Senior Thesis

Horse Play

Optimal Wagers and the Kelly Criterion

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1 Introduction

The fundamental problem in gambling is to find betting opportunities that generate positive returns. After identifying these attractive opportunities, the gambler must decide how much of her capital she intends to bet on each opportunity. This problem is of great interest in probability theory, dating back to at least the 18th c. when The Saint Petersburg Paradox paper by Daniel Bernoulli aroused discussion over the topic [1].

For a rational person, it would seem that they would choose the gambling opportunities that maximized expected return. However the resulting paradox, illustrated by Daniel Bernoulli, was that a gambler should bet no matter the cost. Instead of maximizing returns, economists and probablists (like Bernoulli) have argued that it is some utility function that the gambler should maximize.

Economists use utility functions as a way to measure relative satisfaction. Utility functions are often modeled using variables such as consumption of various goods and services, possession of wealth, and spending of leisure time. All utility functions rely on a diminishing returns. The difference between utility functions and returns arises because people attach a weight of importance to certain monetary values. For example, a bet which involves the loss of one's life savings of $100,000 as contrasted with the possibility of doubling the life savings would never be considered even with a high probability. A rational person attaches more utility to the initial $100,000 then the possibility of the additional value.

As suggested by Daniel Bernoulli, a gambler should not maximize expected returns but rather the utility function $\log x$ where x is the return on the investment. John Larry Kelly Jr., a scientist at Bell Labs, extended Bernoulli's ideas and uncovered some remarkable properties of the utility function $\log x$ [2]. Although his interest lies in transmission of information over a communication channel, his results can be extended to a gambler determining which bets he should make and how much. *In particular, he determined the fixed fraction of ones capital that should be bet in order to maximize the function and provide the gambler with the highest satisfaction given certain conditions.* This is known as the Kelly Criterion.

Currently, the Kelly Criterion's use extends into investment theory. It is of importance to investors to determine the amount they should invest in the stock market. The Kelly Criterion provides a solution although it has it drawbacks namely its long-term orientation.

2 Simple Example

2.1 Maximizing Expected Returns

Suppose you are confronted with an infinitely wealthy opponent who will wager even bets on repeated independent trials of a biased coin. Let the probability of us winning on each trial be $p > \frac{1}{2}$ and conversly losing $q = 1 - p$. Assume we start with an initial capital of X_0 and subsequently X_i will be our capital after i coin tosses. *How much should we bet (B_i) on each toss?* Let $T_i = 1$ be a win on the ith toss and $T_i = -1$ a loss. As a result,

$$X_n = X_0 + \sum_{i=1}^{n} B_i T_i,$$

our initial capital plus or minus the amount we win. The expected value then is

$$\begin{aligned} E[X_n] = X_0 + \sum_{i=1}^{n} E[B_i T_i] &= X_0 + \sum_{i=1}^{n} pE[B_i] - qE[B_i] \\ &= X_0 + \sum_{1}^{n} (p-q)E[B_i] \end{aligned}$$

Hence, we see that in order to maximize our expected return we should maximize the amount we bet on each trial $E[B_i]$. The strategy suggests that on each toss of the coin we should wager our entire capital (i.e. $B_1 = X_0$ and so forth). However, we see that with one loss our capital is reduced to zero and ruin occurs. Ultimately, the strategy would lead to the classic gambler's ruin.

From the example above, we expect that the optimal wager should be a fraction of our total capital. Assuming we bet a fixed fraction, $B_i = fX_{n-1}$ where $0 \leq f \leq 1$, then our capital after n trials is given by

$$X_n = X_0(1+f)^S(1-f)^L,$$

where $S + L = n$ and S is the number of wins and L is the number of losses. In this scenario, gambler's ruin is impossible for $Pr(X_n = 0) = 0$. Finding the optimal fraction to bet is the basis of the Kelly Criterion.

2.2 Introduction to Kelly's Criterion

In order to determine the fixed fractions to bet, Kelly employs the use of a quantity called G. The derivation of G follows from above:

$$\begin{aligned}
\frac{X_n}{X_0} &= (1+f)^S(1-f)^L \\
\log \frac{X_n}{X_0} &= S\log(1+f) + L\log(1-f) \\
\frac{1}{n}\log\left(\frac{X_n}{X_0}\right) &= \frac{S}{n}\log(1+f) + \frac{L}{n}\log(1-f) \\
G = E\left[\frac{1}{n}\log\left(\frac{X_n}{X_0}\right)\right] &= \lim_{n\to\infty}\frac{S}{n}\log(1+f) + \frac{L}{n}\log(1-f)
\end{aligned}$$

In the long run, $\frac{S}{n} = p$ and similarly $\frac{L}{n} = 1 - p = q$. Therefore,

$$G = p\log(1+f) + q\log(1-f) \qquad \text{where } f < 1$$

G measures the expected exponential rate of increase per trial. Originally, Kelly's paper used log base 2 to reflect if $G = 1$ you expect to double your capital, which in turn fits the standards of using $\log_2$ in information theory. However given the relationships between log's of different bases, his results hold if we use a log base e and are easier to compute. For the rest of the paper, log will refer to log base e.

Kelly chooses to maximize G with respect to f, the fixed fraction one should bet at each trial. Using calculus, we can derive the value of f that optimizes G:

$$G'(f) = \frac{p}{1+f} - \frac{q}{1-f} = \frac{p - q - f(p+q)}{(1+f)(1-f)}.$$

Setting this to zero, yields that the optimal value $f^* = p - q$. Now,

$$G''(f) = \frac{-p}{(1+f)^2} - \frac{q}{(1-f)^2}.$$

Since p, q are probabilities, $p, q \geq 0$. Therefore, $G''(f) < 0$, so that $G(f)$ is a strictly concave function for $f \in [0, 1)$. Given that $G(f)$ is concave over $[0, 1)$ and f^* satisfies $G'(f) = 0$, then $G(f)$ has a global maximum at f^* and

$$\begin{aligned}
G_{max} &= p\log(1+p-q) + q\log(1-p+q) \\
&= \log(2) + p\log(p) + q\log(q)
\end{aligned}$$

by Proposition 1

2.2.1 Example of the Kelly Criterion

Elliot plays against an infinitely rich enemy. Elliot wins even money on successive independent flips of a biased coin with probability of winning $p = .75$. Applying $f^* = p - q$, Elliot should bet $.75 - .25 = .5$ or 50% of his current capital on each bet to grow at the fastest rate possible without going broke.

Let's that Elliot doesn't want to bet 50% of his capital ($f \neq .5$). We can see exactly how Elliot's growth rate differs with various f's. Recall that the growth rate is given by

$$G = p\log(1+f) + q\log(1-f)$$

$$G = .75\log(1+f) + .25\log(1-f).$$

Applying this formula numerically with varying f's yields:

Fraction bet (f)	0.25	0.375	**0.5**	0.625	0.75
Growth Rate (G)	9.543715%	12.133939%	**13.081204%**	11.892355%	7.313825%

We see that Elliot's deviation from $f = .5$ results in a lower rate of growth. The further the deviation from the optimal, the greater the difference in growth rate.

Kelly assumes that the gambler will always bet to maximize G since if expected returns are optmized then the gambler will eventually reach ruin. In this simple example, we only bet on one outcome and the odds of winning are even (i.e. 2:1). Kelly examines different scenarios that exist for gamblers with always the same result of maximizing G.

3 Odds and Track Take

Before continuing a brief discussion on the terms *odds* and *track take* is required in order to proceed. Assume there is a betting pool and that you can bet on 4 different teams. You are allowed to bet as much as you want on each team. Let's suppose A dollars are bet on the first team, B on the second, C on the third, and D on the fourth. If team one wins the people who bet on that team share the pot $A+B+C+D$ in proportion to what they each bet. Let

$$\frac{A+B+C+D}{A} = \alpha$$

where for every one dollar bet you get α dollars back if A won. Label $\beta = \frac{A+B+C+D}{B}, \gamma, \delta$ similarly. Therefore, let the "odds" be quoted as α-to-1, β-to-1, etc.

