DISCRETE UNSOLVABILITY FOR THE INVERSE PROBLEM FOR ELECTRICAL NETWORKS

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ABSTRACT. For $n > 1$, we construct graphs X_n such that certain response matrices for X_n correspond to precisely n distinct conductivities on X_n, and we give a general algorithm for obtaining such response matrices from arbitrary given response matrices.

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1. Basic Definitions and Conventions

By a *graph with boundary* we mean an undirected graph X with vertex set V where

(1) X is finite and has no loops
(2) disjoint subsets ∂X and X^o of V are given, where ∂X is nonempty and $V = \partial X \cup X^o$
(3) each connected component of X contains an element of ∂X
(4) an identification of V with the first $|V|$ positive integers is given, such that all elements of ∂X precede all elements of X^o.

Elements of ∂X are called *boundary nodes* (of X), and elements of X^o are called *interior nodes*. When drawing graphs, boundary nodes will be represented by black dots, and interior nodes will be represented by black circles. As all graphs considered in this paper will be graphs with boundary, we will henceforth use the terms graph and graph with boundary interchangeably.

In general, the notation $E(X)$ denotes the edge set of X, and $V(X)$ denotes the vertex set of X. We say that two nodes in X are *adjacent* if there is an edge in X between them (in particular, a node is not adjacent to itself). We say that two distinct edges in X are *parallel* if they have the same endpoints.

When we draw graphs, we may or may not draw the same node more than once. If a given node is drawn more than once, all instances of that node will be labeled with the same number. Thus, the graphs shown in Figures 1 and 2 are in fact the same.

A *conductivity* on a graph is a positive function on the edges of that graph. If X is a graph and γ is a conductivity on X, the pair (X, γ) is called an *electrical network*.

Let (X, γ) be an electrical network. If i and j are distinct nodes in X, we define

$$\gamma_{i,j} = \sum_{\substack{\text{edges } e \\ \text{joining } i \text{ and } j}} \gamma(e), \tag{1}$$

where the empty sum is defined to be 0. If n is the number of nodes in X, we define the *Kirchhoff matrix* of the network (X, γ) to be the $n \times n$ matrix K given by

$$K_{i,j} = \begin{cases} \gamma_{i,j} & i \neq j \\ -\sum_{k \neq i} \gamma_{i,k} & i = j \end{cases}. \tag{2}$$

When multiple networks are under consideration, we will typically add the conductivity or the network as a subscript to K. Thus, $K = K_\gamma = K_{(X,\gamma)}$. If X has m boundary nodes, then K has the following useful block structure

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \tag{3}$$

where A is $m \times m$ and C is $(n-m) \times (n-m)$. We prove in Lemma 1.1 that C is invertible. We define the *response matrix*[1] of (X, γ) to be the $m \times m$ matrix

$$\Lambda = A - BC^{-1}B^T = K/C, \tag{4}$$

that is, Λ is the Schur complement of C in K. Subscripting conventions for Λ are the same as those for K.

Lemma 1.1. *If the Kirchhoff matrix K of a network (X, γ) is decomposed as in (3), then the submatrix C is invertible.*

Proof. Let n be the number of nodes in X, and m the number of boundary nodes, so that K is $n \times n$ and C is $(n-m) \times (n-m)$.

Suppose $y \in \mathbb{R}^{n-m}$ and $y^T C y = 0$. To show that C is invertible, it suffices to show that $y = 0$. Define $x \in \mathbb{R}^n$ by

$$x_i = \begin{cases} 0 & i \leq m \\ y_{i-m} & i > m \end{cases}. \tag{5}$$

Thus, by the definition of x, the hypothesis $y^T C y = 0$, and (3), we have

$$x^T K x = 0^T A 0 + y^T C y = 0. \tag{6}$$

We claim that x is constant on each (connected) component of X. We have

$$\begin{aligned} 0 &= \sum_{i,j} K_{i,j} x_i x_j && \text{by (6)} \\ &= \sum_{i \neq j} \gamma_{i,j} x_i x_j + \sum_i \sum_{j \neq i} (-\gamma_{i,j}) x_i x_i && \text{by (2)} \\ &= \sum_{i \neq j} \gamma_{i,j} (x_i x_j - x_i x_i) \\ &= \sum_{i < j} \gamma_{i,j} (2 x_i x_j - x_i x_i - x_j x_j) && \text{as } \gamma_{i,j} = \gamma_{j,i} \\ &= -\sum_{i < j} \gamma_{i,j} (x_i - x_j)^2. \end{aligned}$$

As $\gamma_{i,j} \geq 0$ by (1), it follows from the above calculation that $x_i = x_j$ if $\gamma_{i,j} > 0$, i.e., if there is an edge joining nodes i and j in X. Thus, x is constant on each component of X. Since each component of X contains a boundary node (by the definition of graph) and x is zero on ∂X (i.e., $x_i = 0$ for $i \leq m$) by the definition of x, it follows that $x = 0$, and thus that $y = 0$. □

If X is a graph, we will say that a matrix M is a Kirchhoff matrix for X if there is a conductivity γ on X with $K_{(X,\gamma)} = M$. If the graph X is understood and γ is a conductivity on X, we will call $K_{(X,\gamma)}$ the Kirchhoff matrix of γ. The same conventions apply for response matrices. The *inverse problem* is then as follows: given a graph X and a matrix L, find all conductivities on X with response matrix L.

It will be convenient to have some terminology regarding how 'well-behaved' a graph is with respect to the inverse problem. We say that a graph X is *recoverable* if *any* matrix L is the response matrix of at most one conductivity on X (so if we know that L is a response matrix for X, we can at least theoretically 'recover' *the*

[1]The term response matrix comes from the following 'physical' characterization of $\Lambda_{(X,\gamma)}$: if $\phi \in \mathbb{R}^n$, we can consider applying a potential to the boundary nodes of X (where the edges of X have conductances given by γ) whose value at a boundary node i is ϕ_i; in this situation, for each i, the ith component of the vector $\Lambda_{(X,\gamma)}\phi$ is the (signed) current out of the boundary node i due to the applied voltage ϕ. See [1] for details.

conductivity on X with response matrix L from the information contained in L). If $n > 1$ is an integer, we say that a graph X is *n-to-1* if *some* matrix is the response matrix of precisely n distinct conductivities on X. Finally, we say that a graph X is *∞-to-1* (read 'infinite to one') if *any* response matrix for X is the response matrix of infinitely many distinct conductivities on X.

Note that nothing we have said thus far precludes the possibility of a particular graph being n-to-1 for more than one value of n, and nothing guarantees that a given graph will be either recoverable, n-to-1 for some n, or ∞-to-1. However, it is immediate from the definitions that a graph cannot be more than one of recoverable, n-to-1 for some n, and ∞-to-1. Also, it is worth remarking (though it may not be obvious from the point of view we have taken, and it is not necessary for the purposes of this paper) that the properties 'recoverable', 'n-to-1', and '∞-to-1' are independent of node-integer identification, in the sense that if X and X' are two graphs which differ only in the identification of their vertices with positive integers, then X is recoverable (resp. n-to-1, ∞-to-1) iff X' is recoverable (resp. n-to-1, ∞-to-1).

2. Historical Motivation

In this section, we sketch how the problem of finding n-to-1 graphs arose. When first considering the inverse problem, Morrow, Curtis, and Ingerman were interested in finding recoverable graphs. Work on this problem is documented at `http://math.washington.edu/~reu/`. A notable result in this direction is that a circular planar graph is recoverable iff it is critical. (Definitions of circular planar and critical (for circular planar graphs) can be found in [1].)

The search for recoverable graphs led naturally to consideration of non-recoverable graphs and ways in which a graph can fail to be recoverable. Certain graphs are obviously not recoverable: perhaps the simplest examples are the so-called series and parallel connections in Figure 3. Simple algebra and the definition of response matrix shows that these graphs are ∞-to-1. Somewhat relatedly, the aforementioned recoverability result was strengthened by Jeff Giansiracusa, who showed that a non-critical circular planar graph is ∞-to-1. Prior to this, Ernie Esser, when considering recoverability of so-called 'annular networks', discovered (by purely symbolic methods) a (rather simple) 2-to-1 graph, shown in Figure 1 (and also in Figure 2).

The existence of this single 2-to-1 graph (and some closely related 2^n-to-1 graphs) together with the fact that a large class of graphs (i.e., circular planar graphs) could not be n-to-1 led to the question of whether or not n-to-1 graphs existed for each n. (For a while, it was in fact conjectured that 2^n-to-1 graphs were the only possibilities.) The first significant progress on this problem was made by Ilya Grigoriev, who succeeded in constructing (albeit largely without proof) a 3-to-1 graph. This paper describes a modification and generalization of his approach, which works for arbitrary n.

3. Preliminary Notions

Denote by $\underline{\underline{G}}$ the space of graphs with the following properties:

- no two interior nodes are adjacent
- every interior node is adjacent to at least three boundary nodes
- no more than one edge joins a given interior node and boundary node.

We define $\underline{\underline{\Gamma}}$ to be the space of electrical networks whose underlying graph is in $\underline{\underline{G}}$.

3.1. **Operation on Graphs.** Throughout this subsection, X will be a fixed but arbitrary graph in $\underline{\underline{G}}$, n will be the number of nodes in X, and m the number of boundary nodes. For $p \in X^o$, we denote by A_pX the set of (boundary) nodes in X adjacent to p.

We define a graph SX as follows. The nodes of SX are the boundary nodes $1, \ldots, m$ of X (identified with positive integers in the obvious way), and each node in SX is a boundary node in SX. If i and j are distinct nodes in SX, then the edges between i and j in SX are those in X together with an additional edge, denoted e^p_{ij}, for each interior node p of X which is adjacent to both i and j. (We may also call this edge e^p_{ji}, $e^p_{j,i}$, or $e^p_{i,j}$.) It is clear that SX meets our requirements for a graph (and moreover that $SX \in \underline{\underline{G}}$).

We call SX the *star-K transformation of* X. We will denote SX by X^* when it is convenient to do so. Note that as X^* has no interior nodes, we have

$$K_{(X^*,\gamma^*)} = \Lambda_{(X^*,\gamma^*)} \tag{7}$$

for any conductivity γ^* on X^*.

We say that a conductivity γ^* on X^* *satisfies the quadrilateral rule*[2] if whenever $p \in X^o$ and i, j, k, l are distinct nodes in A_pX we have

$$\gamma^*(e^p_{ij})\gamma^*(e^p_{kl}) = \gamma^*(e^p_{ik})\gamma^*(e^p_{jl}). \tag{8}$$

This is equivalent to what we call the *triangle condition*: if i is a given element in A_pX, then the quantity

$$\frac{\gamma^*(e^p_{ij})\gamma^*(e^p_{ik})}{\gamma^*(e^p_{jk})} \tag{9}$$

is the same for any choice of j and k in A_pX which are distinct from each other and from i.

We say that a Kirchhoff matrix K for X^* *satisfies the quadrilateral rule* if whenever $p \in X^o$ and i, j, k, l are distinct nodes in A_pX with a unique edge joining i to j, k to l, i to k, and j to l in X^*, we have

$$K_{i,j}K_{k,l} = K_{i,k}K_{j,l}. \tag{10}$$

Note that if γ^* is a conductivity on X^* which satisfies the quadrilateral rule, then K_{γ^*} satisfies the quadrilateral rule. By the preceding sentence and Corollary 3.3,

(11)
any response matrix for X satisfies the quadrilateral rule as a Kirchhoff matrix for X^*.

In particular, Corollary 3.3 implies that any response matrix for X is a Kirchhoff matrix for X^*.

As an example of the star-K transformation, consider the graph Θ in Figure 2. We have computed Θ^* in Figure 4. Edges produced by this process are also labeled according to the above definition. As for the quadrilateral rule, we have the following:

- a conductivity γ^* on Θ^* satisfies the quadrilateral rule iff it satisfies

$$\gamma^*(e^6_{0,1})\gamma^*(e^6_{2,3}) = \gamma^*(e^6_{0,2})\gamma^*(e^6_{1,3}) = \gamma^*(e^6_{1,2})\gamma^*(e^6_{0,3}), \tag{12}$$

$$\gamma^*(e^7_{2,3})\gamma^*(e^7_{4,5}) = \gamma^*(e^7_{2,4})\gamma^*(e^7_{3,5}) = \gamma^*(e^7_{3,4})\gamma^*(e^7_{2,5}), \tag{13}$$

and

$$\gamma^*(e^8_{0,1})\gamma^*(e^8_{4,5}) = \gamma^*(e^8_{0,4})\gamma^*(e^8_{1,5}) = \gamma^*(e^8_{1,4})\gamma^*(e^8_{0,5}); \tag{14}$$

- a Kirchhoff matrix K for Θ^* satisfies the quadrilateral rule iff it satisfies

$$K_{0,2}K_{1,3} = K_{1,2}K_{0,3}, \tag{15}$$

$$K_{2,4}K_{3,5} = K_{3,4}K_{2,5}, \tag{16}$$

and

$$K_{0,4}K_{1,5} = K_{1,4}K_{0,5}. \tag{17}$$

3.2. **Operation on Networks.** We now extend S to an operation from $\underline{\underline{\Gamma}}$ to itself. Given $(X,\gamma) \in \underline{\underline{\Gamma}}$, we define a conductivity $S\gamma$ on SX as follows. For $p \in X^o$, let

$$\sigma_p = \sum_{i \neq p} \gamma_{i,p}. \tag{18}$$

If e is an edge in both SX and X, then we set

$$S\gamma(e) = \gamma(e). \tag{19}$$

If e is an edge in SX but not in X, then e is e_{ij}^p for some $p \in X^o$ and distinct i, j in A_pX, and we set

$$S\gamma(e_{ij}^p) = \frac{\gamma_{i,p}\gamma_{j,p}}{\sigma_p}. \tag{20}$$

Thus, we have $(SX, S\gamma) \in \underline{\underline{\Gamma}}$, and we set $S(X,\gamma) = (SX, S\gamma)$.

