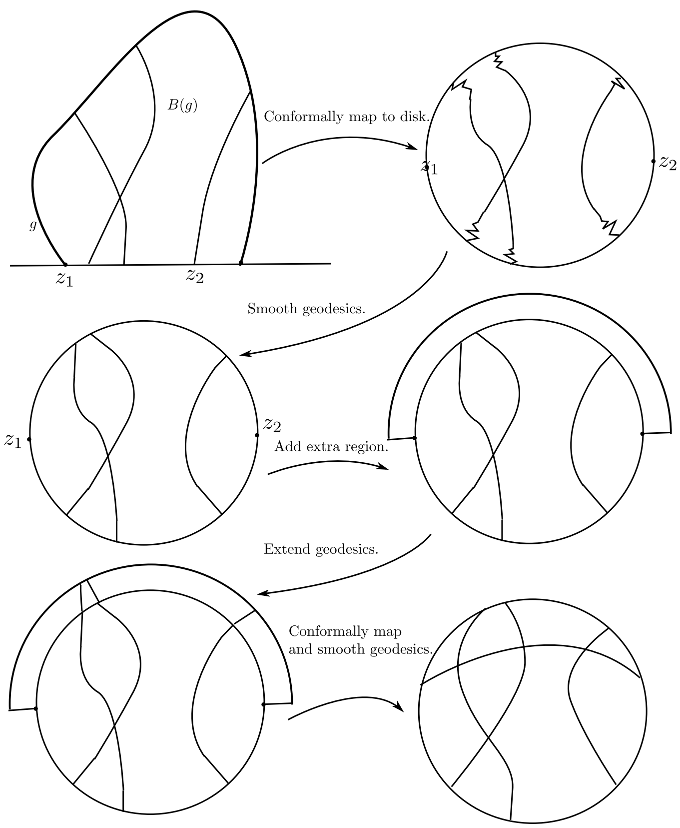


Figure 2.19: An outline of the method of extending $B(g)$ so that $B(g)$ is the complement of a convex subset containing a connected subset of the boundary of a critical circular planar medial graph.

Obviously the medial graph M' satisfies the stated properties. Using Theorem 2.5.2 and the above observations, we get that:

Theorem 2.5.4. If M is an supercritical half planar graph and $X = U(g)$, then there is a finite set of boundary cells $S \subseteq \partial M$ such that M is a simple extension of $X \cup S$. Furthermore, if $c \in B(g) \cap \partial M$ and c borders g, then we can pick S so that $c \in S$.

2.6 Recovery

We are now nearly complete with most of our technical work and we need only to prove some basic facts which will allow for recovery. We first will show how to recover boundary spikes and boundary-to-boundary edges, which will turn out to be sufficient.

2.6.1 Recovering Boundary Spikes

A boundary spike is a pair of vertices $(v_\partial, v_{\text{int}})$ such that $v_\partial \in \partial G$ and $v_{\text{int}} \in \operatorname{int} G$ and there is an edge between v_∂ and v_{int} but there are no other edges connected to v_∂. *Throughout we assume that boundary spikes are never boundary-to-boundary edges*, i.e. every boundary spike consists of a boundary vertex connected to exactly one interior vertex and no other boundary vertices.

Given an electrical network Γ with a boundary spike, we can form the network Γ_0 where we remove the given boundary spike. There is an natural map from Γ_0 into Γ which maps a vertex in Γ_0 to the corresponding vertex in Γ. We will call this map $\alpha : \Gamma_0 \to \Gamma$. We have a simple but important lemma:

Lemma 2.6.1. Let Γ be an infinite electrical network with a boundary spike $(v_\partial, v_{\text{int}})$ and let Γ_0 be the electrical network corresponding to contracting $(v_\partial, v_{\text{int}})$. Let $\phi \in Z(\Gamma)$. If $\phi \in M(\Gamma)$ then $\phi_0 \stackrel{\text{def}}{=} \phi \circ \alpha \in M(\Gamma_0)$. If $\phi \in H(\Gamma)$ and $\phi_0 \in M(\Gamma)$ then $\phi \in M(\Gamma)$.

Proof. Let W be the subset of $Z(\Gamma)$ consisting of functions which are constant on the boundary of Γ and let W_0 be the subset of $Z(\Gamma_0)$ consisting of functions which are constant on the boundary of Γ_0. Notice that there is an obvious embedding of W_0 into W by sending a function u which takes value c on the boundary of Γ_0 to the function which is u on $\operatorname{Im} \alpha$ and takes value c on v_∂. If $u \in W_0$ let $\widetilde{u}$ denote the element of W as described.

Now to proceed with the proof of the lemma, suppose that $\phi \in M(\Gamma) = W^\perp$, then we wish to show that $\phi_0 \in W_0^\perp$. But to do this, we just note that if $u \in W_0$ then $\widetilde{u} \in W$ and hence

$$\begin{aligned}
(\phi_0, u)_{Z(\Gamma_0)} &= \sum_{V(G)\setminus\{v_\partial\} \times V(G)\setminus\{v_\partial\}} \gamma_{vv'}(\phi_0(v) - \phi_0(v'))(u(v) - u(v')) \\
&= 2\gamma_{v_\partial v_{\text{int}}}(\phi(v_\partial) - v_{\text{int}})(\widetilde{u}(v_\partial) - \widetilde{u}(v_{\text{int}})) \\
&\quad + \sum_{V(G)\setminus\{v_\partial\} \times V(G)\setminus\{v_\partial\}} \gamma_{vv'}(\phi_(v) - \phi_(v'))(u(v) - u(v')) \\
&= (\phi, \widetilde{u})_{Z(\Gamma)}
\end{aligned}$$

so that $\phi_0 \in W_0^\perp$. Now to prove the other direction, suppose that $\phi_0 \in W_0^\perp$. Let $u \in W$, without loss of generality, assume that u is 0 on ∂G. Let u_0 denote $u \circ \alpha$. Note that $u = \widetilde{u_0} + \chi_{v_{\rm int}} u(v_{\rm int})$ where $\chi_{v_{\rm int}}$ denotes the indicator function on the set $\{v_{\rm int}\}$. Hence

$$\begin{aligned}(u, \phi)_{Z(\Gamma)} &= (\widetilde{u_0} + \chi_{v_{\rm int}} u(v_{\rm int}), \phi)_{Z(\Gamma)} \\ &= (\widetilde{u_0}, \phi)_{Z(\Gamma)} + (\chi_{v_{\rm int}} u(v_{\rm int}), \phi)_{Z(\Gamma)}.\end{aligned}$$

By the previous computation, we know that $(\widetilde{u_0}, \phi)_{Z(\Gamma)} = (u_0, \phi_0)_{Z(\Gamma_0)}$ which is zero by assumption. On the other hand, we know that $(\chi_{v_{\rm int}} u(v_{\rm int}), \phi)_{Z(\Gamma)}$ is zero since it is exactly the current entering the vertex $v_{\rm int}$ and $\phi \in H(\Gamma)$. Hence $\phi \in W^\perp = M(\Gamma)$ so we are done. □

Lemma 2.6.2. Given the minimal Dirichlet-to-Neumann map $\Lambda_M(\Gamma)$ and a boundary spike with given conductance $\gamma_{v_\partial v_{\rm int}}$, we can find the minimal Dirichlet-to-Neumann map $\Lambda_M(\Gamma_0)$ for the network Γ_0 with the boundary spike contracted.

Proof. Let G_0 denote the graph of Γ_0 (the contracted network) and let $\phi : \partial G_0 \to \mathbb{R}$ be a function such that there exists a $\widetilde{\phi} : V(G_0) \to \mathbb{R}$ of finite power such that $\phi = \hat{\phi}|_{\partial G_0}$. Now define $\hat{\widetilde{\phi}}$ to be $\widetilde{\phi}$ on $V(G_0)$ and define $\hat{\widetilde{\phi}}$ to be the unique γ-harmonic extension of $\widetilde{\phi}$ to the boundary spike. Clearly this exists and is unique. Furthermore, $\hat{\widetilde{\phi}}$ is also clearly finite power. By Lemma 2.6.1, we know that $\hat{\widetilde{\phi}}$ is of minimal power for its boundary voltages. We should note that $\phi : \partial G_0 \to \mathbb{R}$ and $\Lambda_M(G)$ uniquely determines $\hat{\widetilde{\phi}}(v_\partial)$ since

$$\Lambda_M(G)(\hat{\widetilde{\phi}}|_{\partial G})(v_\partial) = \gamma_{v_\partial v_{\rm int}}(\phi(v_{\rm int}) - \hat{\widetilde{\phi}}(v_\partial)),$$

and $\gamma_{v_\partial v_{\rm int}}$ is nonzero so we can just solve for $\hat{\widetilde{\phi}}(v_\partial)$ in terms of known quantities. By γ-harmonicity we know that

$$\Lambda_M(G_0)(\phi)(v_{\rm int}) = \Lambda_M(G)(\hat{\widetilde{\phi}}|_{\partial G}).$$

Furthermore, $\Lambda_M(G)(\hat{\widetilde{\phi}}|_{\partial G})(v) = \Lambda_M(G_0)(\phi)(v)$ for $v \in \partial G$ such that $v \neq v_{\rm int}$. Hence $\Lambda_M(G)$ and the conductivity $\gamma_{v_{\rm int} v_\partial}$ uniquely determine $\Lambda_M(G_0)$. □

Lemma 2.6.3. Given a supercritical half planar electrical network Γ with a boundary spike $(v_\partial, v_{\rm int})$, the minimal Dirichlet-to-Neumann map Λ_M uniquely determines the conductivity $\gamma_{v_\partial v_{\rm int}}$.

Proof. Let x_∂ denote the medial graph cell corresponding to v_∂. We note that x_∂ is a geodesic triangle, i.e. there are two geodesics which bound x_∂ and the other edge of x_∂ is an interval of $\mathbb{R}$. By Lemma 2.3.9 we know that $x_\partial \in B(g)$ for one of the two geodesics which is borders x_∂. We will progressively define the function (ϕ, ψ) where ϕ is a voltage function on Γ and ψ is a covoltage function (on $\Gamma^\dagger$) such that

$$\psi = (\Phi_{\Gamma^\dagger} \circ D_\Gamma)(\phi).$$

Firstly define ϕ and ψ to be 0 on $\partial M \cap U(g)$. By Lemma 2.4.2 we know that ϕ and ψ are zero on $U(g)$. By Lemma 2.5.4, we know that we can pick an $S \subseteq \partial M$ such that $x_\partial \in S$ and M is a simple extension of $X \cup S$. Define ϕ to be 1 on x_∂ and specify ϕ and ψ arbitrarily on the other cells of S. Just as in the finite case, we know that under these conditions we can extend ϕ and ψ to be defined on all of V and $V^\dagger$ such that ϕ is γ-harmonic on $\operatorname{int} G$ and ψ is $\gamma^\dagger$-harmonic on $\operatorname{int} G^\dagger$ and

$$\psi = (\Phi_{\Gamma^\dagger} \circ D_\Gamma)(\phi)$$

(we leave these details to the reader, but it is identical to the finite case, so the interested reader can read [4]). Thus we can find a γ-harmonic function $\phi : V \to \mathbb{R}$ such that $\phi(v_\partial) = 1$ and $\phi(v) = 0$ for $v \in U(g) \cap \partial M$. Further we have that if $\psi = (\Phi_{\Gamma^\dagger} \circ D_\Gamma)(\phi)$ then ψ is (up to a constant) 0 on $U(g) \cap \partial M$. By Lemma 2.4.2 we know that any functions ϕ and ψ which satisfy those conditions will also satisfy $\phi(v_{\text{int}}) = 0$. We note that ϕ is of finite power since it is finitely supported, and furthermore, since it is finitely supported, by Lemma 1.5.2 we know that $\phi \in M(\Gamma)$. Thus we can find a ϕ and ψ satisfying the above conditions such that (using the notation from the previous chapter) that

$$(\partial \circ \Lambda_M)(\phi|_{\partial G}) = \psi|_{\partial G}.$$

Now we are in the same situation as in the finite case, and we know that

$$\Lambda_M(\phi|_{\partial G})(v_\partial) = \gamma_{v_{\text{int}}\, v_\partial} \cdot (1 - 0)$$

which immediately gives us $\gamma_{v_{\text{int}}\, v_\partial}$. □

2.6.2 Recovering Boundary-to-Boundary Edges

We recover boundary-to-boundary edges in a very similar fashion to how we recovered boundary spikes. We define a boundary-to-boundary edge to be an edge $v_1 v_2 \in E(G)$ such that $\gamma_{v_1 v_2} \neq 0$ and $v_1, v_2 \in \partial G$. We note that if we remove a boundary-to-boundary edge, we may be left with a disconnected graph, but that doesn't matter, since we can define medial and dual graphs for disconnected graphs. Similarly all of the results about the minimal boundary value maps were not dependent on the graph being connected.

Lemma 2.6.4. Let Γ be an electrical network with boundary-to-boundary edge $v_1 v_2$ and suppose Γ_0 is the network resulting from removing this edge. There is an obvious map between the vertices in these networks, which we will call $\beta : V(G_0) \to V(G)$ which is essentially just the identity. Let $\phi \in Z(\Gamma)$. Then $\phi \in M(\Gamma)$ iff $(\phi \circ \beta) \in M(\Gamma_0)$.