Notice that in this scenario all the money would be returned to the gamblers. There is no third party that receives a cut of the money. Moreover,

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{A+B+C+D}{A+B+C+D} = 1.$$

Assume now that there exists a bookie who keeps E dollars and returns $A+B+C+D-E$ to the winners. The new odds would be

$$\frac{1}{\alpha} = \frac{A+B+C+D-E}{A}$$

and similarly for β, γ, δ. Also, notice that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{A+B+C+D}{A+B+C+D-E} > 1.$$

We see that the case where

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} < 1$$

is meaningless. This scenario would suggest that more money is returned that what was betted on. In gambling, this is an improbable if not impossible scenario. As a result, Kelly does not consider this option in his analysis.

When Kelly talks about odds, he is referring to the type of odds discussed here. This is in contrast to quoted odds such as 5-to-1 which refer to an event with probability of $\frac{1}{5+1}$. In Kelly's case, he would quote the odds as 6-to-1 for an event with a probability of $\frac{1}{6}$. The difference in the two results from the definition of earnings. In the prior example 5-to-1, the 5 refers to the *additional earnings*, so you receive \$6 but you bet \$1 and thus your earnings is \$5. On the other hand, Kelly looks at the *total earnings* or the pure amount you receive and does not subtract off the dollar that you lost in making the bet.

Note an important observation regarding odds: *Odds do not directly translate into probabilities.* An odd is simply a calculation based on the amount of money in the pool and can/will vary from the probability. Hence, he assumes the odd-makers do not know the underlying true probabilities.

4 Generalizing Kelly Criterion

Kelly explores the case when an information channel has multiple channels transmitting different signals. In stead of using Kelly's notation and definition, we restated his definitions with gambling terminology.

p_s — the probability that an event (i.e. a horse) will win

α_s — the odds paid on the occurrance that the "horse s" wins. α_s represents the number of dollars returned (including the one dollar bet) for a one dollar bet (e.g. $\alpha_s = 2$ says that for you win \$2 for every \$1 bet). This is often stated at "m-to-1" where m is an value greater than or equal to one.

$a(s)$ — the fraction of the gambler's capital that she decides to bet on the "horse s".

b — the fraction of the gambler's capital that is not bet

r — the interest rate earned on the fraction not bet.

α'_s — $\frac{\alpha_s}{1+r}$. Slight modification on the odds that incorporates the rate earned when you don't bet.

σ — $\sum_s \frac{1}{\alpha'_s} = \sigma$ The sum of the reciprocal of the odds over all outcomes (all s). A special remark: if $\sigma > 1$ or $\sum_s \frac{1}{\alpha_s} > \frac{1}{1+r}$ then their exists a track take; if $\sigma = 1$ or $\sum_s \frac{1}{\alpha_s} = \frac{1}{1+r}$, no track take; and if $\sigma < 1$ or $\sum_s \frac{1}{\alpha_s} < \frac{1}{1+r}$ then more money returned than betted on.

We can consider Kelly's information channel as similar to betting on the outcomes of a single game. An extension of the result given by Kelly is to use statistically independent identically distributed games being played at the same time (i.e. the odds are the same for each game).

5 Kelly Criterion

The Kelly Criterion focuses on answering:

1. $a(s)$ the fraction the gambler bets on each s
2. b the fraction the gambler does not bet
3. $s \in \Gamma$ where Γ is the set of indices, s, for which we place a bet (i.e. $a(s) > 0$). Let Γ' be the set of indices, s, in which we don't place a bet.

5.1 Setting up the Maximizing Function

Kelly's focus was on information theory and the transfer of information across channels. For him, the nuiances surrounding money did not play a substantial part of his analysis. For instance, he ignored time value of money because information channels do not earn interest when not in use. When considering his analysis under the concept of money, one logical question to ask is: if I don't bet part of my capital, I could earn interest, so how does that effect the fraction I bet and which "horses" do I bet on? One could say that we are comparing the rate of return on a risk free asset (i.e. Treasury bonds) with the expected rate earned by betting. We expand on Kelly's analysis to include earning a rate on the money not bet.

Let V_N be the total capital after N iterations of the game and V_0 be the gambler's initial capital. Moreover the fraction we don't bet, b, earns an interest rate of r. Lets assume on one iteration we bet on s_1 and s_2, then $V_1 = (a(s_1)\alpha_1 + (1+r)b)V_0$ or $(a(s_2)\alpha_2 + (1+r)b)V_0$ depending upon whether s_1 or s_2 horse won respectively. Occassionally, $a(s)\alpha_s$ will be repeated. Therefore let W_s be the number of times in the sequence that s "wins".

Under Kelly's assumptions, something always wins. It never occurs that no "horse" wins the race. Moreover, the bettor uses up her entire capital either by betting on the horses or with holding some capital not bet. Thus, the gambler loses money on every bet that she made except for the horse that actually won. Hence,

$$V_N = \prod_s^n ((1+r)b + a(s)\alpha_s)^{W_s} V_0.$$

From this, Kelly deduces G by

$$\log\left(\frac{V_N}{V_0}\right) = \log \prod_s ((1+r)b + \alpha_s a(s))^{W_s}$$

$$G = \lim_{N\to\infty} \frac{1}{N} \log\left(\frac{V_N}{V_0}\right) = \sum_s \tfrac{W_s}{N} \log((1+r)b + \alpha_s a(s))$$

$$G = \sum_s^n p(s) \log((1+r)b + \alpha_s a(s)) \quad \text{where in the long run } \lim_{N\to\infty} \tfrac{W_s}{N} = p(s).$$

We will now slightly modify G by changing the odds to $\alpha_s = (1+r)\alpha_s'$. Replacing G with

$$G = \sum_s^n p(s) \log((1+r)b + (1+r)\alpha_s' a(s))$$

$$G = \sum_s^n p(s) \log(b + \alpha_s' a(s)) + \sum_s^n p(s) \log(1+r)$$

$$G = \sum_s^n p(s) \log(b + \alpha_s' a(s)) + \log(1+r) \quad \text{for } \textstyle\sum_s p(s) = 1.$$

Therefore, the function is given by

$$\max \sum_s^n p(s) \log(b + \alpha_s' a(s)) + \log(1+r)$$

over the variables b and $a(s)$. The known parameters are α_s' and $p(s)$ for each s. Note that $\log(1+r)$ is a constant and does not effect the critical values. In Kelly's paper, he simply set $r = 0$ and proceeded with maximizing the function. We are just generalizing his conclusions to include the interest earned when you don't bet.

Constraints
Two general constraints emerge:

1. The total capital is the sum of the fractions bet plus the fraction not bet.

$$b + \sum_{s \in \Gamma} a(s) = 1$$

2. Due to the log, G is defined only when

$$b + \alpha'_s a(s) > 0 \qquad \text{for all } s.$$

   This implies that if $b = 0$ then for all s $a(s) > 0$.

Moreover, $0 \leq b \leq 1$ and $0 \leq a(s) \leq 1$ as the fractions bet and saved must be non-negative. One important observation to note is that

$$\sum_s p(s) = 1.$$

The Optimization Problem
Generally for consistency in optimization problems, optimization problems are written in the form

$$\begin{aligned} &\min f(x) \\ \text{s.t. } &g_i \leq 0 \qquad \text{for } i = 1, \ldots, l \\ &h_j = 0 \qquad \text{for } j = 1, \ldots, k \end{aligned}$$

However any optimization problem can be converted to the form described above (i.e. $\max f(x) = \min -f(x)$. Converting Kelly's constraints, the optimization problem primarily described in Kelly's paper is

$$\min -\sum_s^n p(s) \log(b + \alpha'_s a(s)) + \log(1 + r)$$

$$\begin{aligned} \text{s.t.} \qquad b + \sum_{s=1}^n a(s) &= 1 && && (1) \\ -(b + (1+r)\alpha'_s a(s)) &< 0 && \text{for } s = 1, \ldots, n && (2) \\ -a(s) &\leq 0 && \text{for } s = 1, \ldots, n && (3) \\ a(s) - 1 &\leq 0 && \text{for } s = 1, \ldots, n && (4) \\ -b &\leq 0 && && (5) \\ b - 1 &\leq 0 && && (6) \end{aligned}$$

for the variables b and $a(s)$.

5.2 KKT Criterion

The Karush-Kuhn-Tucker Theorem locates local minimums for non-linear optimization functions subject to constraints. Kelly did not utilize this theorem in constructing his paper. He examines three different optimization problems, distinguishing them based on the value of σ and $p(s)\alpha'_s$. However, the same results can be concluded using the KKT approach with a single optimization problem.