Note that $S\gamma$ satisfies the quadrilateral rule: if p is an interior node of X and i, j, k, l are distinct nodes in A_pX, we have

$$S\gamma(e_{ij}^p)S\gamma(e_{kl}^p) = \frac{\gamma_{i,p}\gamma_{j,p}}{\sigma_p} \cdot \frac{\gamma_{k,p}\gamma_{l,p}}{\sigma_p} = \frac{\gamma_{i,p}\gamma_{k,p}}{\sigma_p} \cdot \frac{\gamma_{j,p}\gamma_{l,p}}{\sigma_p} = S\gamma(e_{ik}^p)S\gamma(e_{jl}^p). \tag{21}$$

3.3. **Some Basic Results.** We now compile some useful properties of the map S.

Lemma 3.1. *If* $(X,\gamma) \in \underline{\underline{\Gamma}}$ *then* $\Lambda_{(X,\gamma)} = \Lambda_{S(X,\gamma)} = K_{S(X,\gamma)}$.

Proof. The assertion $\Lambda_{S(X,\gamma)} = K_{S(X,\gamma)}$ follows immediately from (7), so it is enough to show that $\Lambda_{(X,\gamma)} = K_{S(X,\gamma)}$.

Let n be the number of nodes in X, m the number of boundary nodes, $K = K_{(X,\gamma)}$, and A, B, C as in (3) for K. By the definition of S, for any distinct $i, j \in SX$ we have

$$(K_{S(X,\gamma)})_{i,j} = S\gamma_{i,j} = \gamma_{i,j} + \sum_{p \in X^o} \frac{\gamma_{i,p}\gamma_{j,p}}{\sigma_p} = K_{i,j} + \sum_{p \in X^o} \frac{K_{i,p}K_{j,p}}{\sigma_p}, \tag{22}$$

where the quantity $\frac{\gamma_{i,p}\gamma_{j,p}}{\sigma_p} = \frac{K_{i,p}K_{j,p}}{\sigma_p}$ may very well be zero for some (or all) p.

As $X \in \underline{\underline{G}}$, the submatrix C of K is diagonal, with diagonal values given explicitly by

$$C_{k,k} = K_{m+k,m+k} = -\sigma_{m+k} \text{ for } 1 \leq k \leq n-m, \tag{23}$$

[2] Note that such notions as conductivities and Kirchhoff matrices satisfying the quadrilateral rule on X^* are only well-defined with respect to a fixed choice of 'base' graph X; if X and Y are different graphs with the same star-K transformation, then what it means for, e.g., a conductivity on X^* to satisfy the quadrilateral rule is not in general the same as what it means for a conductivity on Y^* to satisfy the quadrilateral rule, even though X^* and Y^* are the same graph. In practice, it will always be clear from the context (and usually just from the notation) what 'base' graph we have in mind.

so that C^{-1} is diagonal, with values

$$(C^{-1})_{k,k} = (C_{k,k})^{-1} = -\frac{1}{\sigma_{m+k}}. \tag{24}$$

Thus, by the definition of $\Lambda_{(X,\gamma)}$ in (4), for any $i, j \in \partial X$ (not necessarily distinct) we have

$$\begin{align}(\Lambda_{(X,\gamma)})_{i,j} &= (A - BC^{-1}B^T)_{i,j} \tag{25}\\ &= A_{i,j} - \sum_{k=1}^{n-m} B_{i,k}(C^{-1}B^T)_{k,j} \tag{26}\\ &= K_{i,j} + \sum_{k=1}^{n-m} B_{i,k}\frac{1}{\sigma_{m+k}}B^T_{k,j} \tag{27}\\ &= K_{i,j} + \sum_{p\in X^o} \frac{K_{i,p}K_{j,p}}{\sigma_p}. \tag{28}\end{align}$$

Thus, we have $(\Lambda_{(X,\gamma)})_{i,j} = (K_{S(X,\gamma)})_{i,j}$ for $i \neq j$. By the definition of Kirchhoff matrix, $K_{S(X,\gamma)}$ has row sum zero, so to complete the proof we need only show that $\Lambda_{(X,\gamma)}$ has row sum zero. Observe that for any $p \in X^o$, we have

$$\sigma_p = -\sum_{j\in X^o} K_{j,p}, \tag{29}$$

by (18), (2), and the definition of $\underline{\underline{G}}$. Thus, given $i \in \partial X$, we have

$$\begin{align*}\sum_{j\in\partial X}(\Lambda_{(X,\gamma)})_{i,j} &= \sum_{j\in\partial X} K_{i,j} + \sum_{p\in X^o}\frac{K_{i,p}\sum_{j\in\partial X}K_{j,p}}{\sigma_p} && \text{by (28)}\\ &= -\sum_{j\in X^o} K_{i,j} + \sum_{p\in X^o}\frac{K_{i,p}\left(-\sum_{j\in X^o}K_{j,p}\right)}{\sigma_p} && \text{as } K \text{ has row sum zero}\\ &= 0 && \text{by (29)}\end{align*}$$

thus completing the proof. ☐

Lemma 3.2. *If $X \in \underline{\underline{G}}$, then S defines a bijection from conductivities on X to conductivities on SX satisfying the quadrilateral rule.*

Proof. First, we introduce some notation. Let

$$\Gamma_X = \{\text{conductivities on } X\}, \tag{30}$$

and

$$\Gamma'_{SX} = \{\text{conductivities on } SX \text{ satisfying the quadrilateral rule}\}. \tag{31}$$

The claim is that $S : \Gamma_X \to \Gamma'_{SX}$ is a bijection.

If $p \in X^o$ and $i \in A_pX$, we will denote by e^p_i the (unique) edge in X between i and p. If $\delta \in \Gamma'_{SX}$, $p \in X^o$, and $i \in A_pX$ are given, then by the definition of $\underline{\underline{G}}$ there exist nodes $j, k \in A_pX$ which are distinct from each other and from i, and by (9) the quantity

$$\delta^p_i = \sqrt{\frac{\delta(e^p_{ij})\delta(e^p_{ik})}{\delta(e^p_{jk})}}, \tag{32}$$

depends only on δ, p, and i. Observe that for distinct i and j in A_pX, we have

$$\delta_i^p \delta_j^p = \sqrt{\frac{\delta(e_{ik}^p)\delta(e_{ij}^p)}{\delta(e_{kj}^p)}}\sqrt{\frac{\delta(e_{jk}^p)\delta(e_{ji}^p)}{\delta(e_{ki}^p)}} = \delta(e_{ij}^p), \tag{33}$$

where $k \in A_pX$ is distinct from both i and j but is otherwise arbitrary.

Now, we define a map $T : \Gamma'_{SX} \to \Gamma_X$. Let $\delta \in \Gamma'_{SX}$ be given, and define a conductivity $T\delta$ on X as follows. If e is an edge in X which is also in SX, set

$$T\delta(e) = \delta(e). \tag{34}$$

Any other edge e in X is e_i^p for some (unique) p and i, and we define

$$T\delta(e_i^p) = \delta_i^p \sum_{j \in A_pX} \delta_j^p. \tag{35}$$

By (21), S defines a map from Γ_X to Γ'_{SX}. We claim that T is a two-sided inverse for S.

Let $\delta \in \Gamma'_{SX}$ be given; we wish to show that $ST\delta = \delta$. For $p \in X^o$ and distinct $i, j \in A_pX$, observe that

$$\begin{aligned}
ST\delta(e_{ij}^p) &= \frac{T\delta(e_i^p)T\delta(e_j^p)}{\sum_{k\in A_pX} T\delta(e_k^p)} && \text{by (18), (20)}\\
&= \frac{(\delta_i^p \sum_{k\in A_pX}\delta_k^p)(\delta_j^p \sum_{l\in A_pX}\delta_l^p)}{\sum_{k\in A_pX}(\delta_k^p \sum_{l\in A_pX}\delta_l^p)} && \text{by (35)}\\
&= \delta_i^p\delta_j^p && \text{after obvious cancellation}\\
&= \delta(e_{ij}^p) && \text{by (33).}
\end{aligned}$$

As δ and $ST\delta$ agree on edges which are in both X and SX by (19) and (34), it follows that $ST\delta = \delta$.

Next, let $\gamma \in \Gamma_X$; we must show that $TS\gamma = \gamma$. For $p \in X^o$ and $i \in A_pX$, we have

$$\begin{aligned}
TS\gamma(e_i^p) &= \sum_{j\in A_pX} (S\gamma)_i^p (S\gamma)_j^p && \text{by (35)}\\
&= \sum_{j\in A_pX} S\gamma(e_{ij}^p) && \text{by (33)}\\
&= \sum_{j\in A_pX} \frac{\gamma(e_i^p)\gamma(e_j^p)}{\sum_{k\in A_pX}\gamma(e_k^p)} && \text{by (18), (20)}\\
&= \gamma(e_i^p) && \text{after obvious cancellation.}
\end{aligned}$$

As γ and $TS\gamma$ agree on edges which are in both X and SX by (19) and (34), the claim follows. □

Corollary 3.3. *If $X \in \underline{\underline{G}}$ and L is a given matrix, then S defines a bijection from conductivities on X with response matrix L to conductivities on SX which satisfy the quadrilateral rule and have Kirchhoff matrix L.*

Proof. Combine Lemma 3.1 and Lemma 3.2. □

3.4. **The Wedge Product.** We will construct the graphs X_n from simpler graphs via an operation we call the wedge product. Suppose $P_0, \ldots, P_n$ are elements of $\underline{\underline{G}}$. We have the following linear order on $\coprod_0^n V(P_i)$:

- if $a \in \partial P_i$ and $b \in \partial P_j$, then $(a,i) < (b,j)$ iff $i < j$ or $i = j$ and $a < b$ in P_i;
- if $a \in P_i^o$ and $b \in P_j^o$, then $(a,i) < (b,j)$ iff $i < j$ or $i = j$ and $a < b$ in P_i;
- each element of $\coprod_0^n \partial P_i$ precedes each element of $\coprod_0^n P_i^o$.

Suppose that $\sim$ is an equivalence relation on $\coprod_0^n V(P_i)$ satisfying

$$\text{if } a \in \coprod_0^n P_i^o \text{ and } b \in \coprod_0^n V(P_i), \text{ then } a \sim b \text{ iff } a = b \tag{36}$$

and

$$\text{if } a \text{ and } b \text{ are distinct boundary nodes in some } P_j, \text{ then } (a,j) \not\sim (b,j). \tag{37}$$

Let $\pi : \coprod_0^n V(P_i) \to \coprod_0^n V(P_i)/\sim$ be the projection. We define a graph, called the *wedge product of $P_0, \ldots, P_n$ with respect to* $\sim$, and denoted $\bigwedge_0^n P_i$ (with no explicit notational reference to $\sim$), as follows:

- the boundary of $\bigwedge_0^n P_i$ is the set $\pi(\coprod_0^n \partial P_i)$;
- the interior of $\bigwedge_0^n P_i$ is the set $\pi(\coprod_0^n P_i^o)$;
- the edge set of $\bigwedge_0^n P_i$ is the set $\coprod_0^n E(P_i)$, where if an edge $e \in E(P_j)$ joins nodes a and b in P_j then (e,j) joins nodes $[(a,j)]$ and $[(b,j)]$ in $\bigwedge_0^n P_i$;
- $[(a,i)] < [(b,j)]$ in $\bigwedge_0^n P_i$ iff $\inf \pi^{-1}([(a,i)]) < \inf \pi^{-1}([(b,j)])$ in $\coprod_0^n V(P_i)$.

We then have the following result.