Proof. Suppose $\phi \in M(\Gamma) = W(G)^{\perp}$ and let $u \in W(G_0)$. Notice that β is a bijection and $\beta(W(G)) = W(G_0)$. Hence $(\phi, u \circ \beta^{-1})_{Z(\Gamma)} = 0$ by assumption. Denote $u \circ \beta^{-1}$ by $\widetilde{u}$. We simply note that since $\widetilde{u}(v_1) = \widetilde{u}(v_2)$ since v_1 and v_2 are boundary vertices. Hence

$$\begin{aligned}
(\phi, \widetilde{u})_{Z(\Gamma)} &= \sum_{V(G)\times V(G)} \gamma_{vv'}(\phi(v) - \phi(v'))(\widetilde{u}(v) - \widetilde{u}(v')) \\
&= \sum_{(V(G)\times V(G))\setminus\{v_1v_2, v_2v_1\}} \gamma_{vv'}(\phi(v) - \phi(v'))(\widetilde{u}(v) - \widetilde{u}(v')) \\
&= \sum_{V(G_0)\times V(G_0)} \gamma_{vv'}((\phi\circ\beta)(v) - (\phi\circ\beta)(v'))(u(v) - u(v')) \\
&= (\phi\circ\beta, u)_{Z(\Gamma_0)}
\end{aligned}$$

and hence $(\phi \circ \beta, u)_{Z(\Gamma_0)} = 0$. Hence $\phi \circ \beta \in Z(\Gamma_0)$. The other direction follows by reversing the order we presented the above inequalities.

□

Lemma 2.6.5. Given the minimal Dirichlet-to-Neumann map $\Lambda_M(\Gamma)$ and a boundary-to-boundary edge v_1v_2 with conductance $\gamma_{v_1v_2}$, we can find the minimal Dirichlet-to-Neumann map $\Lambda_M(\Gamma_0)$ for the connected network Γ_0 resulting from the removal of the edge v_1v_2.

Proof. By Lemma 2.6.4 we know that the minimal voltage functions on Γ_0 are the same as they are on Γ, and hence given valid boundary data, the Dirichlet solutions are the same. If ϕ is a voltage function, by definition, the current leaving a boundary vertex v is given by the formula

$$\Lambda_M(\Gamma)(\phi)(v) = \sum_{v'\sim_G v} \gamma_{vv'}(\phi(v) - \phi(v')).$$

If v is not v_1 or v_2 this is unchanged and hence $\Lambda_M(\Gamma)(\phi)(v) = \Lambda_M(\Gamma_0)(v)$. For v_1 and v_2, we just compute immediately that

$$\Lambda_M(\Gamma)(\phi)(v_1) = \sum_{v'\sim_G v_1} \gamma_{v_1v'}(\phi(v_1) - \phi(v'))$$

$$= \gamma_{v_1v_2}(\phi(v_1) - \phi(v_2)) + \sum_{v'\sim_{G_0} v_1} \gamma_{v_1v'}(\phi(v_1) - \phi(v'))$$

$$= \Lambda_M(\Gamma_0)(v_1) + \gamma_{v_1v_2}(\phi(v_1) - \phi(v_2)),$$

which gives us a formula for $\Lambda_M(\Gamma_0)(v_1)$. Using an identical argument we can find a nearly identical formula for $\Lambda_M(\Gamma_0)(v_2)$. □

Lemma 2.6.6. Given a supercritical half planar electrical network Γ with a boundary-to-boundary edge v_1v_2, the map $\Lambda_M(\Gamma)$ uniquely determines $\gamma_{v_1v_2}$.

Proof. The proof is essentially the same as in the case of boundary spikes because now we are recovering a boundary spike of the dual graph. We leave the details to the reader, but essentially we find a geodesic g which crosses the boundary spike in the dual graph and such that the boundary vertex of this boundary spike is in $B(g)$. We find boundary values which will guarantee that the interior vertex of this boundary spike will have covoltage zero but such that the boundary vertex will have covoltage 1. We leave the details to the reader since they are identical to before. □

2.6.3 Recovering the Entire Graph

Definition 2.6.7. We will define a **triangular geodesic region** to be a bounded subset of the medial graph whose boundary consists of two exactly geodesic segments and one connected subset of $\mathbb{R}$. We will define a **geodesic triangle** to be a triangular geodesic region R such that there are no geodesics which cross the boundary of R.

Lemma 2.6.8. Let g be a geodesic in a supercritical half planar medial graph and suppose that g intersects at least one other geodesic. Then there is a geodesic triangle (possibly with other geodesics inside of it) in $B(g)$.

Proof. The proof is essentially the same as in [1]. Let x_ℓ^g and x_r^g be the left and right endpoints of g. Let g' be the geodesic which crosses g closest to x_ℓ. One of the endpoints of g' must be in $B(g)$ since the graph is supercritical. This produces a triangular region t. Let g'' be the geodesic which crosses g' closest to this region. Notice that g'' cannot intersect g' again and g'' cannot intersect g at any point in $\bar{t}$ since we assumed that g' was the geodesic which was closest to x. This yields a geodesic triangle $t' \subseteq t \subseteq B(g)$. We summarize in Figure 2.20.

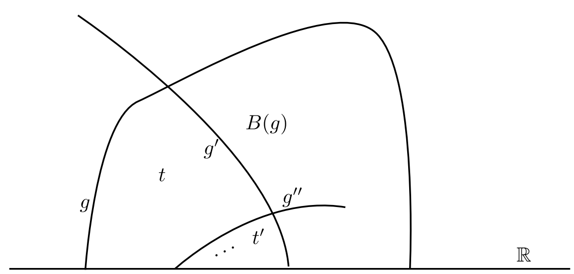


Figure 2.20: A decreasing sequence of triangular geodesic regions in $B(g)$.

We repeat this process to get a descending sequence of triangular geodesic regions which must eventually terminate with a geodesic triangle since $B(g)$ is a finite subset of the medial graph. We note that we don't claim the resulting geodesic triangle is empty. □

Lemma 2.6.9. Removal of a boundary-to-boundary edge, a boundary spike, or removing a finite connected component all preserve supercriticality.

Proof. All operations obviously perserve half-planarity. The first two operations preserve supercriticality since the change in the medial corresponds to just merging two cells as shown in Figure 2.21.

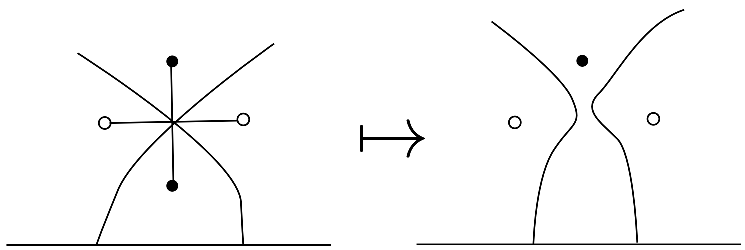


Figure 2.21: The result on the medial graph from a boundary spike contraction. The result of a boundary-to-boundary edge deletion is the same since the two processes are dual to each other.

And hence we will never have two geodesics if they didn't cross before removing a boundary spike or boundary to boundary edge. If there is a finite connected component of the graph, then one there must be a dual cell which has multiple boundary components on $\mathbb{R}$ and hence deleting the finite subset will just correspond to deleting some geodesics, as shown in the below picture. We leave the details to the reader, but the claim is clear. ☐

Lemma 2.6.10. Let G be a supercritical half planar graph and let e be any edge in G. Then e can be removed by sequentially removing boundary spikes, boundary-to-boundary edges or deleting finite connected components.

Proof. Let g be a geodesic which crosses the edge e. we will sequentially remove every edge with a vertex in $B(g)$ (corresponding to removing a cell in the medial graph). So we proceed by induction, showing that we can always remove cells in $B(g)$ or merge cells in $B(g)$ without altering any of the rest of the medial graph. To do this, we note that since e crosses g, there must by another geodesic which crosses e and hence crosses g. Thus we can apply Lemma 2.6.8 to find a geodesic triangle t in $B(g)$. If t is empty, then we can simply perform remove a boundary spike or boundary to boundary edge, which results in just merging two cells as shown in Figure 2.21.

As shown in Lemma 2.6.9 this preserves supercriticality. If t is nonempty then since there are no geodesics which cross t, we know that the vertices of G in t cannot be connected to any of the vertices in $M \setminus t$, and hence the vertices in t correspond to a connected component so we can remove them. As shown in Lemma 2.6.9, we know that this preserves supercriticality and deleting the primal vertices in t correspond to removing the geodesics inside of t which don't cross t. Eventually we must get that no geodesics cross g, which implies that we have removed e. □

Theorem 2.6.11. Given a supercritical half planar network graph G and the minimal Dirichlet-to-Neumann map Λ_M for some electrical network Γ with graph G, we can recover all of the conductivities of Γ.

Proof. Let e be an edge in G. By Lemma 2.6.10 we can remove e from the graph by sequentially removing boundary spikes, boundary-to-boundary edges and finite connected components. Given a boundary spike, or a boundary-to-boundary edge we can recover the conductivity by Lemmas 2.6.3 and 2.6.6 we can recover the conductivity along it. By Lemmas 2.6.2 and 2.6.5 we can find the Dirichlet to Neumann maps of the resulting graphs. Given a finite connected component of G, the Dirichlet-to-Neumann map for the finite component is just the restriction of the minimal Dirichlet-to-Neumann map for the infinite graph, and since the finite component must obviously be critical circular planar we can recover all of the conductivities of the edges in the finite component. The minimal Dirichlet-to-Neumann map for the other components will just be the restriction of the Dirichlet-to-Neumann map restricted to the complement of the finite component. Repeating this process as per Lemma 2.6.10, we will recover the entire graph. □

We note that there there is an obvious analogue for Lemmas 2.6.2, and 2.6.5 for the minimal Neumann-to-Dirichlet map H_M. Similarly, the results from Lemmas 2.6.3 and 2.6.6 hold for Λ_M replaced with H_M since the proof basically carries over without change. Thus we have the proposition, the details are left the reader, but are essentially just as above:

Proposition 2.6.12. Given a supercritical half planar network graph G and the minimal Neumann-to-Dirichlet map H_M for some electrical network Γ with graph G, we can recover all of the conductivities of Γ.

Chapter 3

Determinants and the Cutpoint Lemma.

In this section we will continue our development of infinite networks by proving a determinant connection relation for half planar networks and then using this to prove an infinite version of the cutpoint lemma. Along the way, we generalize many of the notions from finite networks.

3.1 Useful Preliminaries

3.1.1 Maximum Principle for Infinite Networks.

On infinite electrical networks, one can easily cook up examples of finite power harmonic functions which do not satisfy any sort of maximal principle. The crucial problem is that with infinite graphs, it is possible for current to be "absorbed at infinity" or for other such nonsense to happen. Fortunately we can actually develop a maximum principle for minimal functions.

Theorem 3.1.1. Let $\phi \in M(\Gamma)$ satisfy $a \leq \phi(v) \leq b$ for all $v \in \partial G$. Then $a \leq \phi(v) \leq b$ for all $v \in G$.

Proof. Suppose $\Gamma = (G, \gamma)$ is an infinite electrical network and $\phi \in M(\Gamma)$ and further that $m \leq \phi(v) \leq M$ for all $v \in \partial G$. Let $G_1 \subseteq G_2 \subseteq \cdots \subseteq G$ be an increasing chain of finite connected subsets of G such that $\bigcup_j G_j = G$. Also assume that if $v, v' \in G_j$ and the edge $vv' \in G$ then $vv' \in G_j$. Define ∂G_j to be $\partial G \cap G_j$. Define the finite electrical networks $\Gamma_j = (G_j, \gamma_j)$ where $\gamma_j = \gamma|_{(V_j \times V_j)}$. Let ψ_j be the power minimizing function on G_j subject to $\psi_j|_{\partial G_j} = \phi|_{\partial G_j}$. By the maximum principle for finite graphs we know that $m \leq \psi_j(v) \leq M$ for all $v \in G_j$. Let P_j denote the power function for the electrical network Γ_j. Let the vertex set of G_j be denoted by V_j. By definition, we have that if $f : V_j \to \mathbb{R}$ is a function then

$$P_j(f) = \sum_{(v,v') \in V_j \times V_j} \gamma_{vv'} (f(v) - f(v'))^2$$

and hence if $f : V \to \mathbb{R}$ is a vertex function defined on G, then we have that

$$0 \leq P_j(f|_{V_j}) \leq P(f).$$

We note that $\phi|_{\partial G_j}$ is a vertex function on G_j which satisfies $\phi|_{\partial G_j} = \phi|_{\partial G_j}$ and hence

$$P_j(\psi_j) \leq P_j(\phi|_{\partial G_j}) \leq P(\phi). \tag{3.1}$$

Since $m \leq \psi_j \leq M$ we know that we can take a subsequence which converges pointwise for each $v \in V$ (take the standard diagonal subsequence). By passing to a subsequence, suppose that psi_j converges pointwise to a function ψ. Since

$$P_j(\psi_j) = \sum_{V \times V} \gamma_{vv'}(\psi_j(v) - \psi_j(v'))^2 \chi_{V_j}(v)\chi_{V_j}(v')$$

(where χ_{V_j} is the indicator function), we know that $\gamma_{vv'}(\psi_j(v)-\psi_j(v'))\chi_{V_j}(v)\chi_{V_j}(v')$ converges pointwise to $\gamma_{vv'}(\psi_(v) - \psi_(v'))^2$ and is nonnegative, so we just apply Fatou's lemma and Equation (3.1) to see that

$$P(\psi) \leq \liminf_{j \to \infty} P_j(\psi_j) \leq \liminf_{j \to \infty} P(\phi) = P(\phi).$$

Hence ψ is a finite power function. Furthermore, we quickly verify that $\psi|_{\partial G} = \phi|_{\partial G}$ and hence by the uniqueness of a minimal function with given boundary data we know that $\psi = \phi$. But we had that $a \leq \psi_j \leq b$ for all j by the maximum principle for finite graphs. This in equality will be preserved in the limit and hence $a \leq \phi \leq b$. □

Alternate Proof of Theorem 3.1.1. It is sufficient to show that $\phi \in M(\Gamma)$ and $\phi|_{\partial G} \leq 0$ implies that $\phi \geq 0$. Suppose that this weren't the case. Then we can just define the function $\psi : V \to \mathbb{R}$ by

$$\psi(v) = \begin{cases} \phi(v) & \text{if } \phi(v) \geq 0 \\ 0 & \text{if } \phi(v) < 0 \end{cases}.$$

Notice that $\psi|_{\partial G} = \phi|_{\partial G}$. Along any edge between vertices v and v' such that $\phi(v)$ and $\phi(v')$ are both positive, the power of ψ along the edge vv' is the same as the power of ϕ along that edge. If $\phi(v) \geq 0$ and $\phi(v') \leq 0$, then the power along the edge vv' is nonstrictly reduced, and strictly reduced iff $\phi(v') < 0$. Similarly if $\phi(v) \leq 0$ and $\phi(v') \leq 0$, then the power is nonstrictly reduced. Summing over the power on all edges, we see that $0 \leq P(\psi) \leq P(\phi)$ and that $P(\psi) < P(\phi)$ if there is an edge vv' such that $\phi(v) < 0 \leq \phi(v')$. Since G is assumed to be connected (as always!), one easily sees that if there is any vertex v'' such that $\phi(v'') < 0$ then we can find vertices v and v' such that $\phi(v) < 0 \leq \phi(v')$. But since $\phi|_{\partial G} = \psi|_{\partial G}$ this implies that ϕ did not have minimal power for its boundary conditions and hence was not minimal. The statement of the theorem follows.