Applying the KKT condition, the following set of equations hold

$$\sum_{s}^{n} \frac{p(s)}{b+\alpha'_s a(s)} = -\mu_1 + \mu_2 + \lambda \qquad \text{where } \mu_1(b)=0, \mu_2(b-1)=0, \mu_1 \geq 0, \mu_2 \geq 0 \tag{7}$$

$$\text{for all } s \qquad \frac{p(s)\alpha'_s}{b+\alpha'_s a(s)} = -\mu_{s_1} + \mu_{s_2} + \lambda \qquad \text{where } \mu_{s_1}(a(s))=0, \mu_{s_2}(a(s)-1)=0, \mu_{s_1} \geq 0, \mu_{s_2} \geq 0 \tag{8}$$

If a b and set of $a(s)$ exists such that the KKT condition holds, then this is a *necessary* condition for a local minimum. However, G is a concave function. Let $y = b + \alpha'_s a(s)$ and $g(x) = \log(x)$. Because $\log(x)$ is an increasing function and both y and $g(x)$ are concave functions, the composition $g(y)$ is also concave. In addition, the sum of concave functions is also concave, and adding a constant, $\log(1+r)$ preserves its concaveness. Hence, $\sum_s^n p(s)\log(b+\alpha'_s a(s)) + \log(1+r)$ is concave. The significance of this statement is that if there is a x^* that satisfies KKT then x^* is **global maximum of G.**

The goal is to find critical points and optimal values under certain conditions facing a gambler. In particular, when the gambler confronts different $\max(p(s)\alpha'_s)$ and σ. In order to determine whether points are critical, we need to find appropriate $b, \lambda, \mu_k, \mu_{s_j}$, and $a(s)$ that satisfy (7) and (8) as well as the feasibility conditions (1)-(6). Although in Kelly's paper optimization is a central feature, he does not employ the use of KKT. In fact, we are slightly deviating from Kelly by manipulating his three distinct cases and converting it to one optimization problem with three solutions depending on the situation facing the gambler as well as generalizing the results to include the interest rate earned. Let's examine the cases: $\sigma \leq 1$ and $\sigma \geq 1$.

6 $\sum_s \frac{1}{\alpha'_s} \leq 1$

A gambler encountering this situation has two options: either all the money is returned to the winners $\left(\sum_s \frac{1}{\alpha'_s} = 1\right)$ or the "bookie" has injected money into the system so more money is returned than betted on $\left(\sum_s \frac{1}{\alpha'_s} < 1\right)$. In both cases, there exist critical points that satisfy KKT. However, the optimal values depend on the relationship between $p(s)$ and α'_s.

6.1 If $b = 0$

Claim 1. *If $b = 0$, then $a(s) = p(s)$ for all s with an optimal value of*

$$G(0, p(s)) = \sum_s p(s)\log(\alpha'_s p(s)) + \log(1+r)$$

if and only if $\sum_s \frac{1}{\alpha'_s} \leq 1$ or equivalently $\sum_s \frac{1}{\alpha_s} \leq \frac{1}{1+r}$.

Proof. Since $b = 0$ from (2), $a(s) > 0$ for all s. Moreover from (1) and $b = 0$,

$$\sum_s a(s) = 1.$$

These statements provide a bound on $a(s)$,

$$0 < a(s) < 1 \qquad s = 1, \ldots, n.$$

Given this bound, $\mu_{s_j} = 0$ for all s_j and because $b = 0$, the KKT conditions become

$$\frac{p(s)\alpha'_s}{a(s)\alpha'_s} = \lambda \qquad s = 1, \ldots, n \tag{9}$$

$$\sum_s \frac{p(s)}{\alpha'_s a(s)} = \lambda - \mu_1 \tag{10}$$

Hence (9),

$$p(s) = \lambda a(s) \qquad s = 1, \dots, n$$

and summing up over all s

$$\sum_s p(s) = \lambda \sum_s a(s)$$
$$1 = \lambda.$$

Thus,

$$a(s) = p(s)$$

Because $\lambda = 1$ and $a(s) = p(s)$, (10),

$$\sum_s \frac{1}{\alpha'_s} = 1 - \mu_1,$$

must hold at the critical value. So critical points exist if and only if

$$\mu_1 = 1 - \sum_s \frac{1}{\alpha'_s} > 0.$$

Hence $\sum_s \frac{1}{\alpha'_s} \leq 1$ or $\sum_s \frac{1}{\alpha_s} \leq \frac{1}{1+r}$.

Replacing the critical point into the objective function yields:

$$G_{max} = \sum_s p(s) \log(\alpha'_s p(s)) + \log(1+r)$$

$$G_{max} = \sum_s p(s) \log\left(\frac{p(s)\alpha_s}{1+r}\right) + \log(1+r)$$

□

A gambler facing a $\sum_s \frac{1}{\alpha'_s} \leq 1$ or $\sum_s \frac{1}{\alpha_s} \leq \frac{1}{1+r}$ (no track take) will optimize G by betting on everything in proportion to the probability, $a(s) = p(s)$, with $b = 0$. The gambler should ignore the posted odds and bet with the probability and earn

$$G_{\max} = \sum_s p(s) \log(\alpha'_s p(s)) + \log(1+r),$$

or equivalently,

$$G_{\max} = \sum_s p(s) \log\left(\frac{\alpha_s p(s)}{1+r}\right) + \log(1+r).$$

This agrees with Kelly's solution regarding *no track take* case. We don't need to proceed further with the breakdown of $\max\{p(s)\alpha'_s\}$ because the result only depends on σ.

Lets take a closer look at the optimal value G given that $a(s) = p(s)$ and $\sum_s \frac{1}{\alpha'_s} = c$. We want to examine how the "bookies" could limit the growth rate. Since the only control the "bookies" have is the odds, we want to minimize G with respect to $a(s)$ subject to $\sum_s \frac{1}{\alpha'_s} = c$. The critical points would be the odds the "bookies" would like to quote to minimize the return to the gambler.

Lemma 1. *Suppose* $\sum_s \frac{1}{\alpha'_s} = c$ *where* c *is a positive number. Then, the minimum of* G *occurs when* $\alpha'_s = \frac{1}{cp(s)}$ *and*

$$\min G = \sum_s p(s) \log\left(p(s)\frac{1}{cp(s)}\right) + \log(1+r) = -\log(c) + \log(1+r).$$

Furthermore when,

$$\begin{aligned} c < 1 &\implies G_{\min} > \log(1+r) \\ c = 1 &\implies G_{\min} = \log(1+r) \\ c > 1 &\implies G_{\min} < \log(1+r) \end{aligned}$$

Proof. The optimization problem is

$$\min \sum_s p(s) \log(p(s)\alpha'_s) + \log(1+r)$$

$$\text{s.t.} \qquad \sum_s \frac{1}{\alpha'_s} = 1$$

with respect to α'_s.

However instead of optimizing this system, let $\beta_s = \frac{1}{\alpha'_s}$. Hence, the new problem is

$$\min \sum_s p(s) \log\left(p(s)\frac{1}{\beta_s}\right) + \log(1+r) \quad = \quad \min \sum_s p(s)\log(p(s)) - \sum_s p(s) \log(\beta_s) + \log(1+r)$$

$$\text{s.t.} \qquad \sum_s \beta_s = c$$

with respect to β_s.

Given this new system, we see that $-\log(\beta_s)$ is a convex function. Since the sum of convex functions is convex and constants do not change the function, G is convex. Due to convexity, any point that satisfies KKT is a **global minimum**.

Applying Lagrange multiples gives,

$$-p(s) = \lambda\beta_s \quad \text{for all } s.$$

Summing over s, we get

$$-1 = c\lambda.$$

Hence, a global minimum value is

$$\beta_s = cp(s) \quad \text{or} \quad \alpha'_s = \frac{1}{cp(s)}.$$

When $\beta_s = cp(s)$, the global minimum is

$$\begin{aligned} &\sum_s p(s)\log(p(s) - \sum_s p(s)\log(cp(s)) + \log(1+r) \\ &= -log(c)\sum_s p(s) + \log(1+r) \\ &= -\log(c) + \log(1+r) \end{aligned}$$

The last statements follow from substituting in different values for c. $\square$

Corollary 1. *Let $c = 1$. If $\alpha'_s \neq \frac{1}{p(s)}$,*

$$G = \sum_s p(s)\log(p(s)\alpha'_s) + \log(1+r) > \log(1+r).$$

Proof. In Lemma 1, we demonstrated that $\alpha'_s = \frac{1}{p(s)}$ produces a global minimum value of $\log(1+r)$. Hence any slight modification, where $\alpha'_s \neq \frac{1}{p(s)}$ but $\sum_s \frac{1}{\alpha'_s} = 1$, will result in a growth rate greater than the rate earned by placing money in bank. □

In Kelly's paper, he makes a special note of this statement. In his case, he looked at when $r = 0$ and stated that when the odds are unfair (i.e. α_s differs from $\frac{1}{p(s)}$) then the gambler has an advantage and can expect a positive growth rate.