Lemma 3.4. *If $P_0, \ldots, P_n \in \underline{\underline{G}}$ and $\sim$ satisfies* (36) *and* (37)*, then the wedge product of $P_0, \ldots P_n$ with respect to $\sim$ is well-defined as a graph, is an element of $\underline{\underline{G}}$, and satisfies $(\bigwedge_0^n P_i)^* = \bigwedge_0^n P_i^*$, where the wedge product of the P_i^* is with respect to $\sim|_{\coprod_0^n \partial P_i}$.*

Proving Lemma 3.4 is simply an exercise in unraveling the various definitions.

Note that for each j, we have obvious maps $E(P_j) \to E(\bigwedge_0^n P_i)$ and $V(P_j) \to V(\bigwedge_0^n P_i)$, sending $e \in E(P_j)$ to (e,j) and $a \in V(P_j)$ to $[(a,j)]$, respectively. By the definition of the edge set and vertex set of $\bigwedge_0^n P_i$ (as well as (36)), both of these maps are injections. We denote their pair by $P_j \hookrightarrow \bigwedge_0^n P_i$, and call it the *inclusion of P_j into $\bigwedge_0^n P_i$*. By identifying P_j with the image of this inclusion, we can realize P_j as a subobject[3] of $\bigwedge_0^n P_i$. As the images of the inclusions $P_j \hookrightarrow \bigwedge_0^n P_i$ cover $\bigwedge_0^n P_i$ as j ranges from 0 to n, with each edge of $\bigwedge_0^n P_i$ in the image of precisely one of these inclusions, it follows that in order to draw $\bigwedge_0^n P_i$ we need simply draw the image of each inclusion $P_j \hookrightarrow \bigwedge_0^n P_i$, according to our usual convention regarding drawing the same node more than once.

4. The Graphs X_n

4.1. **Construction.** We now apply the wedge product to construct the X_n ($n \geq 3$). The graphs in Figures 5 and 6 which, somewhat without reason, we call A and B, respectively, together with the standard $(n+1)$-star S_{n+1} in Figure 7 will play the

[3]We will not call the image of the map $P_j \hookrightarrow \bigwedge_0^n P_i$ a subgraph of $\bigwedge_0^n P_i$, as the nodes in the image of this map will not in general be identified with some initial segment of the positive integers in $\bigwedge_0^n P_i$. We could of course induce such an identification using the one on P_j, but since we wish to think of the image as living inside $\bigwedge_0^n P_i$, this is somewhat unnatural.

role of building blocks in this process. Note the edge in B labeled $e^B_{4,7}$, which is by definition the unique edge in B between nodes 4 and 7.

Observe that A, B, and S_{n+1} all lie in $\underline{\underline{G}}$. As one might expect, their star-K transformations will be relevant: for A and B, these are shown in Figures 8 and 9, respectively; the star-K transformation of S_{n+1} is simply the complete graph on the $n+1$ boundary vertices of S_{n+1}, which we make no attempt to draw for general n.

Fix $n \geq 3$. Let $P^n_0 = S_{n+1}$. For $1 \leq i \leq \lceil \frac{n}{2} \rceil$, let $P^n_i = A$, and for $\lceil \frac{n}{2} \rceil < i \leq n$, let $P^n_i = B$. Generate an equivalence relation $\sim$ on $\coprod_0^n V(P^n_i)$ by declaring

- for each $1 \leq i \leq n$, the nodes $(0, i)$ and $(0, 0)$ are to be identified,
- for each $1 \leq i \leq n$, the nodes $(1, i)$ and $(i, 0)$ are to be identified,
- all nodes $(2, i)$ for $1 \leq i \leq \lceil \frac{n}{2} \rceil$ or $(6, j)$ for $\lceil \frac{n}{2} \rceil < j \leq n$ are to be identified,
- all nodes $(3, i)$ for $1 \leq i \leq \lceil \frac{n}{2} \rceil$ or $(7, j)$ for $\lceil \frac{n}{2} \rceil < j \leq n$ are to be identified.

It is a simple task to verify that $\sim$ satisfies the hypotheses required for the definition of the wedge product, and so we may define X_n to be the wedge product of $P^n_0, \ldots, P^n_n$ with respect to $\sim$.

4.2. **Notation.** The idea of attaching copies of A and B to S_{n+1} is fairly simple, and the structure of the graphs X_n tends to reflect this. For example, X_3 is shown in Figure 14, and using Lemma 3.4 (or proceeding directly from the definition), we can immediately compute its star-K transformation, as shown in Figure 15. (The edge labels e^i_j are defined below.) In general, by the last paragraph in the Section 3, one could draw X_n by drawing S_{n+1} (with its boundary nodes labeled 0 through n, and its interior node labeled d_0) together with an instance of Figure 10 for each $1 \leq i \leq \lceil \frac{n}{2} \rceil$ and an instance of Figure 11 for each $\lceil \frac{n}{2} \rceil < i \leq n$, where

$$c_i = \begin{cases} n + 1 + 2i & 1 \leq i \leq \lceil \frac{n}{2} \rceil + 1 \\ n - 1 - \lceil \frac{n}{2} \rceil + 4i & \lceil \frac{n}{2} \rceil + 1 < i \leq n \end{cases} \tag{38}$$

and

$$d_i = \begin{cases} 5n + 3 - 2\lceil \frac{n}{2} \rceil & i = 0 \\ 5n + 2 - 2\lceil \frac{n}{2} \rceil + 2i & 1 \leq i \leq \lceil \frac{n}{2} \rceil + 1 \\ 5n - 3\lceil \frac{n}{2} \rceil + 3i & \lceil \frac{n}{2} \rceil + 1 < i \leq n \end{cases}. \tag{39}$$

Applying Lemma 3.4, we can draw X^*_n by drawing S^*_{n+1} (i.e., by drawing the complete graph on $n + 1$ boundary nodes labeled 0 through n) together with an instance of Figure 12 for each $1 \leq i \leq \lceil \frac{n}{2} \rceil$ and an instance of Figure 13 for each $\lceil \frac{n}{2} \rceil < i \leq n$.

By Lemma 3.4, we have $X^*_n = \bigwedge_0^n (P^n_i)^*$ as graphs. For $0 \leq j \leq n$, let ι_j denote the inclusion $(P^n_j)^* \hookrightarrow \bigwedge_0^n (P^n_i)^*$. We label certain edges in X^*_n as follows (where each expression of the form $e^i_j = x = y$ is to be interpreted as defining e^i_j to be equal to x (which will be an edge in $\bigwedge_0^n (P^n_i)^*$), which is also equal to y (which will be an edge in $(\bigwedge_0^n P^n_i)^*$)):

- for $1 \leq j \leq n$,

$$e^0_j \quad = \quad \iota_0(e^{n+1}_{0,j}) \quad = \quad e^{d_i}_{0,i} \tag{40}$$

- for $1 \le i \le \lceil \frac{n}{2} \rceil$,

$$
\begin{array}{lllll}
e_0^i & = & \iota_i(e_{0,1}^6) & = & e_{0,i}^{d_i} \\
e_1^i & = & \iota_i(e_{4,5}^6) & = & e_{c_i,c_i+1}^{d_i} \\
e_2^i & = & \iota_i(e_{4,5}^7) & = & e_{c_i,c_i+1}^{d_i+1} \\
e_3^i & = & \iota_i(e_{2,3}^7) & = & e_{n+1,n+2}^{d_i+1}
\end{array} \tag{41}
$$

- for $\lceil \frac{n}{2} \rceil < i \le n$,

$$
\begin{array}{lllll}
e_0^i & = & \iota_i(e_{0,1}^8) & = & e_{0,i}^{d_i} \\
e_1^i & = & \iota_i(e_{2,3}^8) & = & e_{c_i,c_i+1}^{d_i} \\
e_2^i & = & \iota_i(e_{2,3}^9) & = & e_{c_i,c_i+1}^{d_i+1} \\
e_3^i & = & \iota_i(e_{4,6}^9) & = & e_{n+1,n+2}^{d_i+1} \\
e_4^i & = & \iota_i(e_{4,6}^{10}) & = & e_{n+1,c_i+2}^{d_i+2} \\
e_5^i & = & \iota_i(e_{4,7}^{10}) & = & e_{n+2,c_i+2}^{d_i+2} \\
e_6^i & = & \iota_i(e_{4,7}^{B}) & = & (e_{4,7}^{B}, i) \\
e_7^i & = & \iota_i(e_{6,7}^{10}) & = & e_{n+1,n+2}^{d_i+2}
\end{array} \tag{42}
$$

Intuitively, for $1 \le i \le n$, the subscript on e_j^i increases as one moves farther away from the image of S_{n+1}^* in X_n^* along the image of $(P_i^n)^*$. See for example Figure 15. How these labels appear on the images of the $(P_i^n)^*$ in X_n^* for general n is indicated in Figures 12 and 13.

One can check (from the definition of the X_n) that the edges we have just labeled e_j^i are precisely those edges in X_n^* which are parallel to some other edge in X_n^*. As such, the following convention labels precisely those edges in X_n^* which we have not just labeled e_j^i: if i and j are nodes in X_n^* with a unique edge between them, this edge will be denoted by $e_{i,j}$. In summary, then, the edge set of X_n^* is partitioned as $\{e_j^i\} \cup \{e_{i,j}\}$, where $\{e_j^i\}$ consists precisely of those edges which are parallel to some other edge, and $\{e_{i,j}\}$ consists precisely of those edges which are not parallel to any other edge.

4.3. **The Quadrilateral Rule on X_n^*.** It will be helpful below to have an explicit description of what it means for a Kirchhoff matrix for X_n^* or a conductivity on X_n^* to satisfy the quadrilateral rule, in terms of the notation adopted in the previous subsection. For Kirchhoff matrices, we simply go back to the definition of quadrilateral rule (and tacitly use the symmetry of Kirchhoff matrices) to conclude that a Kirchhoff matrix K for X_n^* satisfies the quadrilateral rule iff the following hold:

- for all distinct $1 \le i, j, k, l \le n$, we have

$$K_{i,j}K_{k,l} = K_{i,k}K_{j,l}; \tag{43}$$

- for all $1 \le i \le \lceil \frac{n}{2} \rceil$, we have

$$K_{0,c_i}K_{i,c_i+1} = K_{0,c_i+1}K_{i,c_i} \tag{44}$$

and

$$K_{n+1,c_i}K_{n+2,c_i+1} = K_{n+2,c_i}K_{n+1,c_i+1}; \tag{45}$$

- for all $\lceil \frac{n}{2} \rceil < i \le n$, we have

$$K_{0,c_i}K_{i,c_i+1} = K_{0,c_i+1}K_{i,c_i} \tag{46}$$

and

$$K_{c_i,c_i+2}K_{n+1,c_i+1} = K_{n+1,c_i}K_{c_i+1,c_i+2}. \tag{47}$$

Similarly, we obtain from the definition that a conductivity γ^* on X_n^* satisfies the quadrilateral rule iff the following hold:

- for all distinct $1 \leq i, j, k, l \leq n$, we have

$$\gamma^*(e_{i,j})\gamma^*(e_{k,l}) = \gamma^*(e_{i,k})\gamma^*(e_{j,l}); \tag{48}$$

- for all distinct $1 \leq j, k, l \leq n$, we have

$$\gamma^*(e_j^0)\gamma^*(e_{k,l}) = \gamma^*(e_k^0)\gamma^*(e_{j,l}); \tag{49}$$

- for all $1 \leq i \leq \lceil \frac{n}{2} \rceil$, we have

$$\gamma^*(e_0^i)\gamma^*(e_1^i) = \gamma^*(e_{0,c_i})\gamma^*(e_{i,c_i+1}) = \gamma^*(e_{0,c_i+1})\gamma^*(e_{i,c_i}) \tag{50}$$

and

$$\gamma^*(e_2^i)\gamma^*(e_3^i) = \gamma^*(e_{n+1,c_i})\gamma^*(e_{n+2,c_i+1}) = \gamma^*(e_{n+2,c_i})\gamma^*(e_{n+1,c_i+1}); \tag{51}$$

- for all $\lceil \frac{n}{2} \rceil < i \leq n$, we have

$$\gamma^*(e_0^i)\gamma^*(e_1^i) = \gamma^*(e_{0,c_i})\gamma^*(e_{i,c_i+1}) = \gamma^*(e_{0,c_i+1})\gamma^*(e_{i,c_i}), \tag{52}$$

$$\gamma^*(e_2^i)\gamma^*(e_3^i) = \gamma^*(e_{c_i,c_i+2})\gamma^*(e_{n+1,c_i+1}) = \gamma^*(e_{n+1,c_i})\gamma^*(e_{c_i+1,c_i+2}), \tag{53}$$

and

$$\gamma^*(e_4^i)\gamma^*(e_{n+2,c_i+3}) = \gamma^*(e_5^i)\gamma^*(e_{n+1,c_i+3}) = \gamma^*(e_7^i)\gamma^*(e_{c_i+2,c_i+3}). \tag{54}$$

5. A Correspondence for Conductivities Satisfying the Quadrilateral Rule on X_n^*

Our goal in this section is to establish an important correspondence for conductivities satisfying the quadrilateral rule on X_n^*. Along the way we introduce some notation which will also be of use in later sections. Let n be fixed but arbitrary for the remainder of this section.