□

We now present a useful theorem that generalizes one of the ideas in the first proof of the maximum principle.

Theorem 3.1.2. Suppose $\Gamma = (G, \gamma)$ is an infinite electrical network and $\phi \in M(\Gamma)$ is a minimal function. Then we can write ϕ as a pointwise limit of γ-harmonic functions on an increasing chain of finite connected subnetworks of Γ which share the same boundary values on ϕ.

Proof. Since G is connected, arbitrarily pick an increasing chain of connected subgraphs $G_0 \subseteq G_1 \subseteq \cdots$ of G. Turn each G_i into an electrical network by defining $\partial G_i = \partial G \cap G_i$ and $\operatorname{int} G_i = V(G_i) \setminus \partial G_i$. Define a conductivity function on each G_i by simply setting $\gamma_i : E(G_i) \to \mathbb{R}^+$ to be the restriction of γ to G_i. Now let $\phi_n : V(G_i) \to \mathbb{R}$ be the unique γ-harmonic function on G_i which satisfies $\phi_n|_{\partial G_i} = \phi|_{\partial G_i}$. As in the proof of the maximum principle we have that

$$P_{G_i}(\phi_i) \leq P_{G_i}(\phi|_{G_i}) \leq P(\phi).$$

Now we claim that for each vertex $v \in G$ the sequence $\{\phi_i(v)\}$ is bounded (for all i which are large enough so that it makes sense). Let $d(v, \partial G)$ denote the minimum number of edges in a path from a vertex $v \in G$ to a vertex in ∂G. We will prove that $\{\phi_i(v)\}$ is bounded by induction on $d(v, \partial G)$. Clearly the claim is true for all vertices v such that $d(v, \partial G) = 0$ since if $d(v, \partial G) = 0$ then $v \in \partial G$ and hence $\phi_i(v) = \phi(v)$ is just constant. Now suppose that $\{\phi_i(v)\}$ is bounded if $d(v, \partial G) \leq k$ and suppose that $d(u, \partial G) = k + 1$. Then there is a vertex $v \in G$ such that there is an edge between vertices u and v and $\phi_i(v)$ is bounded. If $\phi_i(u)$ is unbounded, then clearly the sequence

$$\gamma_{uv}(\phi_i(u) - \phi_i(v))^2$$

is also unbounded since $\phi_i(v)$ is bounded and $\gamma_{uv} > 0$. But since

$$\gamma_{uv}(\phi_i(u) - \phi_i(v))^2 \leq P_{G_i}(\phi_i) \leq P(\phi),$$

this is clearly a contradiction, and hence we know that $\phi_i(u)$ is bounded for all u such that $d(u, \partial G) \leq k + 1$. By induction, since G is connected we know that $\phi_i(v)$ is bounded for all $v \in G$.

Since $\phi_i(v)$ is bounded for each v, by taking an appropriate diagonal subsequence of ϕ_i, we can find a subsequence of G_i such that $\phi_i(v)$ converges for each v. If we let ψ denote the limiting function, applying Fatou's lemma shows that

$$P(\psi) \leq \liminf_{i \to \infty} P_i(\psi_i) \leq \liminf_{i \to \infty} P(\phi) = P(\phi)$$

so ψ is a finite power function with the same boundary values as ϕ, so $\phi = \psi$ by minimality. □

3.2 Connections and Determinants

As in the finite case, we can consider k-connections in an infinite half planar graph. There are several ways to generalize k-connections to the infinite case.

The most useful and general way of doing so is with the notion of a *flowout*, which is just a k-connection between sets of boundary vertices with a certain type of ordering on the boundary.

If v_1 and v_2 are boundary vertices, we define the $q(v_1, v_2)$ by the formula

$$q(v_1, v_2) = (\chi_{v_1}, \Lambda\chi_{v_2}).$$

Note that since Λ is self adjoint we have that $q(v_1, v_2) = q(v_2, v_1)$. To explain what q is, we note that it is just the current that comes of vertex v_2 from the minimal function that has boundary value 1 at v_1 and 0 everywhere else on the boundary.

Theorem 3.2.1. If Γ is any electrical network and v_1 is connected through the interior to v_2, then $q(v_1, v_2) < 0$. If v_1 and v_2 are any distinct vertices, we have that $q(v_1, v_2) \leq 0$.

Proof. Note that $q(v_1, v_2)$ is just the current that leaves vertex v_2 of the function $\Lambda\chi_{v_1}$. By the maximum principle we have that $\Lambda\chi_{v_1}$ takes values in the interval $[0, 1]$. Hence the current leaving vertex v_2 cannot be positive since it is just $\sum_{v\sim v_2} \gamma_{v,v_2}(0 - \Lambda\chi_{v_1}(v))$ and $\Lambda\chi_{v_1}(v) \in [0, 1]$ for all v. Hence we have that $q(v_1, v_2) \leq 0$ (independent of the fact that there is a connection between v_1 and v_2).

Suppose to the contrary of the theorem statement that $q(v_1, v_2) = 0$. All vertices which neighbor v_2 must have voltage 0, since by the maximum principle they must have nonnegative voltage, and hence no voltage can flow from v_2 to any other vertex, and the net current at v_2 is zero. Proceeding by induction along the interior path from v_2 to v_1 shows that the voltage at v_1 must be zero, a contradiction. □

If $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_m)$ are collections of boundary vertices, let $Q(A, B)$ be the matrix $(q(a_i, b_j))_{i,j}$.

There are several "natural" ways to generalize the notion of a k-connection on an infinite network. One obvious notion would be to consider to sets $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$ of boundary vertices such that $a_1 < \cdots a_k < b_1 < \cdots < b_k$ such that there are vertex disjoint paths from a_i to b_i. When one considers a half planar embedding for a finite circular planar embedding, one realizes that this doesn't actually capture all of the original ciricular k-pairs. Hence we make the following definition:

Definition 3.2.2. Let $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$ are collections of k boundary vertices. We will say that there is a flowout from A to B (written $A \rightsquigarrow B$) if $a_1 < a_2 < \cdots < a_k$, and $b_1 < b_2 < \cdots < b_k$, there are no elements of B in the interval $[a_1, a_k] \subseteq \mathbb{R}$ and there is a k-connection through the interior from A to B.

The idea of $A \rightsquigarrow B$ is a natural generalization of A and B being a circular pair in the finite case. The next theorem will end up being very useful.

Remark 3.2.3. Suppose Γ is a finite circular planar graph embedded in the closed unit disk $\overline{\mathbb{D}}$. Let p be some point of $\partial\mathbb{D}$ which does not correspond to a vertex of Γ. If we identify $\mathbb{D} \setminus \{p\}$ with the closed half plane $\mathbb{H}$, then the circular k-connections of Γ given its embedding in $\mathbb{D}$ are in one to one correspondence with the k-flowouts of Γ embedded in $\mathbb{H}$.

Theorem 3.2.4. If $A \rightsquigarrow B$ then $Q(A, B)$ is a nonsingular matrix.

Proof. This will be a proof by induction on the size of the k-connection. The $k = 1$ case is covered by Theorem 3.2.1 This will be a proof by contradiction. Suppose that $A = (a_1, \ldots, a_k)$ and $B = (b_1, b_2, \ldots, b_k)$ form a k-circular pair, i.e. there are paths γ_j from a_j to b_j such that all of the paths γ_j are vertex disjoint. Let J be the maximum index such that b_J is to the left of $[a_1, a_k]$. If none of the b_j are to the left of $[a_1, a_k]$. By possibly reflecting the entire embedding of our graph about some vertical line, we can assume that there are indeed vertices in the collection $b_1, \ldots, b_k$ which are to the left of the closed interval $[a_1, a_k]$.

If $Q(A, B)$ were singular, then there would be a a minimal function ϕ such that the support of $\phi|_{\partial G}$ is contained in $\{a_1, a_2, \ldots, a_k\}$ and $(\phi, \chi_{b_j}) = 0$ for all $j \in \{1, \ldots, k\}$, and furthermore that ϕ is not the zero function. This is just saying that there is a nonzero minimal function $\phi \in M(\Gamma)$ such that ϕ is zero at all boundary vertices except for those in the collection $\{a_1, \ldots, a_k\}$ and the current of ϕ leaving each of the vertices $b_1, \ldots, b_k$ is zero. Since ϕ is not zero, and ϕ is minimal, we know that ϕ cannot be zero on the boundary. By the inductive hypothesis we know that ϕ cannot be zero on any of the vertices $a_1, \ldots, a_k$.

We will now construct $2k$ paths through the interior of Γ. Construct the paths $\alpha_1, \ldots, \alpha_k$ as follows. Let α_j start at a_j. Set α_j to be just the path γ_j until the last vertex v in γ_j which has zero voltage. By what we've said above, this must be an interior vertex. At v, we know that there must be both positive and negative voltages at adjacent vertices by the local maximum principle. Pick the next vertex in the path α_j to be any vertex adjacent to v which has the minimum voltage over all voltages adjacent to v. Now repeat this process either indefinitely or until α_j hits a boundary vertex. By first picking the path α_1 and then the path α_2 and then the path α_3 and so on, we can pick our "adjacent minimums" in a consistent way so that if α_j and α_k intersect at a vertex, then α_j and α_k coincide for all vertices after they intersect. We should note that the voltages on the path α_j form a nonincreasing sequence which is eventually strictly decreasing by the local maximum principle.

Now construct paths β_j using the same construction as the paths α_j except instead of eventually picking minimal adjacent voltages we pick maximal adjacent voltages. A few pictures are in order. The curves γ_i are shown in Figure 3.1. The curves α_i and β_i are shown in Figure 3.2.

We will now try to show that all of the paths α_i and β_i terminate on the boundary. If all of the curves α_i and β_i terminate on the boundary, then a simple planarity argument shows that there must be $k + 1$ vertices on the boundary with nonzero voltage, which is a contradiction.

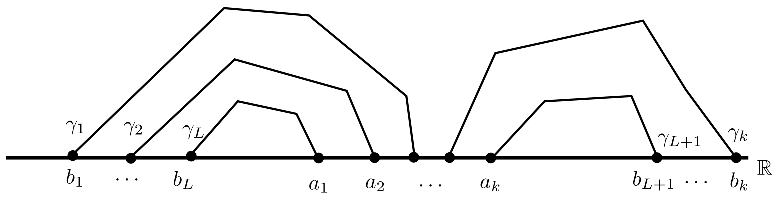


Figure 3.1: The paths γ_j.

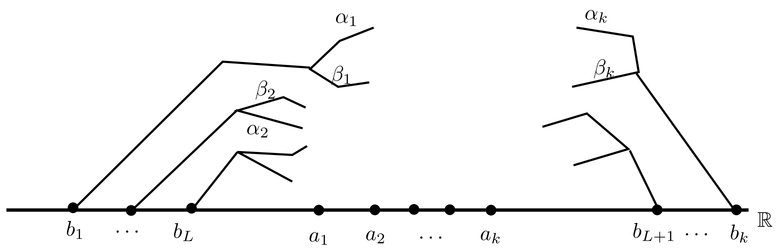


Figure 3.2: The paths α_j and β_j.

We note that each path α_j is disjoint from all of the paths β_k if $j \neq k$ since each γ_j is vertex disjoint and on the vertices of α_j that are not vertices of γ_j, the function ϕ must be strictly negative and on the vertices of β_k that are not part of γ_k, the function ϕ must be strictly positive. Furthermore, if α_j and α_k intersect, then they must be eventually equal by how we "consistently picked" minimum neighboring voltages. The same comment holds for the paths β_j.

We first will show that at least one of the curves α_1 and β_1 must terminate on the boundary. By the Jordan curve theorem, both the sets $\mathbb{H} \setminus \alpha_1$ and $\mathbb{H} \setminus \beta_1$ both consist of exactly two components. Let A_1, A_2 denote the closures of the two components of $\mathbb{H} \setminus \alpha_1$ and let B_1, B_2 denote the closures of the two components of $\mathbb{H} \setminus \beta_1$. By reordering, suppose that A_1 and B_1 do not contain any of the points $a_1, \ldots, a_k$. Note that it's easy to verify that the image of α_1 must be contained entirely in B_1 or entirely in B_2 and that the image of β_1 must be contained entirely in A_1 or entirely in A_2. We note that crossing over the curves α_1 or β_1 we switch components. Thus by looking near the vertex v where β_1 and α_1 diverge, we see that if $\alpha_1 \subseteq B_2$, then $\beta_1 \subseteq A_1$ and vice versa. Hence it is clear that exactly one of $\beta_1 \subseteq A_1$ or $\alpha_1 \subseteq B_1$ is true. By possibly multiplying ϕ by -1. The situation is shown in Figure 3.3.