When Kelly states "unfair odds", he implies that the gambler has *insider information.* In essence, the gambler must know that the real probability of the "horse" winning is different then what the odds say. Hence, the gambler must have insider information; hence if it were public the odds would reflect that information. Therefore, a gambler with insider information can and will gain a positive growth rate.

6.2 If $0 < b \leq 1$

Above we showed that if $\sum_s \frac{1}{\alpha'_s} \leq 1$ then an optimal value exists at $b = 0$ and $a(s) = p(s)$. Recall that there may be multiple critical points but only one optimal value. This is the situation with this case. A gambler only wants to optimize G. Hence from above, any point that satisfies KKT is a global optimal, so if that point fulfills KKT conditions then it is *one way* of achieving the optimal value G. Hence, we found that the global maximum when $\sum_s \frac{1}{\alpha'_s} \leq 1$ would be achieved by $b = 0$ and $a(s) = p(s)$. There may be additional ways to have the same global optimal value, with $0 < b \leq 1$, but it would produce the same optimal value.

6.2.1 Non-Unique Critical Points

Already shown is that a critical point exists at $b = 0$ with $\sum_s \frac{1}{\alpha'_s} \leq 1$. *Is it possible that there are more?* Moreover, does a critical point exist when b is non-zero. The answer is yes. Note that the same optimal value results when

$$a(s) = p(s) - \frac{b}{\alpha'_s}$$

(i.e. $G_{\max} = \sum_s p(s)\log(p(s)\alpha'_s) + \log(1+r)$). However, the hitch is that $a(s) \geq 0$; thus,

$$a(s) = \frac{p(s)\alpha'_s - b}{\alpha'_s} \geq 0 \quad \text{for all } s.$$

Because $\alpha'_s > 0$, then for $a(s) > 0$, $0 < b \leq \min(p(s)\alpha'_s)$. Also, we know that $\sum_s a(s) = 1 - b$. Summing over all s,

$$\sum_s a(s) = \sum_s p(s) - b\sum_s \frac{1}{\alpha'_s}$$
$$1 - b = 1 - b\sum_s \frac{1}{\alpha'_s}$$

This statement will only be true if $\sum_s \frac{1}{\alpha'_s} = 1$.

If $\sum_s \frac{1}{\alpha'_s} = 1$ and the gambler choses any b with $0 < b \leq \min(p(s)\alpha'_s) = \frac{\min(p(s)\alpha_s)}{1+r}$, then the bettor saves b dollars, bets $a(s) = p(s) - \frac{b}{\alpha'_s} = p(s) - \frac{b(1+r)}{\alpha_s}$ dollars, and gets the same expected growth rate

$$\begin{aligned} G_{\max} &= \sum_s p(s) \log(p(s)\alpha'_s) + \log(1+r) \\ &= \sum_s p(s) \log\left(p(s)\frac{\alpha_s}{1+r}\right) + \log(1+r) \end{aligned}$$

Summary

A gambler with $\sum_s \frac{1}{\alpha'_s} = c$ where $c \leq 1$ (no track take) will optimize G by betting on everything in proportion to the probability, $a(s) = p(s)$ and $b = 0$. If

$$\begin{aligned} c < 1 &\implies G = \sum_s p(s) \log(\alpha'_s p(s)) + \log(1+r) > \log(1+r) \\ c = 1 &\implies G = \sum_s p(s) \log(\alpha'_s p(s)) + \log(1+r) = \log(1+r) \end{aligned}$$

Moreover if $\alpha'_s \neq \frac{1}{p(s)}$ for at least some s and $\sum_s \frac{1}{\alpha'_s} = 1$ then $G > \log(1+r)$.

For $\sum_s \frac{1}{\alpha'_s} = 1$, multiple critical points exists. The bettor can save b such that $0 < b \leq \min(p(s)\alpha'_s)$ and bet $a(s) = p(s) - \frac{b}{\alpha'_s}$ and receive the same optimal value of

$$G_{\max} = \sum_s p(s) \log(p(s)\alpha'_s) + \log(1+r).$$

7 $\sum_s \frac{1}{\alpha'_s} > 1$

Now we will look at another case. When $\sum_s \frac{1}{\alpha'_s} > 1$, this is indicative of a track take. Hence, there exists a "bookie" that takes part of the pool off the top. *In this case, how should the gambler bet?* To Kelly, this was the most interesting case. In effect, he derives an algorithm to determine which s to bet and how much. Division into additonal cases is necessary.

7.1 $b = 1$

Assuming $b = 1$, (1) leads to $a(s) = 0$ for all s. This also satisfies the feasibility condition (2). Hence, the optimal value is

$$G(1, 0) = \log(1+r)$$

if $a(s) = 0$ for all s and $b = 1$ is a critical point.

As a result, the KKT conditions, (7) and (8), are

$$\sum_s \frac{p(s)}{b} = \mu_2 + \lambda \qquad \text{where } \mu_1 = 0 \tag{11}$$

$$\text{and} \quad p(s)\alpha'_s = \lambda - \mu_{s_1} \qquad \text{for all } s \tag{12}$$

Letting $b = 1$ and summing over all $p(s)$, (11) becomes

$$1 = \mu_2 + \lambda. \tag{13}$$

From (12),

$$\lambda = p(s)\alpha'_s + \mu_{s_1} \qquad \text{for all } s. \tag{14}$$

Given that $p(s)\alpha'_s > 0$ for all s and feasibility of KKT, $\mu_{s_1} \geq 0$, then (14) implies $\lambda > 0$.

Moreover since at feasibility $\mu_2 \geq 0$ and $\lambda > 0$, from (13)

$$0 < \lambda \leq 1.$$

7.1.1 $\max\{p(s)\alpha'_s\} \leq 1$

Claim 2. *There is a critical point at $a(s) = 0$ for all s and the corresponding optimal value is*

$$G(1, 0) = \log(1 + r)$$

if and only if $\max\{p(s)\alpha'_s\} \leq 1$.

Proof. Now we just need to show there exists $\lambda, \mu_{s_1}, \mu_2$ such that $\mu_{s_1} \geq 0$ and $\mu_2 \geq 0$. If there exists these values, then the above is a critical point.

1. Suppose $b = 1$, $a(s) = 0$ is a critical point, then there is an λ, $0 < \lambda \leq 1$ and $\mu_2 = 1 - \lambda \geq 0$. Moreover, $\mu_{s_1} \geq 0$ such that $\lambda = p(s)\alpha'_s + \mu_{s_1}$ for all s. Hence if $0 < \lambda \leq 1$ and $\mu_{s_1} \geq 0$, implies by (14) $p(s)\alpha'_s \leq 1$ for all s.

2. Suppose $\max\{p(s)\alpha'_s\} \leq 1$ implying that $p(s)\alpha'_s \leq 1$ for all s. Now let $\lambda = 1$. By (13),

   $$\mu_2 = 0,$$

   which satisfies the KKT condition. Moreover letting $\lambda = 1$ and $p(s)\alpha'_s \leq 1$ for all s, (14) is

   $$1 - p(s)\alpha'_s = \mu_{s_1} \geq 0 \qquad \text{for all } s.$$

   Thus, all KKT conditions are fullfilled and $a(s) = 0$ and $b = 1$ is a critical point with optimal value

   $$G(1, 0) = \log(1 + r).$$

□

From the gambler's perspective, if the $\max\{p(s)\alpha'_s\} \leq 1$, then the gambler best growth rate is $\log(1 + r)$. Hence one way of achieving the optimal growth rate with $\max\{p(s)\alpha'_s\} \leq 1$ is by betting 0 on all s.

7.1.2 $\max\{p(s)\alpha'_s\} > 1$

As we showed in Claim 2, there can not exist an optimal value when $\max\{p(s)\alpha'_s\} > 1$ if $b = 1$.