5.1. **The Functions φ.** We make the following (soon-to-be-motivated) definitions, where K is a Kirchhoff matrix for X_n^* satisfying the quadrilateral rule and x is a real parameter:

$$\varphi(K;x;0;1) = x \tag{55}$$

[4]
$$\varphi(K;x;0;j) = \frac{\varphi(K;x;0;1)K_{j,k}}{K_{1,k}} \qquad 1 < j,k \le n,\ k \ne j \tag{56}$$

$$\varphi(K;x;i;0) = K_{0,i} - \varphi(K;x;0;i) \qquad 1 \le i \le n \tag{57}$$

$$\varphi(K;x;i;1) = \frac{K_{0,c_i}K_{i,c_i+1}}{\varphi(K;x;i;0)} \qquad 1 \le i \le n \tag{58}$$

$$\varphi(K;x;i;2) = K_{c_i,c_i+1} - \varphi(K;x;i;1) \qquad 1 \le i \le n \tag{59}$$

$$\varphi(K;x;i;3) = \frac{K_{n+1,c_i}K_{n+2,c_i+1}}{\varphi(K;x;i;2)} \qquad 1 \le i \le \lceil \frac{n}{2} \rceil \tag{60}$$

$$\varphi(K;x;i;3) = \frac{K_{c_i,c_i+2}K_{n+2,c_i+1}}{\varphi(K;x;i;2)} \qquad \lceil \frac{n}{2} \rceil < i \le n \tag{61}$$

$$\varphi(K;x;i;4) = K_{n+1,c_i+2} - \varphi(K;x;i;3) \qquad \lceil \frac{n}{2} \rceil < i \le n \tag{62}$$

$$\varphi(K;x;i;5) = \frac{\varphi(K;x;i;4)K_{n+2,c_i+3}}{K_{n+1,c_i+3}} \qquad \lceil \frac{n}{2} \rceil < i \le n \tag{63}$$

$$\varphi(K;x;i;6) = K_{n+2c_i+2} - \varphi(K;x;i;5) \qquad \lceil \frac{n}{2} \rceil < i \le n \tag{64}$$

$$\varphi(K;x;i;7) = \frac{\varphi(K;x;i;4)K_{n+2,c_i+3}}{K_{c_i+2,c_i+3}} \qquad \lceil \frac{n}{2} \rceil < i \le n \tag{65}$$

We have the following trivial but fundamental result.

Lemma 5.1. *Given K, i, and j, the equations* (55)-(65) *define $\varphi(K;x;i;j)$ as a linear fractional transformation of x (with real coefficients), and the sign of $\partial_x\varphi(K;x;i;j)$ (where it exists) is independent of K.*

Proof. As the reader may readily check,

(66) the entries in K occurring on the right hand sides of (55)-(65) are positive.

Observe that the lemma holds trivially for $\varphi(K;x;0;1) = x$. By (66) and (56), it then holds for all $\varphi(K;x;0;j)$. By (57), it then holds for all $\varphi(K;x;i;0)$. By (66) and (58), it then holds for all $\varphi(K;x;i;1)$. In a similarly trivial manner, one proves the lemma. ☐

For concise reference, we note explicitly that

(67)
$\varphi(K;x;0;j), \varphi(K;x;i;1), \varphi(K;x;i;3),$ and $\varphi(K;x;i;6)$ have positive x partial,

and

(68)
$\varphi(K;x;i;0), \varphi(K;x;i;2), \varphi(K;x;i;4), \varphi(K;x;i;5),$ and $\varphi(K;x;i;7)$ have negative x partial.

We will need some information about which entries in K a given $\varphi(K;x;i;j)$ actually depends on. To this end, we make the following observations:

- for $i = 0$, the functions $\varphi(K;x;i;j)$ depend only on $K_{a,b}$ where $1 \le a, b \le n$,

[4] This is meant to define $\varphi(K;x;0;j)$ by making an arbitrary choice of k such that $1 < k \le n$ and $k \ne j$. By the definition of $\underline{\underline{G}}$, some such k exists, and as the reader may check, the quadrilateral rule (43) for K implies that any two choices of k give the same result.

- for $1 \le i \le \lceil \frac{n}{2} \rceil$, the functions $\varphi(K; x; i; j)$ depend only on $K_{a,b}$ where either $1 \le a, b \le n$ or at least one of a and b lies in $\{c_i, c_i + 1\}$,
- for $\lceil \frac{n}{2} \rceil < i \le n$, the functions $\varphi(K; x; i; j)$ depend only on $K_{a,b}$ where either $1 \le a, b \le n$ or at least one of a and b lies in $\{c_i, c_i + 1, c_i + 2, c_i + 3\}$.

These facts of course follow immediately from the definitions (55)-(65). Since Kirchhoff matrices are symmetric and have row sum zero, we obtain the following corollary.

Lemma 5.2. *Suppose K and K' are Kirchhoff matrices for X_n^* which satisfy the quadrilateral rule.*

(1) *If, for some fixed $1 \le i \le \lceil \frac{n}{2} \rceil$, K and K' agree at all indices (a, b) above the diagonal except possibly those where both a and b lie in $\{n+1, n+2, c_i, c_i+1\}$, then for $k \ne i$ we have $\varphi(K; x; k; j) = \varphi(K'; x; k; j)$.*
(2) *If, for some fixed $\lceil \frac{n}{2} \rceil < i \le n$, K and K' agree at all indices (a, b) above the diagonal except possibly those where both a and b lie in $\{n+1, n+2, c_i, c_i + 1, c_i + 2, c_i + 3\}$, then for $k \ne i$ we have $\varphi(K; x; k; j) = \varphi(K'; x; k; j)$.*

This will be of considerable use in Section 6.

Before moving on, we make a few more definitions and observations. We set $j_i = 3$ if $1 \le i \le \lceil \frac{n}{2} \rceil$ and $j_i = 7$ if $\lceil \frac{n}{2} \rceil < i \le n$ (i.e., j_i is the largest value of j for which $\varphi(K; x; i; j)$ is defined). We also set

$$\Sigma(K; x) = \sum_{i=1}^{n} \varphi(K; x; i; j_i), \tag{69}$$

and

$$\chi(K) = \{x : \varphi(K; x; i; j) > 0 \text{ for all } i \text{ and } j\}. \tag{70}$$

As the φ are linear fractional transformations of x by Lemma 5.1, we have that

$$\chi(K) \text{ is open.} \tag{71}$$

Note also that for given i and j, the function $\varphi(K; x; i; j)$ is defined iff e_j^i is defined (cf. subsection 4.2).

5.2. **Correspondence.** In this subsection, we establish the following result, and note an important corollary.

Theorem 5.3. *Fix a Kirchhoff matrix K for X_n^* which satisfies the quadrilateral rule.*

(1) *If γ^* is a conductivity on X_n^* satisfying the quadrilateral rule with $K_{\gamma^*} = K$, then $\gamma^*(e_1^0) \in \chi(K)$ and $\Sigma(K; \gamma^*(e_1^0)) = K_{n+1,n+2}$. Additionally, γ^* satisfies $\gamma^*(e_j^i) = \varphi(K; \gamma^*(e_1^0); i; j)$ and $\gamma^*(e_{i,j}) = K_{i,j}$ for all i, j, so γ^* is uniquely determined by its value on e_1^0.*
(2) *Conversely, if $a \in \chi(K)$ is given and $\Sigma(K; a) = K_{n+1,n+2}$, then there is a unique conductivity γ^* on X_n^* which satisfies the quadrilateral rule, has $K_{\gamma^*} = K$, and satisfies $\gamma^*(e_1^0) = a$.*

Proof. We first show (1). Given γ^* satisfying the hypotheses in (1), we claim that

$$\gamma^*(e_j^i) = \varphi(K; \gamma^*(e_1^0); i; j) \text{ for all } i \text{ and } j. \tag{72}$$

To begin with, for $i = 0, j = 1$, our claim in (72) is just the definition (55):

$$\gamma^*(e_1^0) = \varphi(K; \gamma^*(e_1^0); 0; 1). \tag{73}$$

For $i = 0$ and $1 < j \leq n$, note that by the quadrilateral rule (49) for γ^* we have

$$\gamma^*(e_j^0)\gamma^*(e_{1,k}) = \gamma^*(e_1^0)\gamma^*(e_{j,k}), \tag{74}$$

where k is distinct from $0, 1, j$ but is otherwise arbitrary, so that by (73), we have

$$\gamma^*(e_j^0) = \frac{\varphi(K; \gamma^*(e_1^0); 0; 1)\gamma^*(e_{j,k})}{\gamma^*(e_{1,k})}. \tag{75}$$

Since $K_{\gamma^*} = K$ and $e_{j,k}$ is the unique edge in X_n^* joining nodes j and k, we have $\gamma^*(e_{j,k}) = K_{j,k}$. Similarly, $\gamma^*(e_{1,k}) = K_{1,k}$. Thus, (75) may be rewritten as

$$\gamma^*(e_j^0) = \frac{\varphi(K; \gamma^*(e_1^0); 0; 1)K_{j,k}}{K_{1,k}}. \tag{76}$$

By (56), this says exactly that

$$\gamma^*(e_j^0) = \varphi(K; \gamma^*(e_1^0); 0; j). \tag{77}$$

Next, since $K_{\gamma^*} = K$, we have $\gamma^*(e_i^0) + \gamma^*(e_0^i) = K_{0,i}$ for each $1 \leq i \leq n$, as e_0^i and e_0^i are precisely the edges in X_n^* between nodes 0 and i. In other words,

$$\begin{aligned}
\gamma^*(e_0^i) &= K_{0,i} - \gamma^*(e_i^0) && (78)\\
&= K_{0,i} - \varphi(K; \gamma^*(e_1^0); 0; i) && \text{by (77)} \quad (79)\\
&= \varphi(K; \gamma^*(e_1^0); i; 0) && \text{by (57),} \quad (80)
\end{aligned}$$

which is (72) for $j = 0$ and $1 \leq i \leq n$.

By the quadrilateral rule (50) for γ^*, for each $1 \leq i \leq n$ we have

$$\gamma^*(e_0^i)\gamma^*(e_1^i) = \gamma^*(e_{0,c_i})\gamma^*(e_{i,c_i+1}). \tag{81}$$

Since $K_{\gamma^*} = K$ and e_{0,c_i} is the unique edge in X_n^* between nodes 0 and c_i, we have $\gamma^*(e_{0,c_i}) = K_{0,c_i}$. Similarly, $\gamma^*(e_{i,c_i+1}) = K_{i,c_i+1}$. Therefore, (81) implies

$$\gamma^*(e_1^i) = \frac{K_{0,c_i}K_{i,c_i+1}}{\gamma^*(e_0^i)}, \tag{82}$$

so by (80) and (58), we have

$$\gamma^*(e_1^i) = \frac{K_{0,c_i}K_{i,c_i+1}}{\varphi(K; \gamma^*(e_1^0); i; 0)} = \varphi(K; \gamma^*(e_1^0); i; 1), \tag{83}$$

which is (72) for $j = 1$ and $1 \leq i \leq n$.

In an entirely similar manner, relying on the facts that $K_{\gamma^*} = K$ and γ^* satisfies the quadrilateral rule, one can easily show that indeed $\gamma^*(e_j^i) = \varphi(K; \gamma^*(e_1^0); i; j)$ for all i and j. In particular, one has

$$\gamma^*(e_{j_i}^i) = \varphi(K; \gamma^*(e_1^0); i; j_i) \tag{84}$$

for each i. Since $K_{\gamma^*} = K$, one also has

$$\sum_{i=1}^{n} \gamma^*(e_{j_i}^i) = K_{n+1,n+2}, \tag{85}$$

as the edges between nodes $n+1$ and $n+2$ in X_n^* are precisely the $e_{j_i}^i$. Combining (69), (84), and (85), we have

$$\Sigma(K; \gamma^*(e_1^0)) = K_{n+1,n+2}. \tag{86}$$

By (72) and the definition of conductivity, $\varphi(K; \gamma^*(e_1^0); i; j)$ is positive for each i and j, i.e., $\gamma^*(e_1^0) \in \chi(K)$. That $\gamma^*(e_{i,j}) = K_{i,j}$ follows immediately from the hypothesis $K_{\gamma^*} = K$ and the definition of the $e_{i,j}$. This completes the proof of item (1).

We now prove (2). Suppose $a \in \chi(K)$ is such that $\varphi(K; a; i; j) > 0$ for all i, j and $\Sigma(K; a) = K_{n+1,n+2}$. We wish to show that there is a unique conductivity γ^* on X_n^* which satisfies the quadrilateral rule, has $K_{\gamma^*} = K$, and has $\gamma^*(e_j^i) = \varphi(K; a; i; j)$ for all i and j. By (1), the only possibility is the conductivity γ^* given by

$$\gamma^*(e_{i,j}) = K_{i,j} \tag{87}$$

and

$$\gamma^*(e_j^i) = \varphi(K; a; i; j), \tag{88}$$

so we need only check that this γ^* satisfies the quadrilateral rule and has Kirchhoff matrix K.