Assume first that $\beta_1 \subseteq A_1$. The case that $\alpha_j \subseteq B_1$ follows a nearly identical argument, so we omit that case. Let G_0 consist of the subgraph consisting of

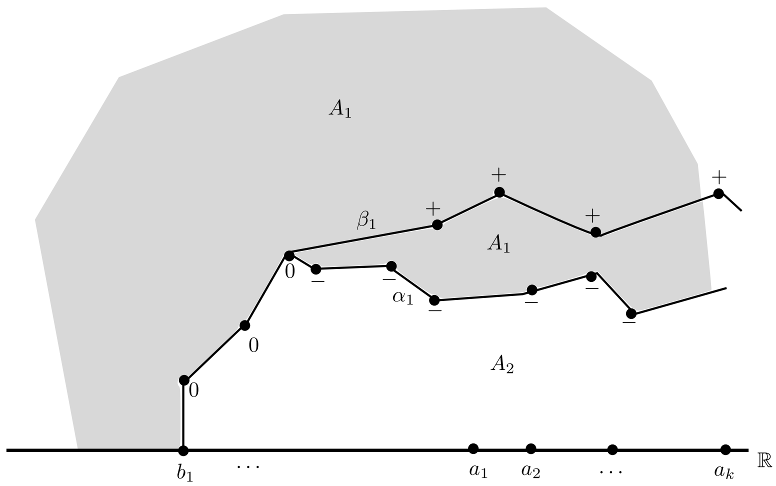


Figure 3.3: The regions A_1, A_2 and the curve α_1 and β_1 in the case that $\beta_1 \subseteq A_1$.

all vertices of G contained in A_1. Define the function $\psi : V \to \mathbb{R}$ by

$$\psi(v) = \begin{cases} \phi(v) & \text{if } v \notin G_0 \\ \phi(v) & \text{if } v \in G_0 \text{ and } \phi(v) \le 0 \\ 0 & \text{if } v \in G_0 \text{ and } \phi(v) > 0 \end{cases} .$$

Note that on $G_0 \cap \partial G$, the function ϕ is zero. Hence the function ψ has the same boundary values as ϕ. We claim that $P(\psi) < P(\phi)$. If an edge $vv' \in G \setminus G$, then the power along vv' of ψ is the same as the power of ϕ along vv'. If $vv' \in G_0$ and $\phi(v) \le 0$ and $\phi(v') \le 0$ then the power along vv' of ψ is the same as for ϕ. If $\phi(v), \phi(v') > 0$ then the power along the edge vv' is nonstrictly decreased. If $\phi(v) > 0$ but $\phi(v') \le 0$, then the power is strictly decreased. Since G is connected and there are vertices $v'', v''' \in G_0$ such that $\phi(v'') > 0$ and $\phi(v''') < 0$, we can find a path in G from v'' to v'''. If this path is contained in G_0, then there must vertices $v, v' \in G_0$ such that $\phi(v) > 0$ and $\phi(v') \le 0$. If the path is not contained in G_0, then it must eventually cross the path α_j. But ϕ is nonpositive along α_j, and hence somewhere along the path from v'' to v''' there must be an edge vv' in G_0 where $\phi(v) > 0$ and $\phi(v') \le 0$. Hence $P(\psi) < P(\phi)$, but ψ and ϕ have the same boundary conditions, contradicting the minimality of ϕ. The case that $\alpha_1 \subseteq B_1$ is handled by switching all the inequalities in the above paragraph. Hence at least one of α_1 and β_1 must terminate at a boundary vertex.

An identical argument shows that if b_k is to the right of the closed interval $[a_1, a_k]$ (i.e. if $J < k$), then at least one of the curves α_k and β_k must terminate at the boundary. Notice that by how we constructed the curves α_j and β_j that if they terminate at the boundary then they must terminate at a vertex where ϕ has nonzero voltage. By how we constructed α_i and β_i, we know that if $i \neq j$,

then we have
$$\alpha_i \cap \alpha_j = \alpha_i \cap \beta_j = \beta_i \cap \beta_j = \varnothing.$$
Since α_1 or β_1 must terminate at some vertex in the collection $\{a_1, \ldots, a_k\}$, by a standard Jordan curve theorem argument, we know that all of the curves $\alpha_2, \beta_2, \ldots, \alpha_J, \beta_J$ must also terminate at the boundary. Similarly, if $J < k$, we know that at least one of the curves α_k and β_k terminate, and hence by a standard Jordan curve theorem argument that we leave to the reader, we know that all of the curves $\alpha_{J+1}, \beta_{J+1}, \ldots, \alpha_{k-1}, \beta_{k-1}$ must also converge.

We note that a simple counting argument combined with a Jordan curve argument shows that the terminating path of the pair α_1 and β_1 must terminate at a_L and the terminating path of α_k and β_k must terminate at a_{L+1}. Since α_j and β_j cannot terminate at the same point for any j, a simple counting argument shows that if all α_j and β_j terminate, then there must be $k+1$ vertices on the boundary where ϕ is nonzero, which would be a contradiction. Hence we just need to show that all of the curves α_j and β_j terminate. By what we have shown already, we now just need to show that the curves α_1 and β_1 and α_k and β_k terminate .

We wish to show that α_1 and β_1 both terminate, and that α_k and β_k both terminate (in the case that $J < k$. There are seven cases to consider:

1. $k = J$, in which case we only need to show that α_1 and β_1 both terminate;
2. β_1 and α_k terminate;
3. α_1 and β_k terminate;
4. α_1 and α_k terminate;
5. β_1 and β_k terminate.
6. everything except α_1 (or β_1) terminates;
7. everything except α_k (or β_k) terminates;

By possibly multiplying ϕ by (-1) we notice that proving case (2) also proves case (3), so we will eliminate case (3). Similarly we don't need to consider case (5) since by multiplying by -1 we can reduce to case (4). Thus we need only consider cases (1), (2), (4), (6) and (7).

Consider case (1). In this case, we have that $b_1, \ldots, b_k$ are all to the left of the closed interval $[a_1, a_k]$. We have already shown that $\alpha_2, \beta_2 \ldots, \alpha_k, \beta_k$ all terminate at the boundary and we know that one of α_1 and β_1 also terminates at the boundary. By possibly multiplying ϕ by -1 we can assume that α_1 terminates at the boundary. Suppose that β_1 does not terminate at the boundary. A simple argument shows that if A_1 is the closure of the region in $\mathbb{H} \setminus \alpha_1$ which doesn't contain any of the points $a_1, \ldots, a_k$, then $\beta_1 \subseteq A_1$. A figure is shown in 3.4.

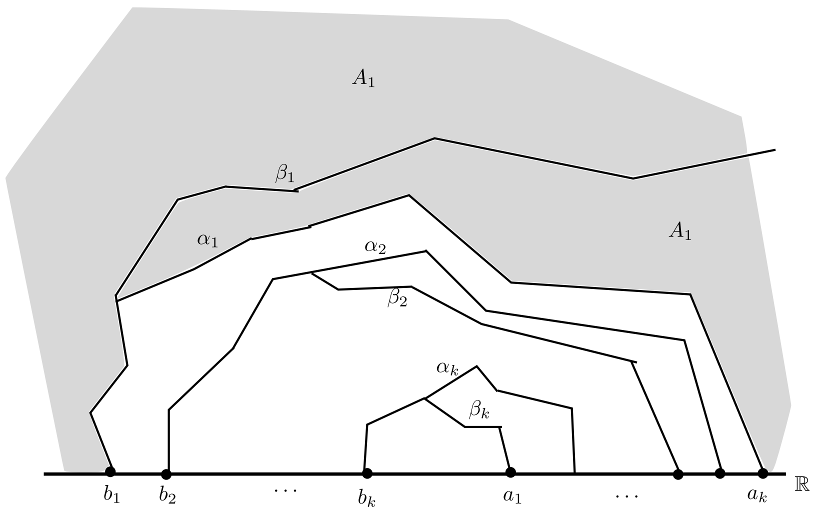


Figure 3.4: Case (1) assuming that α_1 terminates at the boundary.

We do a simple trick that we've done several times before. Let G_0 be the set of vertices in A_1 and define a function $\psi : V \to \mathbb{R}$ by

$$\psi(v) = \begin{cases} \phi(v) & \text{if } v \notin G_0 \\ \phi(v) & \text{if } v \in G_0 \text{ and } \phi(v) \leq 0 \\ 0 & \text{if } v \in G_0 \text{ and } \phi(v) > 0 \end{cases} .$$

As we showed before, since the path β_1 passes through A_1 and our graph is connected, it's straightforward to show that $\psi|_{\partial G} = \phi|_{\partial G}$ but $P(\psi) < P(\phi)$, contradicting our assumption that ψ has minimal power. Hence we know that β_1 must terminate.

Now consider case (2), i.e. β_1 and α_k terminate. Suppose α_1 and β_k do not terminate. A figure is shown in Figure 3.5.

We now need to perform a bit more trickery. We will say a vertex v is **accesible from** α_1 if there is a path from a vertex in α_1 to v such that ϕ is nonpositive at all vertices in the path, and the path contains does not enter the bounded region bounded by β_1. There are two cases:

1. there is a vertex in α_k which is accessible from α_1;

2. there is not a vertex in α_k which is accessible from α_1.

In the first case, we have a path P from a vertex in α_1 to a vertex in α_k. By removing loops in P we can assume that there are no repeated edges or vertices in this path. In this case, a simple argument shows that we can form a Jordan path of vertices in P by starting along α_k until we reach the last vertex in P, then travelling backwards along P until we reach the last vertex in P

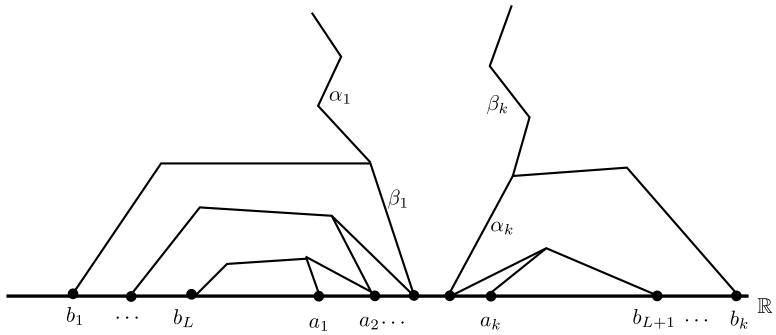


Figure 3.5: Case (2), i.e. $k < J$ and β_1 and α_k terminate at the boundary.

which is also in α_1, and then continuing to ∞ along α_1. The reader checks that this defines a well defined Jordan path ω. Parametrize the images of the edges of ω to make a continuous function $\omega' : [0, \infty) \to \mathbb{H}$. Now extend ω' to a continuous function $\widetilde{\omega}$ from $\mathbb{R}$ to $\mathbb{C}$ by defining $\widetilde{\omega}(t) = (\operatorname{Re} b_k, t) \in \mathbb{R}^2$ for $t \leq 0$. This clearly defines a continuous function, which is also easily seen not to have any self intersections. Since $\widetilde{\omega}(t) \to \infty$ as $t \to \infty$ and $t \to -\infty$, we know that $\mathbb{C} \setminus \operatorname{Im} \widetilde{\omega}$ consists of two components by the Jordan curve theorem. A simple argument shows that all of the vertices of β_k where ϕ is positive are contained entirely in one component of $\mathbb{C} \setminus \operatorname{Im} \widetilde{\omega}$. Let Ω denote the component of $\mathbb{C} \setminus \operatorname{Im} \widetilde{\omega}$ which contains all of the vertices of β_k where ϕ is positive. Note that a simple argument shows that the open set Ω contains none of the vertices $a_1, \ldots, a_k$. A picture is shown in 3.6

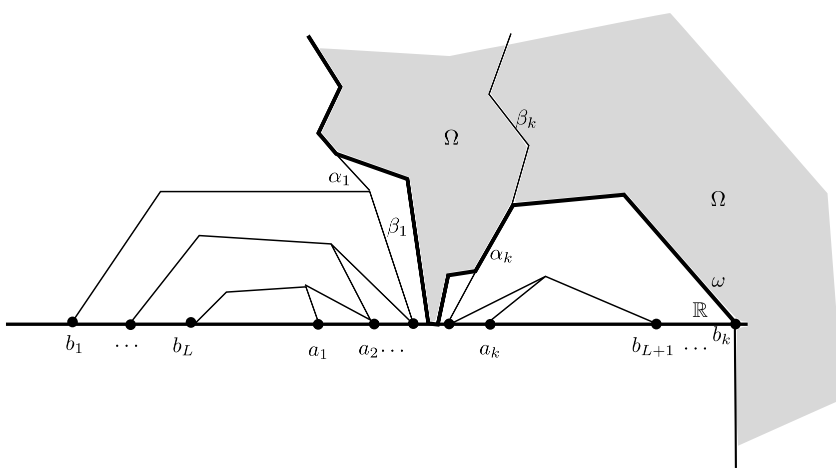


Figure 3.6: The case when a vertex in α_k is accessible from a vertex in α_1.

Now we perform our standard trick. Let G_0 be the set of vertices in which are in $\overline{\Omega}$. Now define the function

$$\psi(v) = \begin{cases} \phi(v) & \text{if } v \notin G_0 \\ \phi(v) & \text{if } v \in G_0 \text{ and } \phi(v) \le 0 \\ 0 & \text{if } v \in G_0 \text{ and } \phi(v) > 0 \end{cases}.$$

The reader verifies that $\psi|_{\partial G}$ and $\phi|_{\partial G}$ have the same boundary values and that, since all the vertices in β_k at which ϕ is positive are in G_0, a simple connectivity argument combined with an argument we've produced many times shows that $P(\psi) < P(\phi)$, a contradiction.

Now we have to consider the second subcase of case 2, namely that no vertex in α_1 is accessible from α_1. Let G_0 be the set of vertices which are accessible from α_1. Define the function

$$\psi(v) = \begin{cases} \phi(v) & \text{if } v \notin G_0 \\ \phi(v) & \text{if } v \in G_0 \text{ and } \phi(v) \ge 0 \\ 0 & \text{if } v \in G_0 \text{ and } \phi(v) < 0 \end{cases}.$$

Notice that none of the vertices $a_1, \dots, a_L$ are accessible from α_1 since we'd have to cross β_1 to get to those vertices. Similarly none of the vertices $a_{L+1}, \dots, a_k$ is accesible from α_1. Hence $\psi|_{\partial G} = \phi|_{\partial G}$. We claim that $P(\psi) < P(\phi)$. One simply checks that the power on each edge is nonstrictly reduced. Similarly, since the graph is connected, a simple argument shows that $P(\psi) < P(\phi)$, which contradicts minimality.