7.2 If $0 < b < 1$

Recall that Γ is the set of all indices in which $a(s) > 0$ and Γ' is the set of all indices in which $a(s) = 0$.

If $0 < b < 1$ and (1) holds, then there exists at least one s where $a(s) > 0$ (Γ is a non-empty set). In addition, $a(s) < 1$ by (1).

With these assumptions, the KKT equations (7) and (8) are

$$\frac{p(s)\alpha'_s}{b + \alpha'_s a(s)} = \lambda \qquad \text{for } a(s) >, \ s \in \Gamma \tag{15}$$

$$\frac{p(s)\alpha'_s}{b} = \lambda - \mu_{s_1} \qquad \text{for } s \in \Gamma', \ \mu_{s_1} \geq 0 \tag{16}$$

$$\sum_s \frac{p(s)}{b + \alpha'_s a(s)} = \lambda \tag{17}$$

and breaking up (17)

$$\sum_{s \in \Gamma'} \frac{p(s)}{b} + \sum_{s \in \Gamma} \frac{p(s)}{b + \alpha'_s a(s)} = \lambda. \tag{18}$$

We will show in Proposition 2 that $\lambda = 1$ and

$$a(s) = p(s) - \frac{b}{\alpha'_s} \qquad \text{for } a(s) > 0.$$

Now that we know $a(s)$, it is still dependent upon knowing the value of b. Computing b requires an introduction of new notation.

Denote

$$\sigma_\Gamma = \sum_{s \in \Gamma} \frac{1}{\alpha'_s}$$

$$p_\Gamma = \sum_{s \in \Gamma} p(s)$$

Note that σ_Γ and σ differ. We can think of σ_Γ as an intermediate value of σ. Put another way, σ_Γ is the sum of the reciprocal odds for all horses we bet on. Contrasting, $\sigma = \sum_s \frac{1}{\alpha'_s}$ is the sum of the reciprocal odds of all the horses. Hence, $\sigma_\Gamma \leq \sigma$. With this notation, a value of b can be computed.

Using (18) with $\sum_s p(s) = 1$,

$$\frac{1}{b}\left(1 - \sum_{s \in \Gamma} p(s)\right) + \sum_{s \in \Gamma} \frac{p(s)}{b + \alpha'_s a(s)} = \lambda$$

Proposition (2) and (23) conclude that

$$\frac{1}{b}\left(1 - \sum_{s \in \Gamma} p(s)\right) + \sum_{s \in \Gamma} \frac{1}{\alpha'_s} = 1.$$

From the definitions,

$$\frac{1 - p_\Gamma}{b} = 1 - \sigma_\Gamma, \tag{19}$$

and hence since $0 \leq p_\Gamma \leq 1$ and $b > 0$ then $\sigma_\Gamma \leq 1$.

At this stage, two possibilities emerge from (19): $\sigma_\Gamma < 1$ or $\sigma_\Gamma = 1$. Kelly considers only one of these cases when $\sigma_\Gamma < 1$. He regards that a positive b only results when a track take occurs ($\sum_s \frac{1}{\alpha'_s} > 1$). By Proposition 3, $\sigma_\Gamma < 1$ only results if $\sigma > 1$ or $\sigma < 1$. If $\sigma \leq 1$, we have handled the cases in Section 6. **However, we are only considering in this section when $\sigma > 1$.**

If $\sigma > 1$, we can say that at optimal $\sigma_\Gamma < 1$ (Proposition 3), so there exists at least one s in which we don't bet on. This implies $p_\Gamma < 1$. The result directly comes from

$$\frac{1 - p_\Gamma}{b} = 1 - \sigma_\Gamma.$$

Because $b > 0$ and $1 > p_\Gamma > 0$, then

$$1 - \sigma_\Gamma > 0$$

and hence $\sigma_\Gamma < 1$ at optimal.

Moreover, we can solve for b

$$b = \frac{1 - p_\Gamma}{1 - \sigma_\Gamma},$$

and with

$$a(s) = p(s) - \frac{b}{\alpha'_s} \quad \text{for } s \in \Gamma$$

we know the critical points.

Because $a(s) > 0$ and from (12), the following must be true at optimum

$$p(s)\alpha'_s > b \qquad \text{for } s \in \Gamma \tag{20}$$
$$p(s)\alpha'_s \leq b \qquad \text{for } s \in \Gamma' \tag{21}$$
$$\sigma_\Gamma < 1 \tag{22}$$

But there is one question:

What is Γ, Γ'?

To solve this problem, let's introduce some new notation.

Notation and assumptions:

1. Order the products $p(s)\alpha'_s$ so that
$$p(s)\alpha'_s \geq p(s+1)\alpha'_{s+1}$$
When the products are equal the ordering is arbitrary. Let t be the index designating the ordering where t is an integer greater than or equal to 0. $t = 1$ is the s corresponding to the $\max\{p(s)\alpha'_s\}$.

2.
$$p(s) > 0, \qquad \sum_{s=1}^{n} p(s) = 1, \qquad p_t = \sum_{s=1}^{t} p(s)$$

3.
$$\alpha'_s > 0, \qquad \sum_{s=1}^{n} \frac{1}{\alpha'_s} \geq 1, \qquad \sigma_t = \sum_{s=1}^{t} \frac{1}{\alpha'_s}$$
Hence if $\sigma_t < 1$ then $t < n$.

4. Let

$$F_t = \frac{1-p_t}{1-\sigma_t}, \qquad F_0 = 1$$

Consider how

$$F_t = \frac{1-p_t}{1-\sigma_t}$$

varies with t. Let t^* be the *critical index*. It is the break point that determines which "horses" we bet or don't bet on (Γ and $\Gamma^{'}$ sets). When we are at optimal, $F_t = b$.

Kelly provides an algorithm for finding the t^* that optimizes G. He computes t^* by

$$t^* = \min\{\min\{t : p(t+1)\alpha'_{t+1} < F_t\}, \max\{t : \sigma_t < 1\}\},$$

as given by Theorem 2. The process starts by computing the first condition with $t = 0$ and increasing t by one until $p(t+1)\alpha'_{t+1} < F_t$. Then repeat but use the second condition, $\sigma_t < 1$. If $t = 0$ then we do not bet on anything.

7.2.1 $\max\{p(s)\alpha'_s\} \leq 1$

Now if $\max\{p(s)\alpha'_s\} = p(1)\alpha'_1 \leq 1$ (i.e. $p(1)\alpha'_1 \leq F_0$), F_t increases with t until $\sigma_t \geq 1$ by Theorem 1. Because $p(1)\alpha'_1 \leq F_0$, then by Theorem 1 $F_0 \leq F_1$ so $p(1)\alpha'_1 \leq F_1$. Therefore, it fails (20). In this case by Theorem 2, $t^* = 0$, $\Gamma = \emptyset$, and $b = 1$. The gambler is better off not betting on anything and hence

$$G_{\max} = \log(1+r).$$

This agrees with the previous case when $b = 1$ and $\max\{p(s)\alpha'_s\} \leq 1$. This demonstrates that there does not exist any critical value points if $0 < b < 1$. By not betting on anything, we reach a contradiction for the only way $0 < b < 1$ is if there exists at least one $a(s) > 0$. In this case, the best option is to resort to the previous case where $b = 1$.

> If $\max\{p(s)\alpha'_s\} \leq 1$ and $\sum_s \frac{1}{\alpha'_s} > 1$ then there are no critical value points. The gambler would use Section 7.1.1 and bet $b = 1$ with
>
> $$G_{\max} = \log(1+r)$$

7.2.2 $\max\{p(s)\alpha'_s\} > 1$

If $\max\{p(1)\alpha'_1\} > 1$ or equivalently $p(1)\alpha'_1 > 1$, F_t decreases with t until $p(t+1)\alpha'_t < F_t$ or $\sigma_t \geq 1$ by Theorems 1 and 2. In essence if $p(1)\alpha'_1 > 1$, then by Theorem 1, $F_0 > F_1$, and hence $p(1)\alpha'_1 > F_1$. This satisfies (20). As long as $p(t)\alpha_t > F_{t-1}$, then 1, $F_{t-1} > F_t$, and it will fulfill (20). At optimum, (20) must hold true, so we need to find all s that meets (20) and (21). By using this procedure, we will find t^*, which indicates $\Gamma = \{t : 0 < t \leq t^*\}$. Now that we know t^*, the optimization problem is solved and critcal values exist.