We first show that $K_{\gamma^*} = K$. Since both K_{γ^*} and K are Kirchhoff matrices for X_n^*, any entry which is zero in one matrix is also zero in the other. Also, being Kirchhoff matrices, both K_{γ^*} and K have row sum zero, and are symmetric. Thus, in order to show that $K_{\gamma^*} = K$, it suffices to assume that $i < j$ are nodes in X_n^* with at least one edge between them and then show that $(K_{\gamma^*})_{i,j} = K_{i,j}$.

Supposing such i and j are given, if it happens that there is a unique edge between i and j, then this edge is precisely $e_{i,j}$, by definition. By (87) and the definition of K_{γ^*}, we immediately obtain $(K_{\gamma^*})_{i,j} = K_{i,j}$. If instead there are at least two edges between i and j, then by the definition of X_n^* we are in (precisely) one of the following cases:

(1) $i = 0$ and $1 \leq j \leq n$
(2) $i = c_k$ and $j = c_k + 1$ for some $1 \leq k \leq n$
(3) $i = n+1$ and $j = c_k + 2$ for some $\lceil \frac{n}{2} \rceil < k \leq n$
(4) $i = n+2$ and $j = c_k + 2$ for some $\lceil \frac{n}{2} \rceil < k \leq n$
(5) $i = n+1$ and $j = n+2$

We proceed to handle each case (though everything follows easily from (88), the definition of the φ, and, for case (5), the hypothesis that $\Sigma(K; a) = K_{n+1,n+2}$).

Suppose that we are in case (1). The edges between nodes i and j in X_n^* are precisely e_j^0 and e_0^j. The values of γ^* on these edges are given by (88):

$$\gamma^*(e_j^0) = \varphi(K; a; 0; j) \tag{89}$$

and

$$\gamma^*(e_0^j) = \varphi(K; a; j; 0). \tag{90}$$

By the definition (57) and the definition of K_{γ^*}, we then have

$$(K_{\gamma^*})_{i,j} = \gamma^*(e_j^0) + \gamma^*(e_0^j) = \varphi(K; a; 0; j) + K_{0,j} - \varphi(K; a; 0; j) = K_{0,j} = K_{i,j}, \tag{91}$$

as desired.

Suppose now we are in case (2). The edges between nodes i and j in X_n^* are precisely e_1^k and e_2^k. The values of γ^* on these edges are given by (88):

$$\gamma^*(e_1^k) = \varphi(K; a; k; 1) \tag{92}$$

and

$$\gamma^*(e_2^k) = \varphi(K; a; k; 2). \tag{93}$$

By the definition (59) and the definition of K_{γ^*}, we then have
(94)
$$(K_{\gamma^*})_{i,j} = \gamma^*(e^k_1)+\gamma^*(e^k_2) = \varphi(K;a;k;1)+K_{c_k,c_k+1}-\varphi(K;a;k;1) = K_{c_k,c_k+1} = K_{i,j},$$
which is what we were trying to show.

The reasoning in case (3) is the same as that in each of the above cases, except that the relevant edges are now e^k_3 and e^k_4, the relevant values of φ are now $\varphi(K;a;k;3)$ and $\varphi(K;a;k;4)$, and the relevant definition of φ is now (62).

The reasoning in case (4) is the same as that in each of the above cases, except that the relevant edges are now e^k_5 and e^k_6, the relevant values of φ are now $\varphi(K;a;k;5)$ and $\varphi(K;a;k;6)$, and the relevant definition of φ is now (64).

The reasoning in case (5) is as follows. The edges between nodes i and j in X^*_n are precisely the $e^k_{j_k}$. For each k, the value of γ^* on $e^k_{j_k}$ is given by (88):

$$\gamma^*(e^k_{j_k}) = \varphi(K;a;k;j_k). \tag{95}$$

By the definition of K_{γ^*}, the definition of $\Sigma(K;a)$, and the hypothesis that $\Sigma(K;a) = K_{n+1,n+2}$, we then have

$$(K_{\gamma^*})_{i,j} = \sum_{k=1}^{n} \gamma^*(e^k_{j_k}) = \sum_{k=1}^{n} \varphi(K;a;k;j_k) = \Sigma(K;a) = K_{n+1,n+2} = K_{i,j}. \tag{96}$$

Having handled all cases, we conclude that $K_{\gamma^*} = K$.

It remains to show that γ^* satisfies the quadrilateral rule. All this involves is checking that the quadrilateral rule conditions (43)-(47) for K together with the definition of γ^* in terms of K and the functions φ imply the quadrilateral rule conditions (48)-(54) for γ^*.

First of all, (87) and (43) immediately imply (48).

Next, let distinct $1 \leq j,k,l \leq n$ be given. By (87), (88), and (56) we have

$$\gamma^*(e^0_j)\gamma^*(e_{k,l}) = \frac{\varphi(K;a;0;1)K_{j,l}K_{k,l}}{K_{1,l}} \tag{97}$$

and

$$\gamma^*(e^0_k)\gamma^*(e_{j,l}) = \frac{\varphi(K;a;0;1)K_{k,l}K_{j,l}}{K_{1,l}}, \tag{98}$$

which immediately yields (49).

Next, we check (50). Given $1 \leq i \leq \lceil \frac{n}{2} \rceil$, we have

$$\begin{aligned}
\gamma^*(e^i_0)\gamma^*(e^i_1) &= \varphi(K;a;i;0)\varphi(K;a;i;1) && \text{by (88)} && (99)\\
&= \varphi(K;a;i;0)\frac{K_{0,c_i}K_{i,c_i+1}}{\varphi(K;a;i;0)} && \text{by (58)} && (100)\\
&= K_{0,c_i}K_{i,c_i+1}. && && (101)
\end{aligned}$$

Now by (87) and (44) we have

$$\gamma^*(e_{0,c_i})\gamma^*(e_{i,c_i+1}) = K_{0,c_i}K_{i,c_i+1} = K_{0,c_i+1}K_{i,c_i} = \gamma^*(e_{0,c_i+1})\gamma^*(e_{i,c_i}) \tag{102}$$

which together with (101) gives (50).

The verification of (51) goes just like that of (50) and may be carried out by the reader. The same is true of (52) and (53).

It remains to check (54). Let $\lceil \frac{n}{2} \rceil < i \le n$ be given. We have

$$\gamma^*(e_5^i)\gamma^*(e_{n+1,c_i+3}) = \varphi(K;a;i;5)K_{n+1,c_i+3} \quad \text{by (87),(88)} \tag{103}$$

$$= \frac{\varphi(K;a;i;4)K_{n+2,c_i+3}}{K_{n+1,c_i+3}}K_{n+1,c_i+3} \quad \text{by (63)} \tag{104}$$

$$= \varphi(K;a;i;4)K_{n+2,c_i+3} \tag{105}$$

$$= \gamma^*(e_4^i)\gamma^*(e_{n+2,c_i+3}) \quad \text{by (87), (88).} \tag{106}$$

Similarly,

$$\gamma^*(e_7^i)\gamma^*(e_{c_i+2,c_i+3}) = \varphi(K;a;i;7)K_{c_i+2,c_i+3} \quad \text{by (87), (88)} \tag{107}$$

$$= \frac{\varphi(K;a;i;4)K_{n+2,c_i+3}}{K_{c_i+2,c_i+3}}K_{c_i+2,c_i+3} \quad \text{by (65)} \tag{108}$$

$$= \varphi(K;a;i;4)K_{n+2,c_i+3} \tag{109}$$

$$= \gamma^*(e_4^i)\gamma^*(e_{n+2,c_i+3}) \quad \text{by (87),(88),} \tag{110}$$

which together with (106), gives (54), and completes the argument that γ^* satisfies the quadrilateral rule. We have thus established (2), and completed the proof of the theorem. □

The following corollary suggests how we will verify that X_n is n-to-1.

Corollary 5.4. *If K is a Kirchhoff matrix for X_n^* which satisfies the quadrilateral rule, then the number of conductivities on X_n with response matrix K is equal to the number of points $a \in \chi(K)$ with $\Sigma(K;a) = K_{n+1,n+2}$.*

Proof. Combine Theorem 5.3 and Corollary 3.3. □

6. Main Lemmas

Corollary 5.4 indicates that it will be helpful to have good simultaneous control over $\chi(K)$ and $\Sigma(K;x)$. To control $\Sigma(K;x)$, it is obviously enough to control each $\varphi(K;x;i;j_i)$ individually. This we do, while maintaining control of $\chi(K)$, by way of the following results.

Lemma 6.1. *Given X_n, a Kirchhoff matrix K for X_n^* which satisfies the quadrilateral rule, $[a,b] \subseteq \chi(K)$, $\epsilon > 0$, $1 \le i \le \lceil \frac{n}{2} \rceil$, and $a < y_0 < x_0 < b$, there exists a Kirchhoff matrix K' for X_n^* which satisfies the quadrilateral rule, such that*

1. *$\varphi(K';x;k;j) = \varphi(K;x;k;j)$ if $k \ne i$ or if $k = i$ and $j = 0,1$,*
2. *$\chi(K') \supseteq [a, x_0)$,*
3. *$\sup_{x\in[a,y_0]} |\partial_x \varphi(K';x;i;j_i)| \le \epsilon$,*
4. *$\varphi(K';x;i;j_i)$ is singular at $x = x_0$.*

Lemma 6.2. *Given X_n, a Kirchhoff matrix K for X_n^* which satisfies the quadrilateral rule, $[a,b] \subseteq \chi(K)$, $C, \epsilon > 0$, $\lceil \frac{n}{2} \rceil < i \le n$, and $a < z_0 < x_0 < b$, there exist a Kirchhoff matrix K' for X_n^* which satisfies the quadrilateral rule and a point $y_0 \in (z_0, x_0)$ such that*

1. *$\varphi(K';x;k;j) = \varphi(K;x;k;j)$ if $k \ne i$ or if $k = i$ and $j = 0,1$,*
2. *$\chi(K') \supseteq [a, y_0)$,*
3. *$\sup_{x\in[a,z_0]} |\partial_x \varphi(K';x;i;j_i)| \le \epsilon$,*
4. *$\varphi(K';z_0;i;j_i) - \varphi(K';y_0;i;j_i) \ge C$*
5. *$\varphi(K';x;i;j_i)$ is singular at $x = x_0$.*

The ideas behind the two proofs are similar; as the proof of Lemma 6.1 is a bit simpler, we handle it first.

Proof of Lemma 6.1. Throughout this proof, it may help to keep Figure 12 in mind. Let notation be as in the statement of the lemma.

Define K' to be the unique Kirchhoff matrix for X_n^* which agrees with K above the diagonal except that

$$K'_{c_i,c_i+1} = \varphi(K; x_0; i; 1) \tag{111}$$

and

(112)

$$K'_{n+1,c_i} = K'_{n+2,c_i+1} = K'_{n+2,c_i} = K'_{n+2,c_i+1} = \sqrt{\frac{\epsilon(\varphi(K; x_0; i; 1) - \varphi(K; y_0; i; 1))^2}{\partial_x \varphi(K; y_0; i; 1)}}.$$

Note first that if such K' exists, it is clearly unique, as a Kirchhoff matrix is symmetric and has row sum zero (and thus is determined by its superdiagonal). To see that K' is well-defined, we need only show that the right hand sides of (111) and (112) are positive (as X_n^* does indeed have edges between each pair of nodes i, j such that $K'_{i,j}$ is being defined in either (111) or (112)). For (111), this is immediate, as $x_0 \in \chi(K)$. For (112), this follows by (67), as $y_0 \in \chi(K)$.

Note that K' satisfies the quadrilateral rule: (43)-(44) and (46)-(47) trivially hold for K' as they hold for K and K' agrees with K at all pairs of indices figuring in these equations; (45) holds for K' by (112).

We claim that

$$\varphi(K'; x; k; j) = \varphi(K; x; k; j) \text{ if } k \neq i \text{ or if } k = i \text{ and } j = 0, 1. \tag{113}$$

For $k \neq i$, this is an immediate consequence of Lemma 5.2. For $k = i$ and $j = 0, 1$, this follows from the definition of K' and the definitions of φ in (57) and (58). Thus, we have item (1) in the statement of Lemma 6.1.