We now consider case (4), the last case. Assume that both α_1 and α_k terminate at the boundary. By the Jordan curve theorem, both the paths α_1 and α_k are the boundaries of a bounded and an unbounded subset of $\mathbb{H}$. Let R be the intersection of the unbounded components formed by α_1 and α_k. The reader checks that $\overline{R}$ contains none of the points $a_1, \dots, a_{L-1}$ and $a_{L+1}, \dots, a_k$. Furthermore, it is clear that the curves β_1 and β_k both have infinitely many vertices in R, and hence there are infinitely many vertices of R on which ϕ is positive. Let G_0 be the set of vertices of G which are contained in $\overline{R}$. Define the function $\psi : V \to \mathbb{R}$ by

$$\psi(v) = \begin{cases} \phi(v) & \text{if } v \notin G_0 \\ \phi(v) & \text{if } v \in G_0 \text{ and } \phi(v) \le 0 \\ 0 & \text{if } v \in G_0 \text{ and } \phi(v) > 0 \end{cases}.$$

Notice that since the only elements of $\overline{R} \cap \{a_1, \dots, a_k\}$ are a_L and a_{L+1}, and we know that $\phi(a_L) < 0$ and $\phi(a_{L+1}) < 0$ since α_1 terminates at a_L and α_k terminates at a_{L+1}, we thus know that $\psi|_{\partial G} = \phi|_{\partial G}$. As we've done many times before, a simple argument shows that $P(\psi) < P(\phi)$, a contradiction.

Finally, cases (6) and (7) follow by nearly identical arguments to what we've done already, so we leave those cases to the reader. □

Proposition 3.2.5. If Γ is a (possibly infinite) electrical network and $A \rightsquigarrow B$ is a k flowout, then

$$(-1)^k \det Q(A, B) > 0.$$

Proof. By Theorem 3.2.4 we know the determinant is nonzero. By analyzing the first proof of Theorem 3.1.1, we see that each $q(a_i, b_i)$ is the limit of corresponding entries in the response matrices of finite subgraphs. Since all large graphs in this sequence of subgraphs will have A, B as a k-circular pair, we know that the sign will be nonnegative by Theorem 3.13 of [1] □

3.3 Formulas For Recovery

Theorem 3.3.1 (Boundary Edge Formula)**.** Let Γ be a half planar network with boundary edge ab. Suppose $A = \{a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$ are two sequences of boundary nodes such that $(P', Q') = ((a, a_1, \ldots, a_k), (b, b_1, \ldots, b_k))$ is a $(k+1)$-circular pair and deleting ab breaks the connection between A' and B'. Then

$$\gamma_{ab} = -Q(a, b) + Q(a, B) \cdot Q(A, B)^{-1} \cdot Q(A, b).$$

Proof. This follows since if the connection exists and is broken in the infinite graph, it will exist and be broken in all large finite subgraphs. By analyzing the proof of 3.1.1, we see that each entry in all of the above matrices is the limit of entries in various response matrices for finite graphs. By Corollary 3.15 of [1] we know the equation is satisfied for finite circular planar networks where the connection exists and is broken. Since $Q(A, B)$ is nonsingular by 3.2.4 we know that in the limit the equality will be preserved. □

Theorem 3.3.2 (Boundary Spike Formula)**.** Suppose Γ is a circular planar resistor network and pr is a boundary spike joining boundary node p to interior node r. Suppose that contracting pr to a single node breaks the connection between a circular pair A, B. Then

$$\gamma(pr) = Q(p, p) - Q(p, B) \cdot Q(A, B)^{-1} \cdot Q(A, p).$$

Proof. The same logic as the previous formula applies since we note the formula holds in the finite case by Corollary 3.16 of [1]. □

3.4 More Facts about Infinite Chord Diagrams

Let Γ be a half planar network such that the regularity conditions from the first paper are satisfied. We note that any geodesic g which intersects $\mathbb{R}$ at two points will uniquely determine two components of $\mathbb{H} \setminus g$, exactly one of which will be bounded. We will say any geodesic which intersects the boundary twice is **finite**. If g is a finite geodesic, write $B(g)$ for the bounded component of $\mathbb{H} \setminus g$ and write $U(g)$ for the unbounded component.

Definition 3.4.1. Let Γ be a half planar network such that the regularity conditions from the first paper are satisfied. We will say that Γ is **pseudocritical** if there are no crossings in the medial graph. We will call Γ **locally bounded** if every vertex in G is in $B(g)$ for some finite geodesic g.

Lemma 3.4.2. Let $g_1, \ldots, g_n$ be a finite collection geodesics in the medial graph of a pseudocritical half planar network Γ. If C_i is a subset of the cells of the medial graph corresponding to exactly one of the components of $\mathbb{H} \setminus g_i$, then $\bigcap_{i=1}^n C_i$ is either empty or there is a pseudocritical half planar network $\Gamma_0 = (G_0, \gamma_0)$ and a medial graph M_0 for Γ_0 and a homeomorphism Φ of $\bigcap_{i=1}^n C_i$ to $\mathbb{H}$ which maps the geodesics that pass through the interior $\bigcap_{i=1}^n C_i$ bijectively onto the geodesics in M_0. Furthermore, the topological boundary of $\bigcap_{i=1}^n C_i$ is given by the image of a Jordan curve which is the concatenation of geodesic segments and the traversal of closed intervals of $\mathbb{R} \subseteq \mathbb{H}$.

Proof. We prove this by induction. The $n = 1$ case is just an application of Carathéodory's theorem about conformal maps. The claim about the bijection between the set of geodesics that pass through the interior of C_1 and the geodesics of M_0 follows since a geodesic can only enter the region C_1 once by pseudocriticality.

Now suppose the claim is true for all intersections of k half planes. Let $C_1, \ldots, C_{k+1}$ be a collection of half planes bounded by geodesics $g_1, \ldots, g_{k+1}$. By assumption there is a homeomorphism $\Phi : \bigcap_{i=1}^k C_i$ to $\mathbb{H}$ which has the desired properties. If g_{k+1} does not enter the interior of the region $\bigcap_{i=1}^k C_i$, then $\bigcap_{i=1}^{k+1} C_i$ is either empty or exactly $\bigcap_{i=1}^k C_i$, in which case the claim follows. If g_{k+1} does enter the interior of $\bigcap_{i=1}^k C_i$, then the claim follows from the $n = 1$ case by first mapping $\bigcap_{i=1}^k C_i$ homeomorphically onto $\mathbb{H}$ and then applying the $n = 1$ result. The reader will verify that all details are satisfied.

The reader also verifies by induction that the boundary of $\bigcap_{i=1}^{k+1} C_i$ is given by a concatenation of geodesic arcs and subintervals of $\mathbb{R}$. □

3.5 Some Miscellaneous Facts

We have that $H(\Gamma) = M(\Gamma) \oplus H_0(\Gamma)$ where $H_0(\Gamma)$ is the set of finite power harmonic functions which are zero on the boundary.

Remark 3.5.1. The subspace of $M(\Gamma)$ consisting of bounded minimal functions is not closed in general.

Proof. We let G denote an infinite ray of conductors in parallel as in Figure 3.7. Let the n^{th} resister have conductivity $1/2^n$ and define all vertices to be boundary vertices. Define $\phi_n : V \to \mathbb{R}$ by

$$\phi_n(v_j) = \begin{cases} j & \text{if } j \leq n \\ n & \text{otherwise.} \end{cases}$$

Let $\phi : V \to \mathbb{R}$ denote $\phi(v_n) = n$. By explicit computation the $\phi, \phi_n \in Z(\Gamma)$. Furthermore, they are clearly minimal since there are no interior vertices. We note by explicit computation however that $P(\phi_n - \phi) \to 0$. We have ϕ_n are bounded but ϕ is not. □

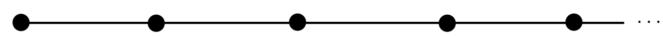


Figure 3.7: The infinite ray of conductors in series, as discussed in Remark 3.5.1.

Proposition 3.5.2. Suppose that Γ is an electrical network with finitely many boundary vertices. Then every function in $M(\Gamma)$ satisfies the condition of having the sum of the currents being zero on the boundary.

Proof. Note that $\chi_{\partial G} = 1 - \chi_{\mathrm{int}\, G} = -\chi_{\mathrm{int}\, G}$ is a finite power function since ∂G is finite. Now since $M(\Gamma) = W^{\perp}$ is a subspace, we know that $\chi_{\partial G} \in W$ and hence by definition we have that $\chi_{\partial G} \in M(\Gamma)^{\perp} = W$. □

Proposition 3.5.3. Let v be a boundary node for an electrical network Γ. Let $\phi = \Lambda \chi_v$. Then we have

$$\sum_{v' \in \partial G, v' \neq v} (\chi_{v'}, \phi) \geq -(\phi, \phi) = (\chi_v, \phi).$$

Proof. By the proof of 3.1.1 we know that ϕ is the pointwise limit of γ-harmonic functions of the same boundary values defined on an increasing chain of finite connected subnetworks G_n. If ϕ_n is the γ-harmonic function on G_n with the same boundary values as ϕ, then by the maximum principle or Theorem 3.2.1 we know that $(\phi_n, \chi_{v'})_n$ will be nonpositive (and $(\phi_n, \chi_{v'})_n$ is just the current leaving v' on the finite subnetwork G_n). By basic facts about finite resistor networks we have that

$$\sum_{v' \in \partial G \cap G_n} (\phi_n, \chi_{v'})_n = -(\phi_n, \chi_v)_n.$$

Taking limits and applying Fatou's lemma gives the desired inequality. □

3.6 Cutpoint Lemma

In this section we present an exposition of the cutpoint lemma. We develop a new proof of the cutpoint lemma for the finite case, which extends naturally to a proof of the Cutpoint lemma in the infinite case. Both of the new proofs that we present represent a departure from the style used to prove the theorem initially. The proof of the cutpoint lemma found in [?p, Morrow]roceeds by uncrossing geodesics. Though this may be possible in the infinite case under for supercritical graphs, the procedure seems intractable for pseudocritical graphs.

Instead, we proceed by developing more machinery related to taking geodesic closures of subsets of the medial graph. The most powerful result, Theorem 3.7.2, generalizes several theorems that we have proven or will prove in this section.

3.6.1 Cutpoint Lemma for Finite Graphs

Definition 3.6.1. Let $\mathbb{D}$ denote the unit disk and let Γ be an electrical network which is embedded in $\mathbb{D}$. Let C denote the unit circle $\partial\mathbb{D}$. Suppose that x, y are distinct points of C which lie in distinct cells of the medial graph. If $x, y \in C$ are distinct points and R is one of the components of $C \setminus \{x, y\}$. We say a triple (x, y, R) is an **directed segment of** Γ if x is counterclockwise from y with respect to R.

Definition 3.6.2. Suppose (x, y, R) is a directed segment. We define the following three quantities:

- $m(R)$ is the maximum integer k such that there is a k-connection between a k-circular pair A, B which respects the cutpoints x, y.
- $r(R)$ is the number of finite geodesics with both endpoints in the arc R.
- $n(R)$ is the number of black intervals which are entirely within R.

Theorem 3.6.3. Let (x, y, R) be a directed segment of $\Gamma = (G, \gamma)$. Then

$$m(R) + r(R) - n(R) = 0.$$

Proof. We will use the rank nullity theorem applied to a particular linear map, which we will define. Let C denote the circle that Γ is embedded into. Let R be one of the connected components of $C \setminus \{x, y\}$. Let M denote the medial graph of G. Let $P_1, \ldots, P_n$ denote the primal cells of the medial graph which are completely contained in the arc R, ordered counterclockwise. Let x be in the cell C_- and let y be in the cell C_+. The setup is shown in Figure 3.8.

Now we define several vector spaces. Let V_1 be the vector space of γ-harmonic functions that are zero at all primal boundary vertices except possibly $P_1, \ldots P_n$. Let V_C be the vector space of values of currents of γ-harmonic functions on the cells in M which are completely contained in R^c. Define a map $L : V_1 \to V_C$ which takes a function $\phi \in V_1$ and maps it to the function in V_C which corresponds to the currents leaving the primal vertices in boundary medial cells which are completely contained in R^c. Note that $\dim V_1$ is just the number of primal boundary primal cells which are completely contained in R, which is just $n(R)$. By the Rank Nullity theorem, we have that

$$\dim V_1 = \dim \ker L + \operatorname{rank} L.$$

We note that the rank L is just $m(R)$ and hence

$$n(R) = m(R) + \dim \ker L.$$

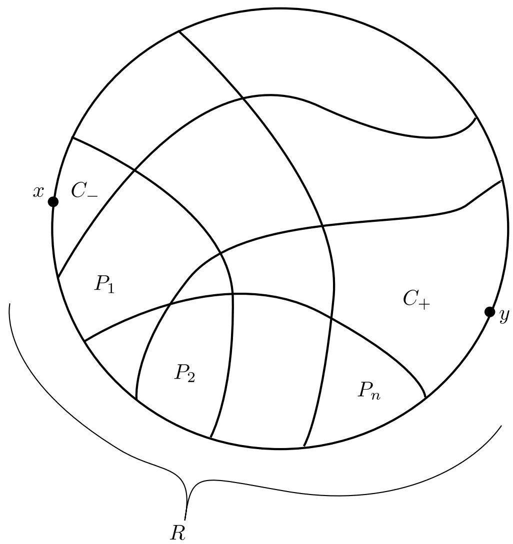


Figure 3.8: The general setup.

If we can show that $\dim \ker L = r(R)$, then we will be done.