> If $\sigma > 1$ and $\max\{p(s)\alpha'_s\} > 1$,
>
> $$G = \sum_{s\in\Gamma} p(s)\log(p(s)\alpha'_s) + \sum_{s\in\Gamma'} p(s)\log(b) + \log(1+r)$$
>
> where $a(s) = p(s) - \frac{b}{\alpha'_s}$, $b = \frac{1-p_{t^*}}{1-\sigma_{t^*}}$ where $\Gamma = \{t : 0 < t \leq t^*\}$ and $t^* = \min\{\min\{t : p(t+1)\alpha'_{t+1} < F_t\}, \max\{t : \sigma_t < 1\}\}$.

Kelly noted that if $\sigma > 1$ and $p(s)\alpha'_s < 1$ then no bets are placed, but if $\max\{p(s)\alpha'_s\} > 1$ some bets might be made for which $p(s)\alpha'_s < 1$ (i.e. the odds are worse than the probabilities and hence the return is less than expected). This contrasts with the classic gambler intuition which is to never bet on such an event.

Summary

1. If $\sigma > 1$ ($\sum_s \frac{1}{\alpha_s} > \frac{1}{1+r}$) and $\max\{p(s)\alpha'_s\} \leq 1$ ($\max\{p(s)\alpha_s\} \leq (1+r)$) then

$$G_{\max} = \log(1+r).$$

   and $b = 1$, $a(s) = 0$ for all s.

2. If $\sigma > 1$ ($\sum_s \frac{1}{\alpha_s} > \frac{1}{1+r}$) and $\max\{p(s)\alpha'_s\} > 1$ ($\max\{p(s)\alpha_s\} > 1 + r$) then

$$G_{\max} = \sum_{s\in\Gamma} p(s)\log(p(s)\alpha'_s) + \sum_{s\in\Gamma'} p(s)\log(b) + \log(1+r)$$

   with $a(s) = p(s) - \frac{b}{\alpha'_s} = p(s) - \frac{b(1+r)}{\alpha_s}$ and $b = \frac{1-p_{t^*}}{1-\sigma_{t^*}}$ and $\Gamma = \{t : 0 < t \leq t^*\}$

8 σ and $\max\{p(s)\alpha'_s\}$ Combinations that Don't Exist

It is important to remark that there exists a few combinations of σ and $\max\{p(s)\alpha'_s\}$ which don't exist.

8.1 $\sigma < 1$ and $\max\{p(s)\alpha'_s\} \leq 1$

Let's consider the situation where a gambler not only has no track take but money is injected into the pool (i.e. $\sum_s \frac{1}{\alpha'_s} < 1$). This scenario will never occur if $\max\{p(s)\alpha'_s\} \leq 1$. In fact,

$$\sum_s \frac{1}{\alpha'_s} < 1 \quad \text{implies} \quad \max\{p(s)\alpha'_s\} > 1.$$

Proof. Assume $\max\{p(s)\alpha'_s\} \leq 1$ then for all s

$$p(s)\alpha'_s \leq 1 \implies p(s) \leq \frac{1}{\alpha'_s}.$$

Summing over all s,

$$1 \leq \sum_s \frac{1}{\alpha'_s},$$

but this is a contradiction for $\sum_s \frac{1}{\alpha'_s} < 1$ and hence $\max\{p(s)\alpha'_s\} > 1$. □

8.2 $\max\{p(s)\alpha'_s\} < 1$ and $\sigma = 1$.

A similar conclusion is reached for a game where $\max\{p(s)\alpha'_s\} < 1$ and $\sum_s \frac{1}{\alpha'_s} = 1$. In this scenario, the winning gamblers receive all of the money in the pool. Intuitively, the statement is untrue because if one $p(s)\alpha'_s < 1$, then another $p(s)\alpha'_s > 1$ to counter it in the summation of the reciprocals of α's. In general,

$$\sum_s \frac{1}{\alpha'_s} = 1 \quad \text{implies} \max\{p(s)\alpha'_s\} \geq 1.$$

Proof. Assume $\max\{p(s)\alpha'_s\} < 1$. Then, for all s

$$p(s)\alpha'_s < 1 \quad \implies p(s) < \frac{1}{\alpha'_s}.$$

Summing over all s,

$$1 < \sum_s \frac{1}{\alpha'_s},$$

but we reach a contradiction as $\sum_s \frac{1}{\alpha'_s} = 1$. Hence, $\max\{p(s)\alpha'_s\} \geq 1$. □

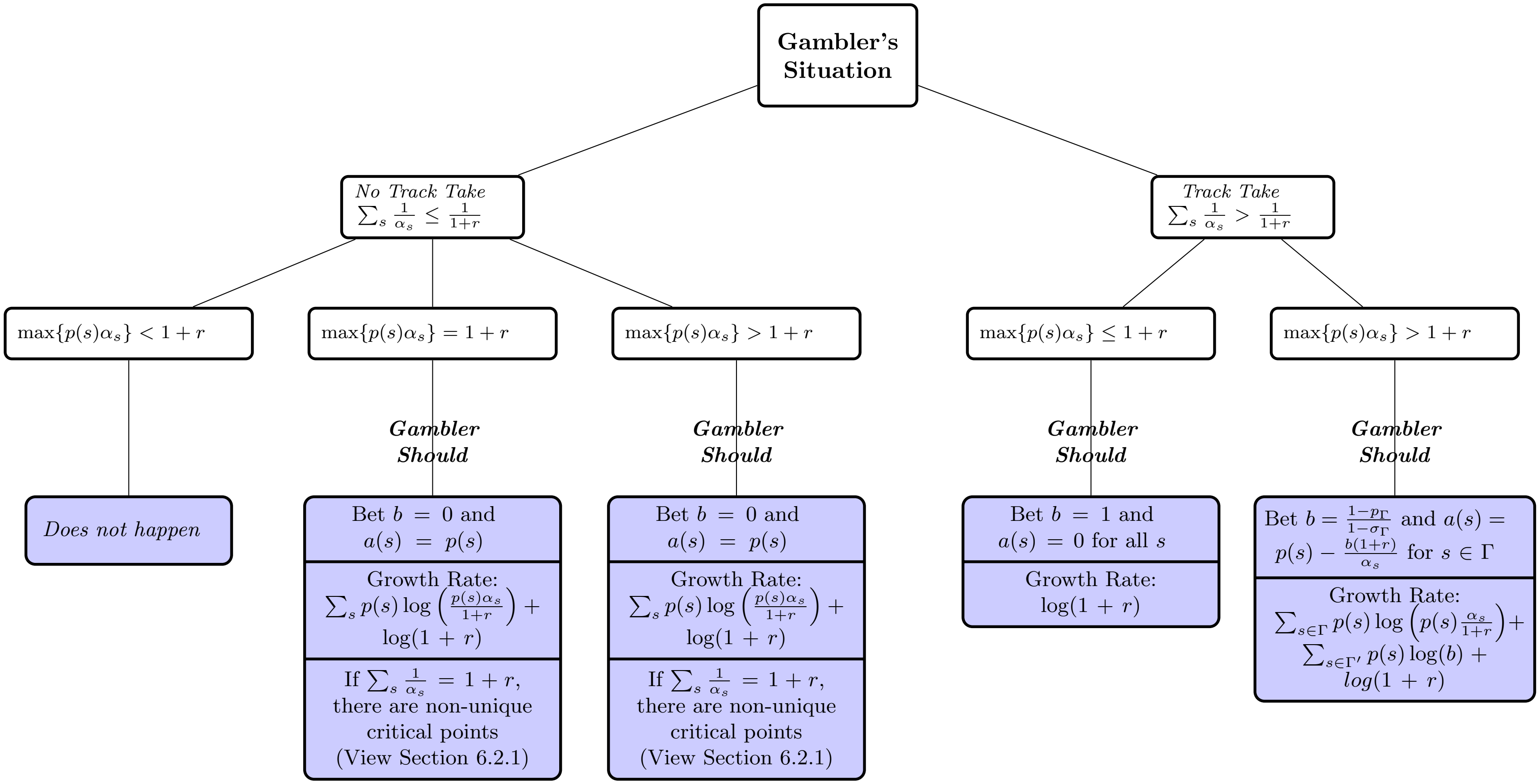
9 Conclusion

When gamblers enters a bet, they have access to only three pieces of information:

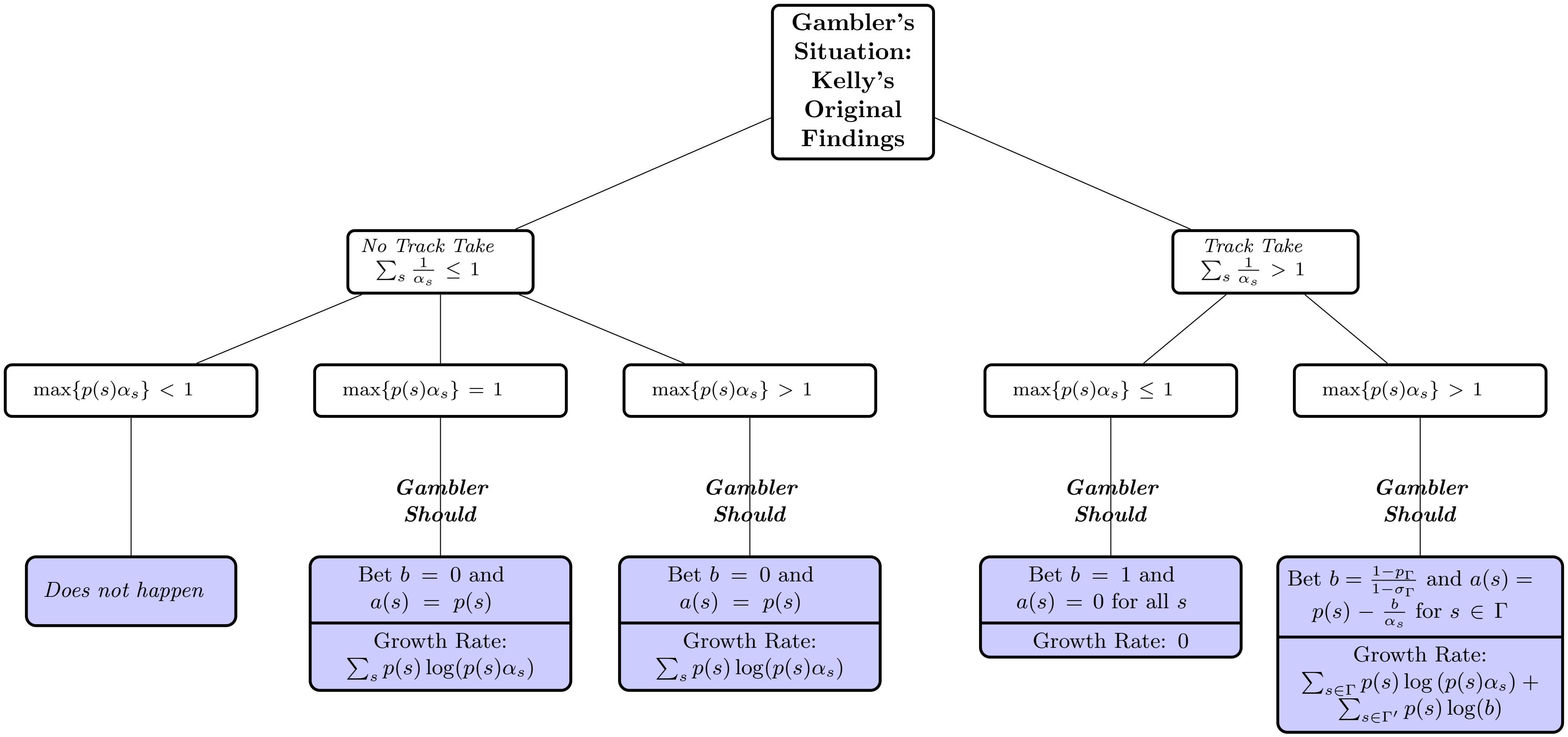
1. The $p(s)\alpha_s$ for all s
2. $\sum_s \frac{1}{\alpha_s}$
3. The rate earned on money not bet, r

Kelly uncovered under the permutations of the above information the fixed fraction to bet, $a(s)$, and b, the fraction not bet to optimize the gambler's growth rate. If the odds are consistent with the probabilities, the gambler can expect a growth rate equal to $\log(1+r)$. Growth rates larger than the "risk free rate" only occur if the gambler has *insider information* (i.e. knows that the true probabilities differ from the odds). Moreover when gamblers experience "track takes" ($\sigma > 1$) and at least one of the probabilities is inconsistent with the odds, the gambler will gain a growth rate greater than the "risk free rate", r. Interestingly as pointed out by Kelly, gamblers will make bets when the odds (i.e. gambler's returns) are worse then the probability ($p(s) < \frac{1}{\alpha_s}$). This is counterintuitive, *why would a gambler bet on something that is expecting a negative return?* Kelly was baffled by this result.

Kelly's original paper does not include the rate earned on money not bet. His research was on information theory and the transfer of signals across channels. Hence, earning an additional rate on the money not bet was not considered in his analysis. We reach the same conclusions as he does by setting $r = 0$. Moreover, Kelly ignored the possibility of multiple critical value points. As we showed in Section 6.2.1, there are non-unique critical points when $\sum_s \frac{1}{\alpha_s} = \frac{1}{1+r}$. The chart below summarizes our findings:


Gambler's Situation
No Track Take $\sum_s \frac{1}{\alpha_s} \leq \frac{1}{1+r}$
Track Take $\sum_s \frac{1}{\alpha_s} > \frac{1}{1+r}$
$\max\{p(s)\alpha_s\} < 1+r$
$\max\{p(s)\alpha_s\} = 1+r$
$\max\{p(s)\alpha_s\} > 1+r$
$\max\{p(s)\alpha_s\} \leq 1+r$
$\max\{p(s)\alpha_s\} > 1+r$
Does not happen
Gambler Should
Bet $b = 0$ and $a(s) = p(s)$
Growth Rate: $\sum_s p(s) \log\left(\frac{p(s)\alpha_s}{1+r}\right) + \log(1+r)$
If $\sum_s \frac{1}{\alpha_s} = 1+r$, there are non-unique critical points (View Section 6.2.1)
Gambler Should
Bet $b = 0$ and $a(s) = p(s)$
Growth Rate: $\sum_s p(s) \log\left(\frac{p(s)\alpha_s}{1+r}\right) + \log(1+r)$
If $\sum_s \frac{1}{\alpha_s} = 1+r$, there are non-unique critical points (View Section 6.2.1)
Gambler Should
Bet $b = 1$ and $a(s) = 0$ for all s
Growth Rate: $\log(1+r)$
Gambler Should
Bet $b = \frac{1-p_\Gamma}{1-\sigma_\Gamma}$ and $a(s) = p(s) - \frac{b(1+r)}{\alpha_s}$ for $s \in \Gamma$
Growth Rate: $\sum_{s\in\Gamma} p(s) \log\left(p(s)\frac{\alpha_s}{1+r}\right) + \sum_{s\in\Gamma'} p(s)\log(b) + log(1+r)$

As stated above, Kelly solved the optimization problem under the assumption that $r = 0$. To conclude, the following exhibits Kelly's original findings.



10 Propositions and Theorems

Proposition 1.

$$G_{max} = \log(2) + p\log(p) + q\log(q)$$

Proof. From the statement, the optimal value for $G(f)$ occurs when $f = p - q$. Substituting into G yield:

$$G_{max} = p\log(1 + p - q) + q\log(1 - p + q)$$

Since $p + q = 1$,

$$G_{max} = p\log(2p) + q\log(2q),$$

which can be reduced further to

$$(p + q)\log(2) + p\log(p) + q\log(q).$$

Hence,

$$G_{max} = \log(2) + p\log(p) + q\log(q).$$

□

Proposition 2. *If* $0 < b < 1$, *then*

$$\lambda = 1 \quad \textit{and} \quad a(s) = p(s) - \frac{b}{\alpha'_s}.$$

Proof. From (15),

$$p(s) = \frac{\lambda(b + \alpha'_s a(s))}{\alpha'_s} \qquad \text{for } s \in \Gamma \tag{23}$$

and substituting into (18)

$$\sum_{s\in\Gamma'} \frac{p(s)}{b} + \sum_{s\in\Gamma} \frac{\lambda}{\alpha'_s} = \lambda.$$

Recall that, $\sum_s p(s) = 1$ and hence

$$\sum_{s\in\Gamma} p(s) = 1 - \sum_{s\in\Gamma'} p(s).$$

Replacing into the above equation and using (23) yields

$$\lambda - \frac{1}{b} = -\sum_{s\in\Gamma} \frac{\lambda}{\alpha'_s} - \sum_{s\in\Gamma} \frac{\lambda a(s)}{b} + \lambda \sum_{s\in\Gamma} \frac{1}{\alpha'_s}$$

Note that from (1), rearranging produces $\frac{1}{b} - 1 = \sum_{s\in\Gamma} \frac{a(s)}{b}$ which substituted in results

$$\lambda - \frac{1}{b} = \lambda\left(1 - \frac{1}{b}\right),$$

so $\lambda = 1$.