As $[a, x_0) \subseteq \chi(K)$, it follows immediately from (113) that

$$\varphi(K; x; k; j) \text{ is positive on } [a, x_0) \text{ if } k \neq i \text{ or if } k = i \text{ and } j = 0, 1. \tag{114}$$

Thus, to establish item (2), it suffices to show that $\varphi(K'; x; i; 2)$ and $\varphi(K'; x; i; 3)$ are positive on $[a, x_0)$. Note that we have

$$\varphi(K'; x; i; 2) = K'_{c_i,c_i+1} - \varphi(K'; x; i; 1) = \varphi(K; x_0; i; 1) - \varphi(K; x; i; 1) \tag{115}$$

where we have used (59), (111), and (113). Since $[a, x_0] \subseteq \chi(K)$ and $\varphi(K; x; i; 1)$ has positive x derivative by (67), it follows from (115) that

$$\varphi(K'; x; i; 2) \text{ is zero at } x = x_0 \text{ and is positive on } [a, x_0). \tag{116}$$

By (60), we have

$$\varphi(K'; x; i; 3) = \frac{K'_{n+1,c_i} K'_{n+2,c_i+1}}{\varphi(K'; x; i; 2)}. \tag{117}$$

In other words, $\varphi(K'; x; i; 3)$ is a positive constant divided by $\varphi(K'; x; i; 2)$. It then follows immediately from (116) that

$$\varphi(K'; x; i; 3) \text{ is positive on } [a, x_0) \text{ and is singular at } x_0. \tag{118}$$

This completes the argument for item (2), and also establishes item (4).

It remains to show item (3). We have

$$
\begin{aligned}
\varphi(K'; x; i; 3) &= \frac{K'_{n+1,c_i} K'_{n+2,c_i+1}}{\varphi(K'; x; i; 2)} && \text{by (59)} \\
&= \frac{K'_{n+1,c_i} K'_{n+2,c_i+1}}{K'_{c_i,c_i+1} - \varphi(K'; x; i; 1)} && \text{by (115)} \\
&= \frac{K'_{n+1,c_i} K'_{n+2,c_i+1}}{K'_{c_i,c_i+1} - \varphi(K; x; i; 1)} && \text{by (113)} \\
&= \frac{\epsilon}{\partial_x \varphi(K; y_0; i; 1)} \cdot \frac{(\varphi(K; x_0; i; 1) - \varphi(K; y_0; i; 1))^2}{\varphi(K; x_0; i; 1) - \varphi(K; x; i; 1)} && \text{by (111), (112).}
\end{aligned}
$$

As $y_0 \in \chi(K)$ and $x_0 \neq y_0$, the previous equation and the Chain rule yield

$$
\partial_x \varphi(K'; y_0; i; 3) = \epsilon \cdot \frac{\partial_x \varphi(K; y_0; i; 1)}{\partial_x \varphi(K; y_0; i; 1)} \cdot \frac{(\varphi(K; x_0; i; 1) - \varphi(K; y_0; i; 1))^2}{(\varphi(K; x_0; i; 1) - \varphi(K; y_0; i; 1))^2} = \epsilon. \tag{119}
$$

Since $\varphi(K'; x; i; 3)$ is singular at x_0 by (118), its x partial is just a constant multiple of $(x - x_0)^{-2}$. Thus, since $a < y_0 < x_0$ and we have just shown that $|\partial_x \varphi(K'; y_0; i; 3)| \leq \epsilon$, it follows that $\sup_{x \in [a, y_0]} |\partial_x \varphi(K'; y_0; i; 3)| \leq \epsilon$, which is item (3). □

Proof of Lemma 6.2. Figure 13 is relevant here. In this proof, it will be convenient to get from K to K' via several intermediate Kirchhoff matrices (each of which will satisfy the quadrilateral rule), so we adopt the notation $K^1 = K$. In general, K^{i+1} will be obtained from K^i by modification of a few entries.

First, we define K^2 to be the unique Kirchhoff matrix for X_n^* which agrees with K^1 above the diagonal except that

$$
K^2_{c_i, c_i+1} = \varphi(K^1; x_0; i; 1). \tag{120}
$$

Since $x_0 \in \chi(K^1)$, the right hand side of (120) is positive, and since X_n^* has edges between nodes c_i and $c_i + 1$, the Kirchhoff matrix K^2 is indeed well-defined. Since K^1 satisfies the quadrilateral rule and K^2 agrees with K^1 on all relevant pairs of vertices (cf. (43)-(47)), K^2 also satisfies the quadrilateral rule.

We claim that

$$
\varphi(K^2; x; k; j) = \varphi(K^1; x; k; j) \text{ if } k \neq i \text{ or if } k = i \text{ and } j < 2. \tag{121}
$$

For $k \neq i$, this is Lemma 5.2, and for $k = i$ and $j < 2$ this follows from inspection of the definition of K^2 and of φ in (57) and (58).

We have

$$
\begin{aligned}
\varphi(K^2; x; i; 2) &= K^2_{c_i, c_i+1} - \varphi(K^2; x; i; 1) && \text{by (59)} \qquad (122) \\
&= \varphi(K^1; x_0; i; 1) - \varphi(K^2; x; i; 1) && \text{by (120)} \qquad (123) \\
&= \varphi(K^1; x_0; i; 1) - \varphi(K^1; x; i; 1) && \text{by (121).} \qquad (124)
\end{aligned}
$$

In particular, $\varphi(K^2; x_0; i; 2) = 0$, by (124). By (61), it follows that $\varphi(K^2; x; i; 3)$ is singular at x_0. By (62), then,

$$
\varphi(K^2; x; i; 4) \text{ is singular at } x_0. \tag{125}
$$

Note that since $[a, x_0] \subseteq \chi(K^1)$, the function $\varphi(K^1; x; i; 1)$ is increasing on $[a, x_0]$ by (67). Thus, by (124), the function $\varphi(K^2; x; i; 2)$ is positive on $[a, x_0)$. By (61), it follows that $\varphi(K^2;x;i;3)$ is also positive on $[a,x_0)$. Thus, by (121) and the fact that $[a,x_0)\subseteq\chi(K^1)$, we have that

$$\varphi(K^2;x;k;j) \text{ is positive on } [a,x_0) \text{ if } k\neq i \text{ or if } k=i \text{ and } j<4. \tag{126}$$

Next, we let K^3 be the unique Kirchhoff matrix for X_n^* which agrees with K^2 above the diagonal except that

$$K^3_{c_i+2,c_i+3}=\epsilon^{-1}|\partial_x\varphi(K^2;z_0;i;4)|K^2_{n+2,c_i+2}. \tag{127}$$

One checks as usual that K^3 is well-defined. It also satisfies the quadrilateral rule, as K^2 does. Note that

$$K^3_{n+2,c_i+3}=K^2_{n+2,c_i+3}, \tag{128}$$

which will be used below.

We claim that

$$\varphi(K^3;x;k;j)=\varphi(K^2;x;k;j) \text{ if } k\neq i \text{ or if } k=i \text{ and } j\neq 7. \tag{129}$$

For $k\neq i$, this is Lemma 5.2. For $k=i$ and $j<7$ this follows from inspection of the definition of K^3 and the relevant definitions of the φ. In particular, by (126), we have that

$$\varphi(K^3;x;k;j) \text{ is positive on } [a,x_0) \text{ if } k\neq i \text{ or if } k=i \text{ and } j<4. \tag{130}$$

Note that

$$\begin{aligned}
\varphi(K^3;x;i;7) &= \frac{K^3_{n+2,c_i+3}}{K^3_{c_i+2,c_i+3}}\varphi(K^3;x;i;4) && \text{by (65)} \qquad (131)\\
&= \frac{K^2_{n+2,c_i+3}}{K^3_{c_i+2,c_i+3}}\varphi(K^3;x;i;4) && \text{by (128)} \qquad (132)\\
&= \frac{K^2_{n+2,c_i+3}}{K^3_{c_i+2,c_i+3}}\varphi(K^2;x;i;4) && \text{by (129)} \qquad (133)\\
&= \epsilon\frac{\varphi(K^2;x;i;4)}{|\partial_x\varphi(K^2;z_0;i;4)|} && \text{by (127).} \qquad (134)
\end{aligned}$$

By (125) and (134), we have that

$$\varphi(K^3;x;i;7) \text{ is singular at } x_0. \tag{135}$$

In particular, $\varphi(K^3;x;i;7)$ is differentiable (with respect to x) at z_0, and by (134) we have

$$\partial_x\varphi(K^3;z_0;i;7)=\epsilon\frac{\partial_x\varphi(K^2;z_0;i;4)}{|\partial_x\varphi(K^2;z_0;i;4)|}=-\epsilon, \tag{136}$$

as $\varphi(K^2;x;i;4)$ has negative x derivative, by (68). Since $\varphi(K^3;x;i;7)$ is singular at x_0 by (135), we know that $\partial_x\varphi(K^3;x;i;7)$ is a constant multiple of $(x-x_0)^{-2}$, so since $a<z_0<x_0$ by hypothesis and $|\partial_x\varphi(K^3;z_0;i;7)|\le\epsilon$ by (136) it follows immediately that

$$\sup_{x\in[a,z_0]}|\partial_x\varphi(K^3;x;i;7)|\le\epsilon. \tag{137}$$

By (135) and (68), there is a point $y_0\in(z_0,x_0)$ such that

$$\varphi(K^3;z_0;i;7)-\varphi(K^3;y_0;i;7)\ge C. \tag{138}$$

Now, we define K^4 to be the unique Kirchhoff matrix for X_n^* which agrees with K^3 above the diagonal except that

$$K^4_{n+1,c_i+2} = \varphi(K^3; y_0; i; 3) \tag{139}$$

and

$$K^4_{n+2,c_i+2} = 1 + \frac{K^3_{n+2,c_i+3}}{K^3_{n+1,c_i+3}}(\varphi(K^3; y_0; i; 3) - \varphi(K^3; a; i; 3)). \tag{140}$$

(The additive 1 on the right hand side of (140) could be replaced by any positive number.) Since $[a, y_0] \subseteq [a, x_0)$ by the definition of y_0, it follows from (130) that

$$\varphi(K^3; x; i; 3) \text{ positive on } [a, y_0]. \tag{141}$$

Thus, the right hand side of (139) is positive. By (141) and (67),

$$\varphi(K^3; x; i; 3) \text{ is increasing on } [a, y_0]. \tag{142}$$

Thus, the right hand side of (140) is positive. It follows that K^4 is well-defined as a Kirchhoff matrix for X_n^*. As K^3 satisfies the quadrilateral rule, so too does K^4, as they agree at all relevant indices. Note that

$$K^4_{n+2,c_i+3} = K^3_{n+2,c_i+3} \text{ and } K^4_{n+1,c_i+3} = K^3_{n+1,c_i+3}, \tag{143}$$

which will be used below.

We claim that

$$\varphi(K^4; x; k; j) = \varphi(K^3; x; k; j) \text{ if } k \neq i \text{ or if } k = i \text{ and } j < 4. \tag{144}$$

As usual, this follows from Lemma 5.2 for $k \neq i$ and from inspection of the relevant definitions for $k = i$ and $j < 4$. By (121), (129), and (144), we have that

$$\varphi(K^4; x; k; j) = \varphi(K^1; x; k; j) \text{ if } k \neq i \text{ or if } k = i \text{ and } j < 2. \tag{145}$$

We claim that

$$[a, y_0) \subseteq \chi(K^4). \tag{146}$$

By (62), (139), and (144), we have

$$\varphi(K^4; x; i; 4) = K^4_{n+1,c_i+2} - \varphi(K^4; x; i; 3) = \varphi(K^3; y_0; i; 3) - \varphi(K^3; x; i; 3). \tag{147}$$

From (142), then, it follows that

$$\varphi(K^4; x; i; 4) \text{ is positive on } [a, y_0). \tag{148}$$

By (63) and (65), then,

$$\text{both } \varphi(K^4; x; i; 5) \text{ and } \varphi(K^4; x; i; 7) \text{ are positive on } [a, y_0). \tag{149}$$

As for $\varphi(K^4; x; i; 6)$, we have by (64) and (140) that

$$\varphi(K^4; x; i; 6) = 1 + \frac{K^3_{n+2,c_i+3}}{K^3_{n+1,c_i+3}}(\varphi(K^3; y_0; i; 3) - \varphi(K^3; a; i; 3)) - \varphi(K^4; x; i; 5). \tag{150}$$

By (62) and (63), we have

$$\varphi(K^4; x; i; 5) = \frac{K^4_{n+2,c_i+3}}{K^4_{n+1,c_i+3}}(K^4_{n+1,c_i+2} - \varphi(K^4; x; i; 3)). \tag{151}$$

Therefore, by (139), (144), and (143), we have

$$\varphi(K^4; x; i; 5) = \frac{K^3_{n+2,c_i+3}}{K^3_{n+1,c_i+3}}(\varphi(K^3; y_0; i; 3) - \varphi(K^3; x; i; 3)), \tag{152}$$

so by (150) we have

$$\varphi(K^4;x;i;6) = 1 + \frac{K^3_{n+2,c_i+3}}{K^3_{n+1,c_i+3}}(\varphi(K^3;x;i;3) - \varphi(K^3;a;i;3)), \tag{153}$$

so that

$$\varphi(K^4;x;i;6) \text{ is positive on } [a,y_0), \tag{154}$$

by (142). Thus, by (130), (144), (148), (149), and (154), we have (146).