To find the dimension of the kernel, we will actually explicitly construct it. This is where we need to use results from [4]. We will construct $r(R)$ γ-harmonic functions which are in $\ker L$ and we will show that they are linearly independent and span $\ker L$. Let g be a reentrant geodesic. We will construct a function ϕ_g defined on the cells of the medial graph in the following way. Let R_g denote the subset of the medial graph which is bounded by g which contains boundary cells which do not completely border R. Define ϕ_g to be zero on R_g (voltage and covoltage equal to zero). There are two boundary cells of M which are not in R_g but which neighbor R_g. Let b be the one which is closer to x with respect to the ordering of R. This is shown in Figure 3.9.

Now let b be a boundary cell of the medial graph which is not in R_g but which touches R_g. Define ϕ_g to be 1 at b. By Lemma 9.13 of [4] the set $R_g \cup \{b\}$ is a *safe* cellset with connected closure, and by Theorem 8.7 of [4] ϕ_g extends in exactly one way to $\overline{R_g \cup \{b\}}$. Now pick any boundary cell in $c \in M \setminus \overline{R_g \cup \{b\}}$. Define ϕ_g arbitrarily at c (for definiteness pick ϕ to be zero) and again by the exact same reasoning there is a unique extension of ϕ_g to $\overline{\{c\} \cup \overline{R_g \cup \{b\}}}$. Repeating this process until we exhaust all of M yields a well defined function ϕ_g (defined on the cells of the medial graph) which induces a well defined γ-harmonic function on the primal graph. For each g let $\widetilde{\phi_g}$ denote the restriction of ϕ_g to the cells of the primal graph.

There are $r(R)$ such functions ϕ_g. Let $n = r(R)$ and let $g_1, \ldots, g_n$ be the reentrant geodesics, ordered so that the if $I(g_i)$ denotes the first intersection of g_i with C (where first is with respect to the direction given by (x, y, R)), then

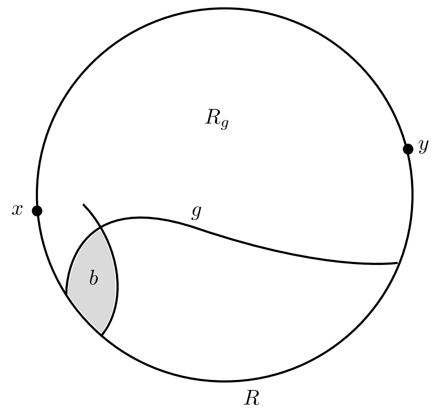


Figure 3.9: Picking the cell b.

$I(g_1) < I(g_2) < \cdots < I(g_n)$. The situation is shown in Figure 3.10.

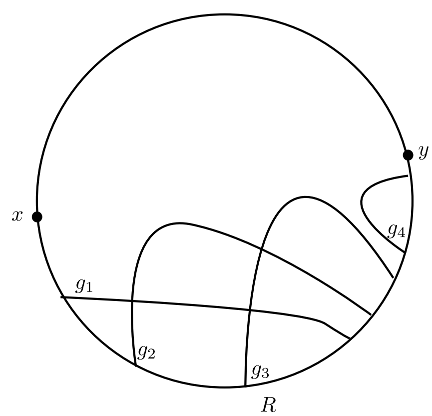


Figure 3.10: The labelling scheme for reentrant geodesics.

We first will show that the functions $\widetilde{\phi_g}$ are actually in the kernel of L. Notice that they are zero on every primal boundary cell which is not completely contained in R, so they are in V_1. If p is a dual primal cell which is completely contained in $C \setminus R$, then neither of the two adjacent dual cells can be completely contained in R. Hence by definition ϕ_g has covoltage 0 at both of those cells and hence the current leaving p is $0 - 0 = 0$, so ϕ_g is actually in the kernel.

We now need to show that they are linearly independent. Suppose that $a_1\widetilde{\phi_{g_1}} + \cdots + a_n\widetilde{\phi_{g_n}} = 0$. This implies $\phi = a_1\phi_{g_1} + \cdots a_n\phi_{g_n}$ is constant on the primal graph, and constant on the dual graph. If x and y are both in primal cells, then there must be a dual cell completely contained in $C \setminus R$ and hence there is at least one primal vertex and at least one dual vertex where ϕ is zero.

If one of x and y is in a dual cell and the other is in a primal cell, then we know that ϕ is also zero at at least one primal cell and at least one dual cell. Similarly, if x and y are both in dual cells, then there must be a primal cell completely contained in $C \setminus R$, and in particular there have to be at least one primal cell and at least one dual cell where ϕ is zero. In all cases we see that ϕ is zero everywhere and hence $a_1\phi_{g_1} + \cdots a_n\phi_{g_n} = 0$. Let b_g the cell which is in $M \setminus R_{g_1}$ which neighbors R_{g_1} and is closest to x with respect to R. We note that ϕ_{g_1} is the only function in the collection $\{\phi_{g_i}\}$ which is nonzero at b_{g_1}. Hence

$$0 = (a_1\phi_{g_1} + \cdots a_n\phi_{g_n})(b_{g_1}) = a_1 \cdot \phi_{g_1}(b_{g_1}) = a_1$$

and hence $a_1 = 0$. Proceeding in this manner shows that $a_i = 0$ for all i and hence the functions ϕ_{g_i} are all linearly independent.

We now need to show that the functions ϕ_{g_i} span all of $\ker L$. Let $\phi_0 \in \ker L$. This means that the currents leaving the boundary vertices in $C \setminus R$ are all zero. Let ϕ be a function defined on the cells of the medial graph which is equal to ϕ_0 on the primal cells, and which is equal to a covoltage function of ϕ_0 on the dual cells. Since we can change the definition of ϕ by a constant function on the dual graph, assume that ϕ is equal to zero at D_-. Since $\phi_0 \in \ker L$ we know by the definition of the covoltage function associated to a γ-harmonic function that ϕ is zero at both C_-, C_+ and all boundary cells whose intersection with R is empty. Note that ϕ_0 is by assumption equal to zero at all of the boundary primal cells which are not in R. Let B denote the set of boundary medial cells which are either C_-, C_+ or which which don't intersect R. Note that by results of [4], we know that the closure of B is equal to the intersection of all half planes that contain it. But this is just $\bigcap_{i=1}^n R_{g_i}$. Hence in particular we know that ϕ is supported at most on

$$\left(\bigcap_{i=1}^n R_{g_i}\right)^c = \bigcup_{i=1}^n R_{g_i}^c.$$

Let b_{g_i} be defined as above. Note that the function

$$\phi - \phi(b_{g_1})\phi_{g_1}$$

is zero on $\overline{B} \cup \{b_{g_1}\}$. Since $\phi - \phi(b_{g_1})\phi_{g_1}$ is a well defined function which is γ-harmonic on the primal graph and on the dual graph, by results in [4] we know that $\phi - \phi(b_{g_1})\phi_{g_1}$ is equal to zero on $\overline{\overline{B} \cup \{b_{g_1}\}}$. But $\overline{\overline{B} \cup \{b_{g_1}\}}$ is just the intersection of half planes which contain $\overline{B} \cup \{b_{g_1}\}$, but this is obviously just

$$\left(\bigcap_{i=2}^n R_{g_i}\right)^c = \bigcup_{i=2}^n R_{g_i}^c.$$

Proceeding in this way by induction, we find real numbers a_i such that

$$\phi - (a_1\phi_{g_1} + \cdots a_n\phi_{g_n})$$

is supported nowhere on M and hence

$$\phi = (a_1\phi_{g_1} + \cdots a_n\phi_{g_n}),$$

which shows that $\phi_{g_1}, \ldots, \phi_{g_n}$ span all of $\ker L$. The result follows.

$\square$

3.6.2 Cutpoint Lemmas for Infinite Graphs

Here we present versions of the cutpoint lemma for infinite graphs.

Theorem 3.6.4. If $\Gamma = (G, \gamma)$ is a pseudocritical halfplanar network and (x, y, R) is a directed segment of $\mathbb{R}$, then we have that

$$n(R) \geq m(R) + r(R).$$

Proof. As for the proof of the finite case, we define a function $\widetilde{L} : V_1 \to V_C$ which takes a vertex function supported only on the boundary primal vertices whose primal cells are completely contained in R and maps them to the vector space of currents of minimal functions which attains the given boundary values. Restricting the codomain of $\widetilde{L}$ to the image of $\widetilde{L}$ we get a map $L : V_1 \to \operatorname{Im} \widetilde{L}$ which is a map between finite dimensional vector spaces. We now just apply the rank nullity theorem on L to get that

$$n(R) = \operatorname{rank} L + \dim \ker L.$$

Once again by Theorem 3.2.4 we know that $\operatorname{rank} L = m(R)$. One sees just as in the finite case that every reentrant geodesic g of R yields a function ϕ_g and that the $r(R)$ such functions generated this way are all linearly independent. Hence $\dim \ker L \geq r(R)$ so we have that

$$n(R) = m(R) + \dim \ker L \geq m(R) + r(R),$$

as we wanted. $\square$

It turns out that for supercritical networks, we actually have equality in the cutpoint lemma. To prove this however, we will need several lemmas.

Lemma 3.6.5. Suppose that $X \subseteq M$ is a connected, closed subset. Then X contains no degenerate corners.

Proof. Let v be a degenerate corner of X. Let c_1 and c_2 be the cells of X at the degenerate corner v. Then there is a path of cells $P = p_1 p_2 \ldots p_n$ of cells in X such that $c_1 = p_1$ and $c_2 = p_n$. By the Jordan curve theorem, P bounds a unique finite collection of cells. Let C be the set of medial cells in $M \setminus X$ which are in the compact region bounded by P. A picture is shown in Figure 3.11

By local finiteness, we know that C is a finite collection of cells. Each component of C must have three corners. By using a standard argument, if A is the adjacency graph for the set of components of C (where two components are adjacent if there is a degenerate corner between them), since X is closed, we know that no component of C can have an actual corner, and hence we know that A is a graph such that every vertex has valence at least three, except possibly the vertex corresponding to the component adjacent to c_1 and c_2. But since A is a finite graph, we know that A must have a loop, which is impossible by the Jordan Curve Theorem since X is connected. $\square$

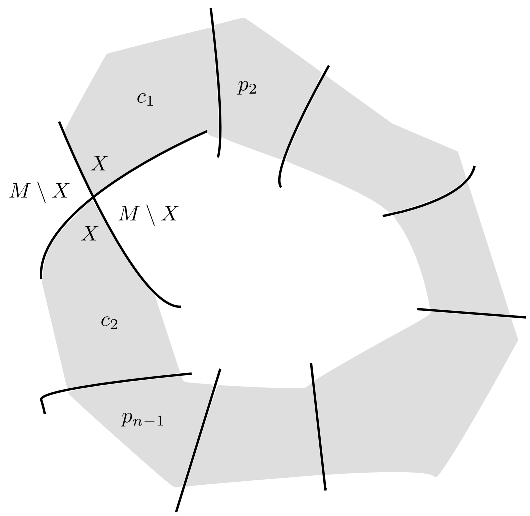


Figure 3.11: The cells c_1, c_2 and the path between them.

Lemma 3.6.6. Let Γ be a supercritical half planar electrical network with a medial graph M. Let $R = R_1 \sqcup R_2$ where R_1 and R_2 are disjoint connected subsets of ∂M such that R_1 extends infinitely to the left and R_2 extends infinitely to the right. Then $\overline{R_1 \cup R_2} = \widetilde{R_1 \cup R_2}$ (i.e. the closure of of $R_1 \cup R_2$ is equal to the intersection of all half planes which contain it).

Proof. We note that we clearly have that

$$R_1 \cup R_2 \subseteq \overline{R_1} \cup \overline{R_2} \subseteq \overline{R_1 \cup R_2} \subseteq \widetilde{R_1 \cup R_2}.$$

Our first goal will be to show that $M \setminus (\overline{R_1} \cup \overline{R_2}) = (\overline{R_1} \cup \overline{R_2})^c$ is finite. By DeMorgan's law, we have that

$$(\overline{R_1} \cup \overline{R_2})^c = \overline{R_1}^c \cap \overline{R_2}.$$

By Theorem 2.3.42 we have that $\overline{R_1} = \widetilde{R_2}$ and $\overline{R_2} = \widetilde{R_2}$. Define

$$\mathfrak{R}_i = \{g : g \text{ doesn't enter the interior of } R_i\}.$$

Since $\widetilde{R_i}$ is the intersection of all half planes which intersect R_i, and we know that R_i is not contained in the bounded region of any geodesic, we know that Then we have that

$$\widetilde{R_i} = \bigcap_{g \in \mathfrak{R}_i} U(g).$$

Hence

$$(\widetilde{R_i})^c = \bigcup_{g \in \mathfrak{R}_i} U(g)^c = \bigcup_{g \in \mathfrak{R}_i} B(g).$$

Hence we have that

$$(\overline{R_1} \cup \overline{R_2})^c = \left(\bigcup_{g \in \mathfrak{R}_1} B(g) \right) \cap \left(\bigcup_{g \in \mathfrak{R}_2} B(g) \right).$$

We can rewrite this as

$$\bigcup_{g \in \mathfrak{R}_1, \widetilde{g} \in \mathfrak{R}_2} (B(g) \cap B(\widetilde{g})).$$

We now claim that only finitely many sets in the above union are nonempty. Let $S \subseteq \mathbb{R}$ denote the interval of $\mathbb{R}$ which doesn't intersect any cell in R_1 or R_2. Suppose that $g_1 \in \mathfrak{R}_1$ and $g_2 \in \mathfrak{R}_2$ are geodesics which down intersect S. Then if a_i is the left endpoint of g_i and b_i is the right endpoint of g_i, then clearly we must have that $a_1 < b_1 < a_2 < b_2$. By a simple Jordan Curve Theorem argument we see immediately that $B(g_1) \cap B(g_2) = \varnothing$. Hence

$$\bigcup_{g \in \mathfrak{R}_1, \widetilde{g} \in \mathfrak{R}_2} (B(g) \cap B(\widetilde{g})),$$

can actually be written as a finite union of finite sets, and is hence finite. Since $(\overline{R_1} \cup \overline{R_2})^c$ is contained inside of that set, we know that $(\overline{R_1} \cup \overline{R_2})^c$ is finite.