Due to $\lambda = 1$ and (15),

$$a(s) = p(s) - \frac{b}{\alpha'_s} \qquad \text{for } a(s) > 0.$$

□

Proposition 3.

$$\sigma_\Gamma = 1 \textit{ if and only if } \sigma = 1$$

$$\sigma_\Gamma < 1 \textit{ if and only if } \sigma > 1 \textit{ or } \sigma < 1$$

Proof. By (19),

$$\frac{1-p}{b} = 1 - \sigma_\Gamma.$$

If $\sigma_\Gamma = 1$ and $b > 0$ then

$$1 - p = 0$$

Hence, $p = 1$. As a result because $\sum_s p(s) = 1$, $p = 1$ implies that at critical points we bet on all s. Thus, $\sigma_\Gamma = \sigma$. This will only be true if and only if $\sigma = 1$ to start with. For any other σ value, $\sigma_\Gamma \neq 1$. $\square$

Theorem 1. *Let $T = \{t : \sigma_t < 1\}$ and assume $t + 1 \in T$ and $\sigma_1 < 1$. Then $F_1 > 0$ and for $t \geq 1$*

$$p(t+1)\alpha_{t+1} = F_t \iff F_{t+1} = F_t \tag{24}$$

$$p(t+1)\alpha_{t+1} > F_t \iff F_{t+1} < F_t \tag{25}$$

$$p(t+1)\alpha_{t+1} < F_t \iff F_{t+1} > F_t \tag{26}$$

Proof. The assumption that $\sigma_1 < 1$ directly results from the basics of odds. By definition,

$$\sigma_1 = \frac{1}{\alpha'_1}$$

and $\sigma_1 > 1$ only if $\alpha'_1 < 1$. In terms of odds, $\alpha'_1 < 1$ indicates that the gamblers receives less money than they bet. This scenario is improbable; odds will never be less than 1. Hence, the fact that $F_t > 0$ under the stated assumptions is clear and probable.

I will prove (25) and the two other proofs for (26) and (24) are similar. Since $0 < \sigma_t < \sigma_{t+1} < 1$, then $1 - \sigma_t > 0$ and $1 - \sigma_{t+1} < 0$. Also $1 - p_t > 0$ since $t < n$. Thus, $F_{t+1} < F_t$ is equivalent to the following equalities

$$\frac{1 - p_{t+1}}{1 - \sigma_{t+1}} = \frac{1 - p_t - p(t+1)}{1 - \sigma_t - \frac{1}{\alpha'_{t+1}}} < \frac{1 - p_t}{1 - \sigma_t}$$

$$1 - \sigma_t - p_t + p_t\sigma_t - p(t+1) + p(t+1)\sigma_t < 1 - \sigma_t - \frac{1}{\alpha'_{t+1}} + p_t\sigma_t + p_t\frac{1}{\alpha'_{t+1}} - p_t$$

Cancelling out $1, \sigma_t, p_t\sigma_t, p_t$, this becomes

$$-p(t+1) + p(t+1)\sigma_t < \frac{-1}{\alpha'_{t+1}} + \frac{p_t}{\alpha'_{t+1}}$$

$$-p(t+1)(1 - \sigma_t) < \frac{-1}{\alpha'_{t+1}}(1 - p_t))$$

Hence because $(1 - \sigma_t) > 1$,

$$p(t+1)\alpha_{t+1} > \frac{1 - p_t}{1 - \sigma_t} = F_t$$

$\square$

Theorem 2. *There exists a unique index t^* such that*

$$p(s)a(s) > b \qquad \textit{for } s \leq t^* \tag{27}$$

$$p(s)a(s) \leq b \qquad \textit{for } s > t^* \tag{28}$$

where $b = F_{t^*}$. This t^* represents the critical index that determines Γ and Γ', the sets you bet on and don't bet on respectively. Hence, $\Gamma = \{t : 1 \leq t \leq t^*\}$ and $\Gamma' = \{t : t > t^*\}$ where the sets maximize G.

Use the following procedure to find t^*:

$$t^* = \min\{\quad \min(t : p(t+1)\alpha'_{(t+1)} < F_t), \quad \max(t : \sigma_t < 1) \quad \}$$

In words, start with $t = 0$ and test the first condition. Increase t until $p(t+1)\alpha'_{(t+1)} < F_t$. Then repeat for σ_t until $\sigma_t \geq 1$. Compare the two t values and choose the minimum to be t^*.

Assume $\sigma_t < 1$ and

1. *If $p(1)\alpha'_1 \leq 1$ then $t^* = 0$.*
2. *If $p(1)\alpha'_1 > 1$ then $t^* \geq 1$ where t^* is chosen by the method above.*

Proof. From Proposition (2),

$$a(s) = p(s) - \frac{b}{\alpha'_s} \qquad \text{for } s \in \Gamma.$$

Substituting into G,

$$\sum_{s\in\Gamma} p(s) \log\left(b + \alpha'_s\left(p(s) - \frac{b}{\alpha'_s}\right)\right) + \sum_{s\in\Gamma'} p(s)\log(b)$$

$$\sum_{s\in\Gamma} p(s) \log\left(\alpha'_s p(s)\right) + \sum_{s\in\Gamma'} p(s)\log(b).$$

Hence, we want to find all s such that

$$p(s)\alpha'_s > b$$

for these s will maximize G. By reordering the s values as described above,

1. If $p(1)\alpha'_1 < 1 = F_0$, then by Theorem (1) Equation (26), $F_1 > F_0$ so $p(1)\alpha'_1 < F_1$. Moreover $p(2)\alpha'_2 < p(1)\alpha'_1$; thus by the same logic $F_0 < F_1 < F_2$ and again $p(2)\alpha'_2 < F_2$. By induction, we can continue for all t and conclude

   $$p(t)\alpha'_t < F_t.$$

   Because at maximum $F_t = b$ and assuming $\sigma_t < 1$, none of these t values are candidates for no t value satisfies the condition that

   $$p(s)\alpha'_s > b$$

   Hence, $t = 0$ which implies $b = 1$ by (2). However, this is a contradiction for we assumed that $0 < b < 1$ and therefore this does not occur.

2. If $p(1)\alpha'_1 > 1$, then $F_1 < F_0$. This satisfies the conditions at optimal (i.e. $p(s)\alpha_s > b$). Therefore, we bet on at least on s, $s = 1$. From Theorem 1 if $p(t+1)\alpha'_{t+1} > F_t$ then $F_{t+1} < F_t$. Lets assume $t = 2$ satisfies this condition then $F_2 < F_1 < F_0$. Because $p(2)\alpha'_2 > F_1$ then $p(2)\alpha'_2 > F_2$ and $p(1)\alpha_1 > F_2$. As a result, this satisfied the conditions at optimality. By induction, we can see that as long as $p(t+1)\alpha'_{t+1} > F_t$ then for all t less than or equal to $t+1$ will satisfy the optimality condition $p(s)\alpha'_s > F_{t+1}$. Thus, we should bet on these outcomes.

   If $p(t+1)\alpha'_{t+1} \leq F_t$ then by Theorem 1 $F_t \leq F_{t+1}$. Once this occurs, $F_{t+1} \geq p(t+1)\alpha'_{t+1}$. However at optimality, we need $F_{t+1} < p(t+1)\alpha'_{t+1}$ and therefore we do not include $t+1$ as part of Γ.

   With these two statements, it allows us to find t^*. We note from Proposition 3 that at optimality $\sigma_\Gamma < 1$. Hence t^* will be computed by

   $$\min\{\min(t : p(t+1)\alpha'_{(t+1)} < F_t, \max(t : \sigma_t < 1).$$

   As shown above, computing t^* with this method guarantees that the conditions at optimality hold.

□

References

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