Next, we observe that

$$\begin{aligned}
\varphi(K^4;x;i;7) &= \frac{K^4_{n+2,c_i+3}}{K^4_{c_i+2,c_i+3}}(K^4_{n+1,c_i+2} - \varphi(K^4;x;i;3)) && \text{by (62), (65)}\\
&= \frac{K^3_{n+2,c_i+3}}{K^3_{c_i+2,c_i+3}}(K^4_{n+1,c_i+2} - \varphi(K^3;x;i;3)) && \text{by (144), (143)}\\
&= \frac{K^3_{n+2,c_i+3}}{K^3_{c_i+2,c_i+3}}(K^4_{n+1,c_i+2} - \varphi(K^3;x;i;3) + K^3_{c_i+2,n+1} - K^3_{c_i+2,n+1}) && \text{trivially}\\
&= \frac{K^3_{n+2,c_i+3}}{K^3_{c_i+2,c_i+3}}(K^3_{n+1,c_i+2} - \varphi(K^3;x;i;3)) + \frac{K^3_{n+2,c_i+3}}{K^3_{c_i+2,c_i+3}}(K^4_{n+1,c_i+2} - K^3_{n+1,c_i+2}) && \text{trivially}\\
&= \varphi(K^3;x;i;7) + \frac{K^3_{n+2,c_i+3}}{K^3_{c_i+2,c_i+3}}(K^4_{n+1,c_i+2} - K^3_{n+1,c_i+2}) && \text{by (62), (65).}
\end{aligned}$$

In other words, $\varphi(K^4;x;i;7)$ differs from $\varphi(K^3;x;i;7)$ by an additive constant. It then follows immediately from (135) that $\varphi(K^4;x;i;7)$ is singular at $x = x_0$, from (138) that $\varphi(K^4;z_0;i;7) - \varphi(K^4;y_0;i;7) \geq C$, and from (137) that $\sup_{x\in[a,z_0]} |\partial_x \varphi(K^4;x;i;7)| \leq \epsilon$. Thus, by (145) and (146), we may take $K' = K^4$, and thereby complete the proof. □

7. The Main Result

We are now in a position to prove that X_n is n-to-1. To simplify notation, we make the following definition: given a Kirchhoff matrix K for X_n^* which satisfies the quadrilateral rule, and an integer $1 \leq k \leq n$, set

$$\sigma(K;x;k) = \sum_{1\leq i\leq \lceil \frac{k}{2}\rceil} \varphi(K;x;i;j_i) + \sum_{\lceil \frac{n}{2}\rceil+1\leq i\leq \lceil \frac{n}{2}\rceil + \lfloor \frac{k}{2}\rfloor} \varphi(K;x;i;j_i), \tag{155}$$

where x is a real parameter. For $1 \leq k \leq n$, define

$$l_k = \begin{cases} \frac{k+1}{2} & k \text{ odd} \\ \lceil \frac{n}{2}\rceil + \frac{k}{2} & k \text{ even} \end{cases}, \tag{156}$$

so that for $1 \leq k \leq n$,

$$\sigma(K;x;k) = \sum_{1\leq i\leq k} \varphi(K;x;l_i;j_{l_i}), \tag{157}$$

and in particular, for $1 \leq k < n$,

$$\sigma(K;x;k+1) = \sigma(K;x;k) + \varphi(K;x;l_{k+1}, j_{l_{k+1}}). \tag{158}$$

The following easy lemma regarding the $\sigma(K;x;k)$ will simplify matters below.

Lemma 7.1. *Suppose that for some K and k, the function $\sigma(K;x;k)$ is not constant, and that it assumes some value c_k at k distinct points $x_1^k,\dots,x_k^k$. Then $\sigma(K;x;k)$ assumes the value c_k at precisely these k points, and $\partial_x\sigma(K;x_i^k;k)$ is nonzero for each i.*

Proof. Since $\sigma(K;x;k)$ is by definition a sum of k (real) linear fractional transformations and is by hypothesis not constant, there exist (real) polynomials p and q, neither of which is zero, with $\deg q \le k$, $\deg p \le k$, and $(p,q)=1$, such that

$$\sigma(K;x;k) = \frac{p(x)}{q(x)}. \tag{159}$$

Since $\sigma(K;x;k)$ is not a constant, the equation

$$\frac{p(x)}{q(x)} = c_k \tag{160}$$

is not satisfied for all x. Hence, the equation

$$p(x) - c_k q(x) = 0 \tag{161}$$

is not satisfied for all x. Hence the function

$$p(x) - c_k q(x) \tag{162}$$

is a nonzero polynomial (of degree at most k). It therefore has at most k zeroes, counting multiplicity. Note that by (159), the function in (162) has a zero at x_i^k for all $1 \le i \le k$. It follows that these are precisely the zeroes of this function, and hence the x_i^k are precisely the points where $\sigma(K;x;k)$ assumes the value c_k. Also, the function in (162) has a simple zero at each of these x_i^k. If, for some $1 \le i \le k$, we had $\partial_x\sigma(K;x_i^k;k)=0$, then by (159) and the Chain Rule we would have

$$\frac{p'(x_i^k)q(x_i^k) - p(x_i^k)q'(x_i^k)}{q(x_i^k)^2} = 0, \tag{163}$$

i.e., after multiplying both sides by $q(x_i^k)$ and applying (159) and the definition of x_i^k,

$$p'(x_i^k) - c_k q'(x_i^k) = 0, \tag{164}$$

which says that the function in (162) has a zero of order at least two at x_i^k, which is a contradiction. We conclude that $\partial_x\sigma(K;x_i^k;k) \neq 0$ for each $1 \le i \le k$. $\square$

That X_n is n-to-1 is an easy consequence of the following result.

Lemma 7.2. *For each $1 \le k \le n$, there exist a Kirchhoff matrix K^k for X_n^* which satisfies the quadrilateral rule and a positive number c_k such that*

1. *$\sigma(K^k;x;k)$ assumes the value c_k at precisely k distinct points $x_1^k < \cdots < x_k^k$ in $\chi(K^k)$,*
2. *the x_i^k all lie in the same connected component of $\chi(K^k)$,*
3. *$\partial_x\sigma(K^k;x_1^k;k) > 0$.*

Proof. We proceed by induction. For $k=1$, let K^1 be any response matrix for X_n. By Corollary 3.3 and Theorem 5.3, $\chi(K^1)$ is nonempty. (Here, we are also using (11).) Choose any $x_1^1 \in \chi(K^1)$, and set $c_1 = \sigma(K^1;x_1^1;1)$. Note that $\sigma(K^1;x;1) = \varphi(K^1;x;1;j_1)$, by definition. Thus, item (1) in the statement of the theorem follows as $\varphi(K^1;x;1;j_1)$ is a linear fractional transformation of x, item (2) is trivially satisfied, and item (3) follows by (67).

Suppose inductively that for some $1 \leq k < n$ we have produced K^k, c_k, and $x_1^k, \ldots, x_k^k$ as in the statement of the theorem. By Lemma 7.1, item (3) in the statement of the theorem, and elementary calculus, we have $\operatorname{sgn} \partial_x \sigma(K^k; x_i^k; k) = (-1)^{i+1}$. We may therefore choose intervals $x_i^k \in [a_i^k, b_i^k] \subseteq \chi(K^k)$ and a positive number η such that

(1) $[a_i^k, b_i^k] \cap [a_j^k, b_j^k] = \varnothing$ if $i \neq j$,
(2) $\sigma(K^k; b_i^k; k) - \sigma(K^k; x_i^k; k) = (-1)^{i+1}\eta$,
(3) $\sigma(K^k; x_i^k; k) - \sigma(K^k; a_i^k; k) = (-1)^{i+1}\eta$.

As $\chi(K^k)$ is open and the x_i^k all lie in the same component of $\chi(K^k)$, there exists an interval $[u, v] \subseteq \chi(K^k)$ which properly contains each $[a_i^k, b_i^k]$.

To finish the inductive argument, we consider two cases: k even and k odd.

Suppose that k is even. Apply Lemma 6.1 to K^k with $a = u, b = v, \epsilon = \frac{\eta}{2(v-u)}, i = l_{k+1}, y_0 = b_k^k$ and $x_0 = \frac{b_k^k + v}{2}$, and denote by K^{k+1} the resulting Kirchhoff matrix for X_n^* (which satisfies the quadrilateral rule). By item (1) in the statement of Lemma 6.1 and (157), we have $\sigma(K^{k+1}; x; k) = \sigma(K^k; x; k)$, and hence by (158), we have

$$\sigma(K^{k+1}; x; k+1) = \sigma(K^k; x; k) + \varphi(K^{k+1}; x; l_{k+1}; j_{l_{k+1}}). \tag{165}$$

Define

$$c_{k+1} = c_k + \varphi(K^{k+1}; a; l_{k+1}; j_{l_{k+1}}). \tag{166}$$

We claim that $\sigma(K^{k+1}; x; k+1)$ assumes the value c_{k+1} at precisely $k+1$ distinct points in $\chi(K^{k+1})$, all of which lie in $[a, x_0)$, which is a subinterval of $\chi(K^{k+1})$ by item (2) in Lemma 6.2. Note that for $x, y \in [a, x_0)$, we have

$$|\varphi(K^{k+1}; x; l_{k+1}; j_{l_{k+1}}) - \varphi(K^{k+1}; y; l_{k+1}; j_{l_{k+1}})| \leq \frac{\eta}{2} \tag{167}$$

by item (3) in the statement of Lemma 6.1, and our choice of ϵ. Thus, by (165), (167), (166), and the definition of the a_i^k and b_i^k, for i odd we have

$$\sigma(K^{k+1}; b_i^k; k+1) - c_{k+1} \geq \frac{\eta}{2} \tag{168}$$

and

$$c_{k+1} - \sigma(K^{k+1}; a_i^k; k+1) \geq \frac{\eta}{2}, \tag{169}$$

while if i is even we have

$$\sigma(K^{k+1}; b_i^k; k+1) - c_{k+1} \leq -\frac{\eta}{2} \tag{170}$$

and

$$c_{k+1} - \sigma(K^{k+1}; a_i^k; k+1) \leq -\frac{\eta}{2}. \tag{171}$$

In other words, by the Intermediate Value Theorem, for $1 \leq i \leq k$ there exists a point $x_i^{k+1} \in (a_i^k, b_i^k)$ such that $\sigma(K^{k+1}; x_i^{k+1}; k+1) = c_{k+1}$. Note that (170) shows that $\sigma(K^{k+1}; b_k^k; k+1) < c_{k+1}$. On the other hand, by item (4) in Lemma 6.1 and (67), we have $\varphi(K^{k+1}; x; l_{k+1}; j_{l_{k+1}}) \to +\infty$ as $x \to x_0^-$. Since $\sigma(K^k; x; k)$ is continuous on $[a, b] \supsetneq [a, x_0]$, it follows that $\sigma(K^{k+1}; x; k+1) \to +\infty$ as $x \to x_0^-$. Hence, by the Intermediate Value Theorem again, there is a point $x_{k+1}^{k+1} \in (b_k^k, x_0)$ with $\sigma(K^{k+1}; x_{k+1}^{k+1}; k+1) = c_{k+1}$.

We have thus produced $k+1$ distinct points in the subinterval $[a, x_0)$ of $\chi(K^{k+1})$ at which $\sigma(K^{k+1}; x; k+1)$ assumes the value c_{k+1}. Observe that $\sigma(K^{k+1}; x; k+1)$ is not a constant, as, e.g., $\sigma(K^{k+1}; x; k+1)$ is finite on $\chi(K^{k+1})$ but tends to $+\infty$ as $x \to x_0^-$. Thus, by Lemma 7.1, our choice of K^{k+1}, c_{k+1}, and x_i^{k+1} satisfy (1) and (2) in the statement of the lemma. Item (3) is also satisfied, for (168) shows that $\sigma(K^{k+1}; a_1^k; k+1) < c_{k+1}$ while (169) shows that $\sigma(K^{k+1}; b_1^k; k+1) > c_{k+1}$, and since $\partial_x \sigma(K^{k+1}; x_1^{k+1}; k+1)$ cannot be zero by Lemma 7.1, it follows that it must be positive. This completes the inductive argument if k is even.