We now will show that $\overline{R_1 \cup R_2}$ is connected. Since $M \setminus (\overline{R_1} \cup \overline{R_2})^c$ is finite, we know that it is contained in a finite union of bounded halfplanes by Theorem 2.3.9. Call this finite union of halfplanes Q. The complement of Q is completely contained in $\overline{R_1} \cup \overline{R_2}$. Furthermore Q^c is a finite intersection of half planes, which is connected by Theorem 3.4.2. Since Q is finite, it contains only finitely many boundary cells and hence Q^c must contain boundary cells in both R_1 and R_2. Hence there is a cellular path P in $Q^c \subseteq \overline{R_1 \cup R_2}$ from a cell in R_1 to a cell in R_2. Since R_1 and R_2 are connected, we know that $R_1 \cup R_2 \cup P$ is connected. Clearly since $P \subseteq \overline{R_1 \cup R_2}$ we have that

$$\overline{R_1 \cup R_2 \cup P} = \overline{R_1 \cup R_2}$$

and hence $\overline{R_1 \cup R_2}$ is the closure of a connected set and is hence connected.

By Theorem 2.3.42 we know that the closure of a connected set is equal to the intersection of half planes containing it. Hence $\overline{\overline{R_1 \cup R_2}}$ is equal to the intersection of half planes which contain $\overline{R_1 \cup R_2}$. Since the set of half planes which contain $\overline{R_1 \cup R_2}$ is a subset of the half planes which contained $R_1 \cup R_2$, we know that

$$\begin{aligned} \overline{R_1 \cup R_2} &= \overline{\overline{R_1 \cup R_2}} \\ &= \overline{\widetilde{R_1 \cup R_2}} \\ &\supseteq \widetilde{R_1 \cup R_2}. \end{aligned}$$

Hence $\overline{R_1 \cup R_2} \supseteq \widetilde{R_1 \cup R_2}$. Since we already knew that $\overline{R_1 \cup R_2} \subseteq \widetilde{R_1 \cup R_2}$, we know that

$$\overline{R_1 \cup R_2} = \widetilde{R_1 \cup R_2}$$

□

The following corollary is a useful theorem which I proved unsuccessfully to prove with other methods, but which follows from the last result.

Corollary 3.6.7. Let M be a supercritical half planar medial graph and let $X \subseteq M$ be any subset such that $M \setminus X$ contains only finitely many boundary cells. Then $M \setminus \overline{X}$ is finite. If each component of X is infinite, then $\overline{X}$ is connected. If each component of X is infinite, then $\overline{X} = \widetilde{X}$.

Proof. Let c_1 and c_2 be the leftmost and rightmost boundary medial cells of $X \setminus M$ respectively, and let R_1 and R_2 denote the the boundary cells to the left of c_1 and to the right of c_2 respectively. We note that $R_1 \subseteq X$ and $R_2 \subseteq X$. Hence $\overline{R_1 \cup R_2} \subseteq \overline{X}$. Since $\overline{R_1 \cup R_2}$ is cofinite by Theorem 3.6.6, we know that $\overline{X}$ is cofinite as well.

Now suppose that each component of X is infinite. We reproduce a similar argument as in the proof of Theorem 3.6.6. Each cell c of $M \setminus \overline{X}$ is contained in $B(g_c)$ for some g_c. Hence

$$M \setminus X \subseteq \bigcup_{c \in M \setminus X} B(g_c)$$

and hence

$$\bigcap_{c \in M \setminus X} U(g_c) \subseteq \overline{X}.$$

By Theorem 3.4.2, $\bigcap_{c \in M \setminus \overline{X}} U(g_c)$ can be identified with a supercritical medial graph and is hence connected. Since each component of X is infinite, and $\bigcup_{c \in M \setminus \overline{X}} B(g_c)$ is finite, we know that each component of X must have a cell in $\bigcap_{c \in M \setminus X} U(g_c)$. Hence there is a path in $\overline{X}$ from every component of X to every other component, and hence as before we conclude that $\overline{X}$ is connected. Since $\overline{X}$ is connected, we know that

$$\widetilde{X} \subseteq \widetilde{\overline{X}} = \overline{\overline{X}} = \overline{X}.$$

Since we trivially have that $\overline{X} \subseteq \widetilde{X}$ we conclude finally that $\overline{X} = \widetilde{X}$. $\square$

Lemma 3.6.8. Let $\Gamma = (G, \gamma)$ be a (possibly infinite) electrical network. Suppose that $\phi \in M_f(\Gamma)$, i.e. ϕ is a minimal function with finite support. Then the sum of the boundary currents of ϕ is equal to zero.

Proof. Let V be the set of vertices where ϕ is nonzero or which are adjacent to a vertex where ϕ is nonzero. Let V' be the set of all vertices that are in V or which are adjacent to a vertex in V. Notice that no vertex in $V' \setminus V$ is adjacent to a vertex where ϕ is nonzero. Define a graph G_0 with vertex set V' by setting two vertices $v, v' \in V'$ being connected in G_0 iff they are connected in Γ. Now define an electrical network $\Gamma_0 = (G_0, \gamma_0)$ by simply taking γ_0 to be the restriction of γ to G_0. Define ∂G_0 to be the union of $(\partial G) \cap G_0$ taken together with all of all the vertices $v \in V'$ with the property that some neighboring vertex of v is not in ∂G.

Consider the function $\phi_0 = \phi|_{G_0}$. We claim that ϕ_0 is a γ_0-harmonic function on Γ_0. This follows since if $v \in \operatorname{int} G_0$, then $v \in \operatorname{int} G$ and all of the neighboring vertices of v are in G_0 and hence ϕ_0 is γ_0-harmonic at v simply because ϕ was γ-harmonic at v. Hence by basic facts we know that the sum of the currents on the boundary of ∂G_0 is zero. If $v \in \partial G_0$, then either $v \in \partial G \cap G_0$ or v is not adjacent to any vertex on which ϕ is supported. If v is not adjacent to any vertex in the support of ϕ, then we know that the sum of the currents of ϕ_0 at v is equal to zero (since it's a sum where every summand is zero). Hence the sum over the currents of ϕ_0 over DG_0 is equal exactly to the sum of the currents of ϕ over $\partial G \cap G_0$. Since the sum of the currents of ϕ_0 over ∂G_0 is zero, we know that the sum of the currents of ϕ over $\partial G \cap G_0$ is zero. If $v \in \partial G \setminus G_0$, then $\phi(v) = 0$ and $\phi(v') = 0$ for any v' adjacent to v. Hence the sum of the currents of ϕ over ∂G is exactly equal to the sum of the currents of ϕ over $\partial G \cap G_0$, which is zero. □

We are now situated to prove the cutpoint lemma for supercritical networks.

Theorem 3.6.9 (Supercritical Cutpoint Lemma)**.** If $\Gamma = (G, \gamma)$ is a supercritical halfplanar network and (x, y, R) is a directed segment of $\mathbb{R}$, then we have that

$$n(R) = m(R) + r(R).$$

Proof. Since every supercritical halfplanar network is pseudocritical, by the proof of the cutpoint inequality for pseudocritical graphs, we know that

$$n(r) = m(R) + \dim \ker L$$

and that each reentrant geodesic yields a function ϕ_g which is in the kernel of L. We need to show that the functions ϕ_g generate $\ker L$. Let $\phi \in \ker L$ and let $\widetilde{\phi}$ be the voltage-covoltage function for ϕ defined on the primal and dual cells of the medial graph. There are at most two connected components of the boundary cells which are not completely contained in R. Let R_1 and R_2 denote the sets of boundary cells which are not completely contained in R and which meet the boundary to the left and to the right respectively of R. Note that it is possible that $R_1 \cap R_2 \neq \varnothing$. Since ϕ is in the kernel of L, we know that $\widetilde{\phi}$ is zero on the primal cells of $R_1 \cup R_2$, and is equal to some constant on the dual cells of R_1 and is equal to a (possibly different) constant on R_2. We will show that $\widetilde{\phi}$ is in fact equal to the same constant on the dual cells of R_1 and R_2. By basic facts about unique extensions, we know that ϕ is zero on all of the primal cells of $\overline{R_1}$ and $\overline{R_2}$, and hence is equal to zero on the primal cells of $\overline{\overline{R_1} \cup \overline{R_2}} = \overline{R_1 \cup R_2}$. Furthermore, we know that $\widetilde{\phi}$ is constant on the dual cells of $\overline{R_1}$. Similarly we know that $\widetilde{\phi}$ is constant on the dual cells of $\overline{R_2}$. Since $M \setminus \overline{R_1 \cup R_2}$ is finite, we know that ϕ is a finitely supported minimal function, and hence by Lemma 3.6.8, we know that the sum of the currents on the boundary of ϕ is equal to zero. If c_1 is any dual cell in R_1 and c_2 is any dual cell in R_2, we know that the current leaving any vertex to the left of c_1 or to the right of c_2 is zero, and hence the sum of the currents leaving primal boundary vertices between c_1 and

c_2 is zero. By definition of the dual function of ϕ, we know that $\widetilde{\phi}(c_1) - \widetilde{\phi}(c_2)$ is just the sum of the currents leaving primal vertices on the boundar between c_1 and c_2, and hence $\widetilde{\phi}(c_1) = \widetilde{\phi}(c_2)$. Hence $\widetilde{\phi}$ is constant on the dual cells of $\overline{R_1} \cup \overline{R_2}$. Without loss of generality, we may assume that $\widetilde{\phi}$ is equal to zero on all cells (both dual and finite) in $\overline{R_1} \cup \overline{R_2}$ and hence $\widetilde{\phi}$ must be zero on $\overline{\overline{R_1} \cup \overline{R_2}} = \overline{R_1 \cup R_2}$.

By an identical argument to the one in the finite case, we know that $\widetilde{\phi}$ can be written as a linear combination of the functions $\widetilde{\phi_g}$ where g ranges over the reentrant geodesics of the directed segment R. Hence $\dim \ker L \leq r(R)$, so

$$n(R) = m(R) + \dim \ker L \leq m(R) + r(R).$$

Since the other direction of the inequality is proven in Theorem 3.6.4, we know that

$$n(R) = m(R) + r(R).$$

□

3.7 Important Generalizations of Repeated Computations for Supercritical Graphs

Our goal of this section is to prove Theorem 3.7.2, which is a very important generalization of Theorem 3.6.6 and Lemma 2.4.2.

We recall that given a supercritical half planar medial graph M, if $X \subseteq M$ is connected, then we have that $\overline{X} = \widetilde{X}$. The following is a fact that we've used several times in proving various theorems.

Lemma 3.7.1. Let M be a supercritical medial graph and $X \subseteq M$. Then $\overline{X} = \widetilde{X}$ iff $\overline{X}$ is connected.

Proof. If $\overline{X}$ is connected, then by a previous theorem we have that $\widetilde{\overline{X}} = \overline{\overline{X}} = \overline{X}$. But

$$\widetilde{X} \subseteq \widetilde{\overline{X}} = \overline{X}$$

and hence $\widetilde{X} \subseteq \overline{X}$. Since $\overline{X} \subseteq \widetilde{X}$ for all subsets of the medial graph, we clearly have that $\overline{X} = \widetilde{X}$.

Now supposing that $\overline{X} = \widetilde{X}$. By Theorem 2.3.33 we note that $\widetilde{X}$ is always connected. □

Theorem 3.7.2. Let M be a supercritical half planar medial graph and let $X \subseteq M$ be a subset of M such that every connected component of X contains infinitely many cells. Then $\overline{X} = \widetilde{X}$.

Proof. Our strategy will be to prove that $\overline{X}$ is connected. This is sufficient since if $\overline{X}$ is connected then we would have that $\widetilde{X} \subseteq \widetilde{\overline{X}} = \overline{X}$ since $\overline{X}$ is connected. Since we always have that $\overline{X} \subseteq \widetilde{X}$, we would thus have that $\overline{X} = \widetilde{X}$.

To make matters simpler, we will assume that X has exactly two connected components. To see that this is sufficient, suppose $X = \bigcup_{i \in I} X_i$ where each X_i is a connected component of X and I is an index set for the set of connected components. Suppose $\overline{X_i \cup X_j}$ is connected for all $i, j \in I$. Then there is some path P_{ij} in $\overline{X_i \cup X_j}$ from some cell in X_i to a cell in X_j. Clearly we have that

$$Y = \left(\bigcup_{i \in I} X_i \right) \cup \left(\bigcup_{i,j \in I} P_{ij} \right)$$

is a connected subset of $\overline{X} = \overline{\bigcup_i X_i}$ which contains X. The reader verifies that $X \subseteq Y \subseteq \overline{X}$ implies that $\overline{Y} = \overline{X}$. But Y is connected, so we would have that $\overline{Y} = \overline{X}$ is connected. Hence $\widetilde{X} \subseteq \widetilde{\overline{X}} = \overline{X}$. Since we always have that $\overline{X} \subseteq \widetilde{X}$, we would thus have that $\overline{X} = \widetilde{X}$.