The case k odd is similar (except that it uses Lemma 6.2) and for the time being is left to the reader. □

Corollary 7.3. *X_n is n-to-1.*

Proof. Take $k = n$ in Lemma 7.2, so that $\Sigma(K^n; x)$ assumes some positive value c_n at precisely n points in $\chi(K^n)$. Let K be the unique Kirchhoff matrix for X_n^* which agrees with K^n above the diagonal except that

$$K_{n+1,n+2} = c_n. \tag{172}$$

It is trivial to check that K satisfies the quadrilateral rule as K^n does, and by Lemma 5.2 we have $\varphi(K; x; i; j) = \varphi(K^n; x; i; j)$ for all i and j. In particular, $\Sigma(K; x) = \Sigma(K^n; x)$ and $\chi(K) = \chi(K^n)$, so by Corollary 5.4, we are done. □

A few remarks are in order. First, the proof of Lemma 7.2 (together with that of Corollary 7.3) contains the algorithm advertised in the abstract for $n \geq 3$. Second, as $\chi(K)$ is in fact connected for any Kirchhoff matrix K for X_n^* which satisfies the quadrilateral rule (although we have not shown this), item (2) in the statement of Lemma 7.2 is superfluous.

8. An Annoying Extra Case: $n = 2$

References

[1] Curtis, Edward B. and James A. Morrow, Inverse Problems for Electrical Networks, Series on Applied Mathematics - Vol. 13, World Scientific, New Jersey, 2000.

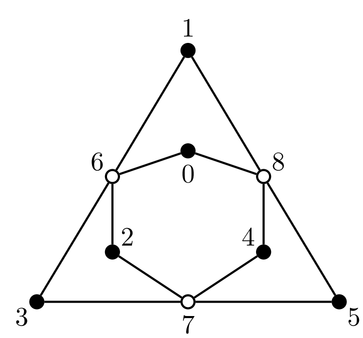

Figure 1. A drawing of the graph Θ.

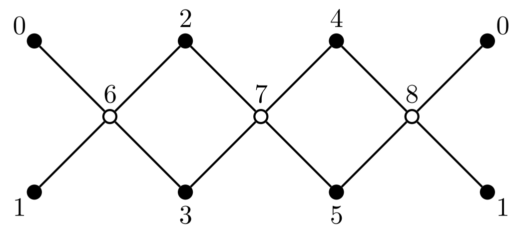

FIGURE 2. Another drawing of the graph Θ.

FIGURE 3. The parallel and series connections, respectively.

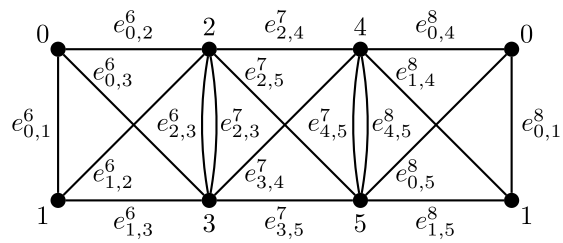

FIGURE 4. The graph Θ^*.

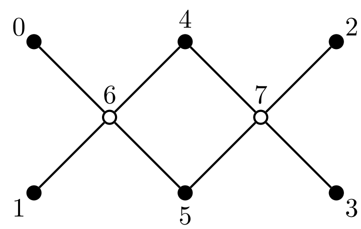

FIGURE 5. The graph A.

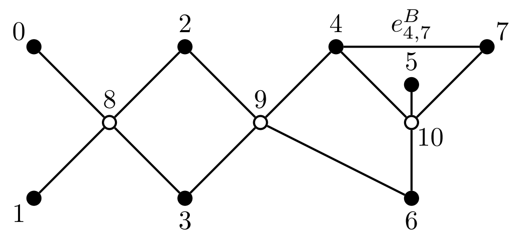

FIGURE 6. The graph B.

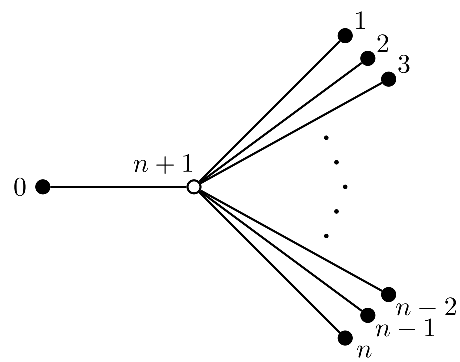

FIGURE 7. The graph S_{n+1}.

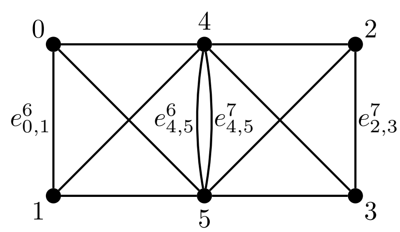

FIGURE 8. The graph A^*.

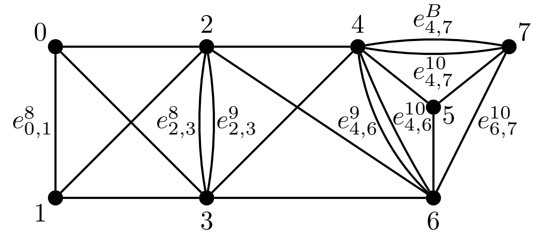

FIGURE 9. The graph B^*.

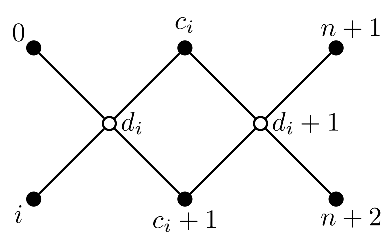

FIGURE 10. For $1 \leq i \leq \lceil \frac{n}{2} \rceil$, the image of the inclusion $P_i^n \hookrightarrow \bigwedge_0^n P_j^n$

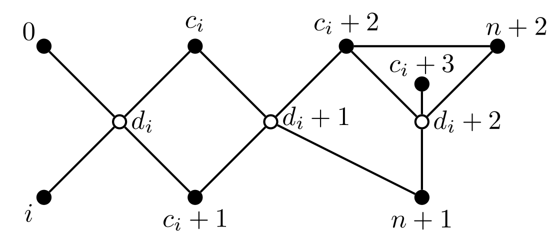

FIGURE 11. For $\lceil \frac{n}{2} \rceil < i \leq n$, the image of the inclusion $P_i^n \hookrightarrow \bigwedge_0^n P_j^n$.

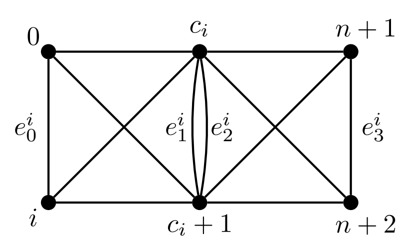

FIGURE 12. For $1 \leq i \leq \lceil \frac{n}{2} \rceil$, the image of $(P_i^n)^* \hookrightarrow \bigwedge_{j=0}^n (P_j^n)^*$.

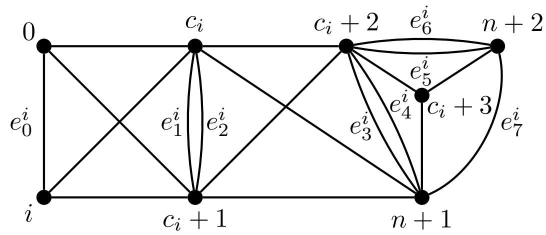

FIGURE 13. For $\lceil \frac{n}{2} \rceil < i \leq n$, the image of $(P_i^n)^* \hookrightarrow \bigwedge_{j=0}^{n}(P_j^n)^*$.

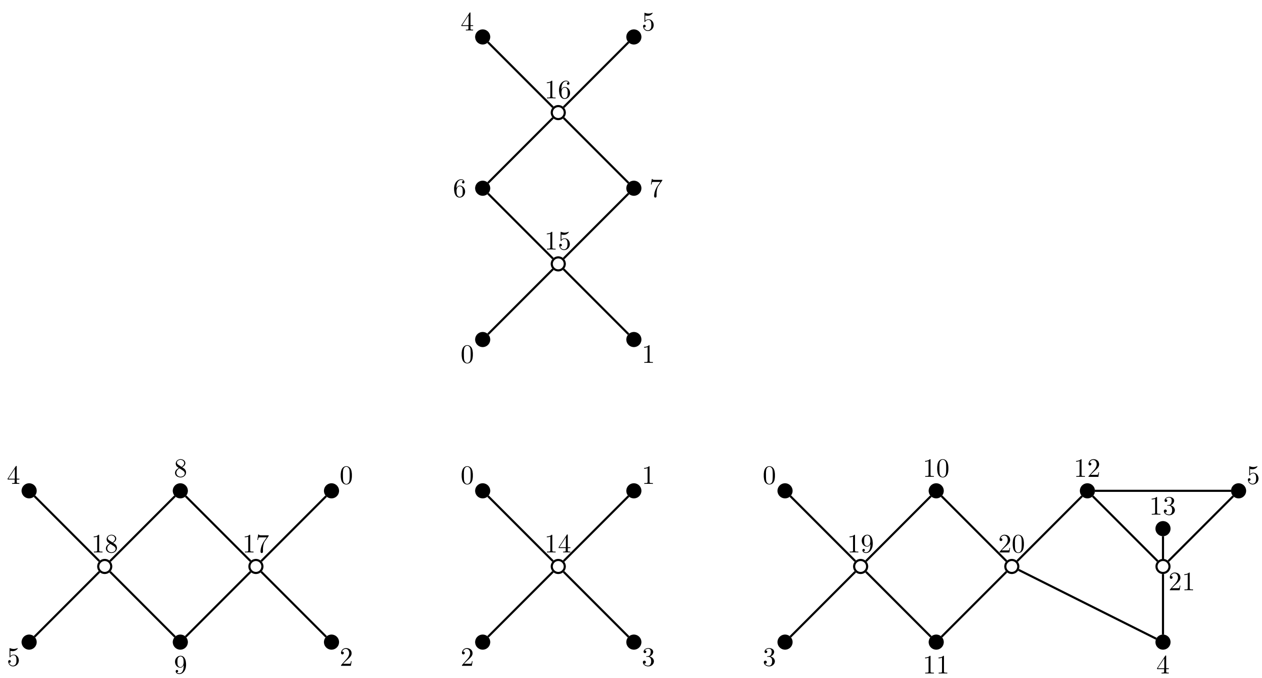

Figure 14. The graph X_3.

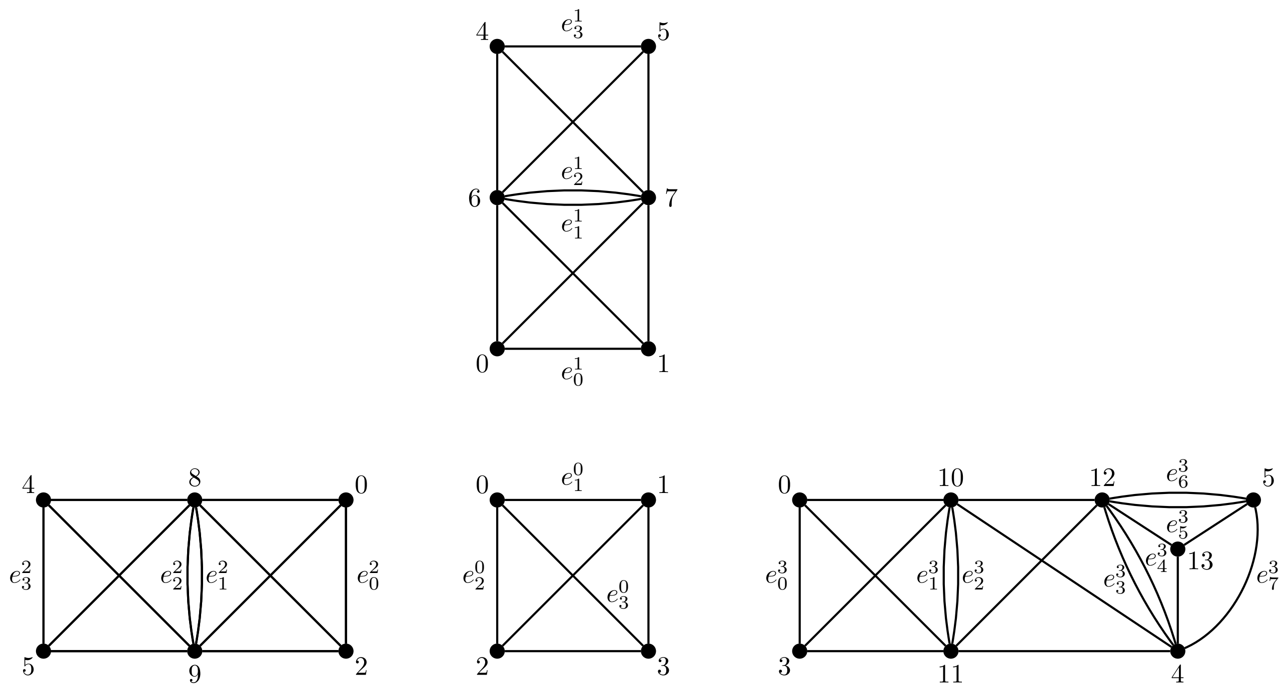

FIGURE 15. The graph X_3^*, with vertices and certain edges labeled.