Thus we assume that $X = X_1 \cup X_2$ where X_1, X_2 are the connected components of X and we will try to show that $\overline{X}$ is connected. Let c_1 and c_2 be arbitrary cells in X_1 and X_2 respectively. A simple argument shows that there are infinite paths of cells $P_1 = p_1^1 p_2^1 \cdots$ and $P_2 = p_1^2 p_2^2 \cdots$ in X_1 and X_2 respectively which have no repeated cells and such that $p_1^j = c_j$. Let g_1, g_2 be geodesics such that $c_i \in B(g_i)$. By Theorem 3.4.2, the set $U(g_1) \cap U(g_2) = M \setminus (B(g_1) \cup B(g_2))$ is naturally identified with a supercritical medial graph. By Theorem 3.4.2, there is a curve γ whose image is the topological boundary of $U(g_1) \cap U(g_2)$ as a subset of $\mathbb{C}$. Let p_j^1 denote the last cell in P_1 to be inside of $B(g_1) \cup B(g_2)$, and let p_k^2 denote the last cell of P_2 to be inside of $B(g_1) \cup B(g_2)$. Clearly p_{j+1}^1 and p_{k+1}^2 are both in $U(g_1) \cap U(g_2)$ and both border the image of the curve γ. Upon identification with a supercritical medial graph $M' = U(g_1) \cap U(g_2)$, we know that $X_1 \cap M'$ and $X_2 \cap M'$ both have cells along the boundary of M'. If we can show that the closure of $(X_1 \cup X_2) \cap M'$ in M' is connected then this will clearly imply that the closure of $X_1 \cup X_2$ in M is connected since any sequence of filling in anticorners in M' correspond to a sequence of filling in anticorners in M, and similarly any path in M' corresponds to a path in M. Since we picked the cells p_{j+1}^1 and p_{k+1}^2 such that all of the cells which followed in the paths P_1 and P_2 were in $U(g_1) \cap U(g_2)$, by picking the component of $X_1 \cap U(g_1) \cap U(g_2)$ which contains p_{j+1}^1 and picking the component of $X_2 \cap U(g_1) \cap U(g_2)$ which contains p_{k+1}^2, we may reduce to the case that X_1 and X_2 are connected infinite subsets of M and that X_1 and X_2 both contain cells on the boundary of M.

Hence assume that X_1 and X_2 are infinite connected subsets of M which both contain a cell of ∂M. We will show that $\overline{X_1} \cup \overline{X_2}$ is connected. If we view each X_i as a closed subset of $\mathbb{H}$, then a simple argument shows that each is an unbounded, path connected subset of $\mathbb{H}$. Hence we can find curves $\gamma_i : [0, 1) \to X_i$ (where X_i is viewed as viewed here as a closed subset of $\mathbb{H}$) with no self intersection such that $\gamma_i(0) \in \mathbb{R}$ and $\gamma_i(0)$ is not at a vertex of the medial graph, $\gamma_i(t) \in \operatorname{int} X_i$ (where $\operatorname{int} X_i$ denotes the topological interior of X_i) and $|\gamma_i(t)| \to \infty$ as $t \to 1^-$. Since X_1 and X_2 are disjoint (since otherwise $\overline{X_1} \cup \overline{X_2}$ would trivially be connected), and the curves γ_i pass through the interior of the sets X_i and since our assumptions imply that $\gamma_1(0) \neq \gamma_2(0)$, we know that the images of γ_1 and γ_2 are disjoint. Hence by a Jordan curve theorem argument, we know that $\mathbb{H}\setminus(\operatorname{Im}\gamma_1\cup\operatorname{Im}\gamma_2)$ contains exactly three connected components, and exactly one component has bounded intersection with the real axis. A picture is shown in Figure 3.12.

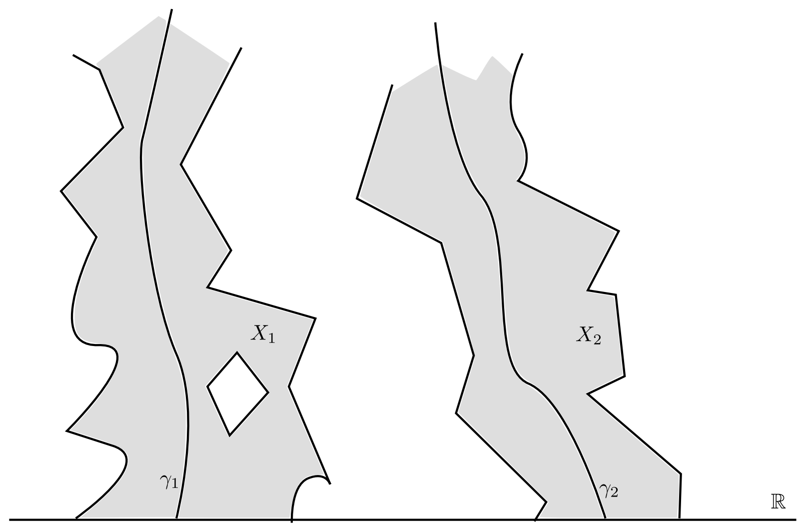


Figure 3.12: The sets X_i and the curves γ_i.

Let R_1, R_2 and R_3 be the three components of $\mathbb{H}\setminus(\operatorname{Im}\gamma_1\cup\operatorname{Im}\gamma_2)$, ordered so that $\overline{R_1}$ (topological closure) contains $\operatorname{Im}\gamma_1$, while $\overline{R_2}$ contains $\operatorname{Im}\gamma_1$ and $\operatorname{Im}\gamma_2$, and finally $\overline{R_3}$ contains only $\operatorname{Im}\gamma_2$. A picture is shown in Figure 3.13.

We will now show that the set of cells in $(M\setminus(\overline{X_1}\cup\overline{X_2}))$ that have nonempty intersection with R_2 is finite. We will essentially repeat an argument that we produced earlier. Let $\mathfrak{R}_1$ denote the set of geodesics which do not intersect X_1 and let $\mathfrak{R}_2$ denote the set of geodesics which do not intersect X_2. Since X_1 and X_2 are connected, we have that $\overline{X_i}=\widetilde{X_i}$. If g is a geodesic which doesn't intersect X_i, then X_i must be in either $B(g)$ or $U(g)$. Since $B(g)$ is finite and X_i is infinite, we must have that $X_i\subseteq U(g)$. Hence

$$\overline{X_i}=\bigcap_{g\in\mathfrak{R}_i}U(g).$$

Hence we have that

$$\begin{aligned}M\setminus(\overline{X_1}\cup\overline{X_2}) &= \left(\bigcup_{g\in\mathfrak{R}_1}B(g)\right)\cap\left(\bigcup_{g\in\mathfrak{R}_2}B(g)\right)\\ &= \bigcup_{g\in\mathfrak{R}_1,g'\in\mathfrak{R}_2}B(g)\cap B(g').\end{aligned}$$

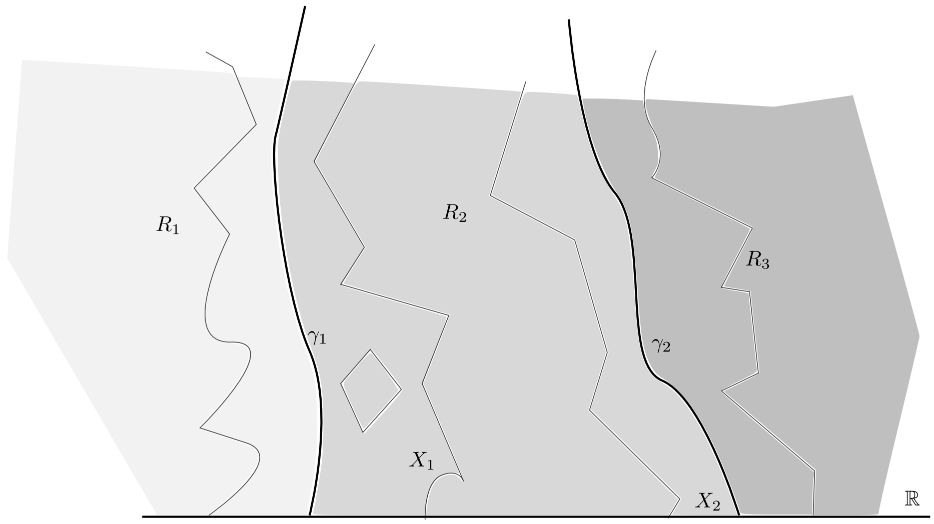


Figure 3.13: The regions R_1, R_2 and R_3 are the regions of varying shades of gray.

Let Y denote $\bigcup_{g \in \mathfrak{R}_1, g' \in \mathfrak{R}_2} B(g) \cap B(g')$. We claim that the set of cells in Y which have nonempty intersection with R_2 is finite. Suppose that $g \in \mathfrak{R}_1$ and $g' \in \mathfrak{R}_2$ are geodesics such that $B(g) \cap B(g')$ is nonempty and contains a cell with nonempty intersection with R_2. By definition, g cannot enter into the interior of the set X_1 and g' cannot enter into the interior of the set X_2. In particular, we know that

$$\operatorname{Im} g \cap \operatorname{Im} \gamma_1 = \varnothing$$

and

$$\operatorname{Im} g' \cap \operatorname{Im} \gamma_2 = \varnothing,$$

i.e. g cannot cross γ_1 and g' cannot cross γ_2. We wish to show that both g and g' have at least one endpoint between $\gamma_1(0)$ and $\gamma_2(0)$. We can simply break this into many cases. If $g(0)$ and $g(1)$ are both in R_1, then since g cannot cross γ_1, one easily sees that $B(g)$ contains no cells whose intersection with R_2 is nonempty. Similarly, $g'(0)$ and $g'(1)$ cannot both be in R_3. Now if both $g(0)$ and $g(1)$ are in R_3, then if $B(g) \cap B(g')$ is nonempty, then either g' must cross g or $B(g') \supseteq B(g)$. If $B(g') \supseteq B(g)$ then one sees that $g'(0) < g(0) < g(1) < g'(1)$ (where we use the convention that geodesics are parametrized from their left endpoint to their right). But since g' cannot cross γ_2, this implies that $\gamma_2(0) < g'(0)$. Since g' does not cross γ_2, this implies that $R_2 \cap B(g') = \varnothing$. Hence we cannot have that $B(g') \supseteq B(g)$, and we must have that g and g' intersect. Since geodesics can only intersect once, this implies that

$$g'(0) < g(0) < g'(1) < g(1)$$

or

$$g(0) < g'(0) < g(1) < g'(1).$$

Since we know that $g'(0)$ and $g'(1)$ cannot both be in R_3, and we're assuming that $g(0)$ and $g(1)$ are both in R_3, we know that

$$g'(0) < g(0) < g'(1) < g(1).$$

Hence $g'(1)$ is in R_3 (since $g(0)$ is in R_3). But we've already shown that $g'(0)$ cannot be in R_3, so we have that $g'(0)$ is in R_2 but $g'(1)$ is in R_3, which implies that g' crosses γ_2, which is a contradiction. Hence g cannot have both endpoints in R_3. Since clearly g cannot have an endpoint in R_1 and an endpoint in R_3 (since g cannot cross γ_1, we know that g must have at least one endpoint in R_2. Similarly one sees that if $B(g) \cap B(g')$ has a cell with nonempty intersection with R and $g \in \mathfrak{R}_1$ and $g' \in \mathfrak{R}_2$, then g' must also have an endpoint in R_2. Since there are only finitely many geodesics with endpoints in R_2, we know that the subset of $M \setminus (\overline{X_1} \cup \overline{X_2})$ of cells which have nonempty intersection with R_2 is finite.

We now claim that this implies that $\overline{X_1} \cup \overline{X_2}$ is connected. Let $c_1, \ldots, c_n$ be the cells of $M \setminus (\overline{X_1} \cup \overline{X_2})$ which have nonempty intersection with R. By Theorem 2.3.9, we know that each c_i is in $B(g_i)$ for some geodesic g_i. We observe that $\bigcap_{i=1}^n U(g_i)$ contains no cells in $M \setminus (\overline{X_1} \cup \overline{X_2})$ which have nonempty intersection with R_2. By Theorem 3.4.2 we know that $\bigcap_{i=1}^n U(g_i)$ can be identified with a supercritical half planar medial graph. Hence in particular, $\bigcap_{i=1}^n U(g_i)$ is connected. Let a_1 be any cell in $\bigcap_{i=1}^n U(g_i)$ which also has nonempty intersection with R_1, and let a_3 be any cell in $\bigcap_{i=1}^n U(g_i)$ which has nonempty intersection with R_3 (for example, one could just pick boundary cells which were in $\bigcap_{i=1}^n U(g_i)$ and were separated by $\bigcup_{i=1}^n B(g_i)$, which one can do since there are infinitely many boundary cells). Now since $\bigcap_{i=1}^n U(g_i)$ is connected, find any path $P = p_1 p_2 \cdots p_n$ of cells in $\bigcap_{i=1}^n U(g_i)$ from a_1 to a_2. A simple Jordan curve theorem argument shows that the topological closure of some cell in this path must have nonempty intersection with the image of γ_1 and some cell must have nonempty intersection with the image of γ_2. Let p_j be the last cell in the path P whose topological closure has nonempty intersection with R_1. We note that p_j cannot be equal to p_n (the last cell in P), since then $p_j = p_n$ would be in R_1 and R_3. By Jordan curve reasoning, we would know that the images of γ_1 and γ_2 would both have to intersect (the topological closure of) p_n. By construction of the curves γ_1 and γ_2, we would know that p_n is in both X_1 and X_2, contradicting the assumption that $X_1 \cap X_2 = \varnothing$.

We now claim this implies that p_j is in X_1. This follows since the curve γ_1 travels through the topological interior of X_1. Hence if p_j had nonempty intersection with R_1 and p_{j+1} did not, then by a Jordan curve theorem argument, we know that (the topological closure of) p_j would have to intersect the image of γ_1, which implies that $p_j \in X_1$.

Now let p_k be the first cell after p_j to be in R_3. Notice that by an argument we made above, we cannot have $p_j \in R_3$ so we have that $j < k$. Now if i is an index such that $j < i < k$, then by construction, p_i cannot intersect R_1 and p_i

cannot intersect R_3, and hence p_i must be completely contained in R_2. Since $p_i \in \bigcap_{i=1}^{n} U(g_i)$, we know that $p_i \notin M \setminus (\overline{X_1} \cup \overline{X_2})$, and hence $p_i \in \overline{X_1} \cup \overline{X_2}$. Hence the path $P' = p_j p_{j+1} \cdots p_k$ is a path in $\overline{X_1} \cup \overline{X_2}$ from a cell in $X_1 \subseteq \overline{X_1}$ to a cell in $X_2 \subseteq \overline{X_2}$. Hence $\overline{X_1} \cup \overline{X_2}$ is connected and hence

$$\overline{X_1 \cup X_2} = \overline{\overline{X_1} \cup \overline{X_2}}$$

is also connected. Since we reduced the general case to proving this, we know that the theorem statement follows.

□

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