

For Whom the Booth Tolls

February 7, 2005

Abstract

Many states find it necessary to put tolls on their highways as a form of usage-based taxation. We would like to minimize the amount of time motorists spend waiting to get through a toll plaza. The only parameter we can control when designing a plaza is the number of toll booths to incorporate. We find that if few booths are used, traffic will build up at the entrance to the plaza as motorists wait in line to pay. If too many booths are available, massive congestion will result on the exit side of the plaza due to the number of lanes required to merge. In this paper we first write a simulation to examine the nature of these traffic jams. This simulation shows that the optimal number of booths depends only on the processing time of the booth, not the traffic density. However, the simulation seems to break down for high levels of traffic. We use the data from this simulation to construct a traffic flow model that predicts more accurate estimates for the mean wait time to pass through a plaza. We then build a model that actually describes the behavior of traffic between the various lanes available. This model combines stochastic and dynamical elements and incorporates the likelihood of motorists choosing from different toll booths based on the number of people waiting in line. In addition, this model describes the competition between motorists as they attempt to merge back to their original lanes. Included in this paper is a section on the numerical methods used to solve the systems of differential equations in our models. In conclusion, we apply these models to predict the optimal number of toll booths to use on the Garden State Parkway.

Contents

1	Introduction	3
1.1	The Problem	3
1.2	Our Approach	4
2	Division into Five Zones	4
3	Traffic Simulation	5
3.1	The Algorithm	5
3.2	Data	7
3.3	Analysis	7
4	Ensemble Model	9
4.1	Analysis of the Model	11
5	Behavioral Model	12
5.1	Assumptions	12
5.2	Which Toll Booth to Choose?	12
5.3	Some Rates of Traffic	13
5.4	The Behavior of Merging	15
5.5	Solving the System of Equations	16
5.6	Numerics	16
6	Conclusion	17
6.1	The Garden State Parkway	19



Figure 1: An artistic rendition of a toll plaza [1].

1 Introduction

The two things people dislike most about driving are:

1. Getting stuck in traffic
2. Paying fines

Unfortunately, toll plazas combine both these elements into an unpleasant experience for everyone from daily commuters to traveling tourists. Since the toll fees are a necessary evil, we focus our attention on minimizing the delay that toll plazas incur on motorists.

1.1 The Problem

Consider a motorist heading Eastbound on a four-lane highway. Suppose this highway has a toll plaza with four toll booths. Then the Western side of the plaza is likely to become congested, since drivers (such as our hypothetical motorist) will have to wait for each car in their lane to slowly pass through the toll booth in the corresponding lane. This suggests that waiting times could be decreased by having a greater number of booths than lanes. Our motorist approaches the

plaza, sees a booth with a short line, heads towards that booth, fumbles for some change but ultimately pays the fee and passes through the booth, then simply merges to his original lane and is back on track to reaching his destination. However, as most drivers are aware, merging lanes is not a simple task. Indeed, the greater the number of lanes required to merge simultaneously, the more delays one should expect when merging. Thus as the number of toll booths increases, the amount of congestion on the Western side of the plaza decreases, but the congestion on the Eastern side increases. A philosopher would call this a paradox, but a mathematician refers to it as an optimization problem!

This leads to our initial formulation of the problem:

Problem Statement: Given a specified traffic volume and number of lanes, find the optimal number of toll booths so that the average delay induced by the plaza is minimized.

1.2 Our Approach

Below is an outline of our method to approach this problem:

- Divide the toll plaza into five *zones* to simplify analysis.
- Develop a computer program to simulate traffic flow through the plaza.
- Analyze the data from this simulation and use it to motivate an *ensemble* traffic flow model.
- Consider the interplay between lanes and produce a *behavioral* model.
- Use these models to determine the optimal number booths for the Garden State Parkway.

2 Division into Five Zones

In this paper we consider a highway with L lanes and a toll plaza consisting of n booths, where $L \leq n$. For concreteness, let us say that traffic comes in from the West and proceeds to the East.

Zone 1: *The stretch of highway West of the toll plaza.* This is where the cars that must pass the plaza come from. The only relevant parameter from this zone is *traffic density*, since this specifies the rate at which cars enter the plaza.

Zone 2: *The queue leading up to the booths.* This is the area in which cars idle while waiting for a toll booth to become available. Under favorable conditions this zone is indistinguishable from Zone 1, but more typically there is congestion in the form of a queue.

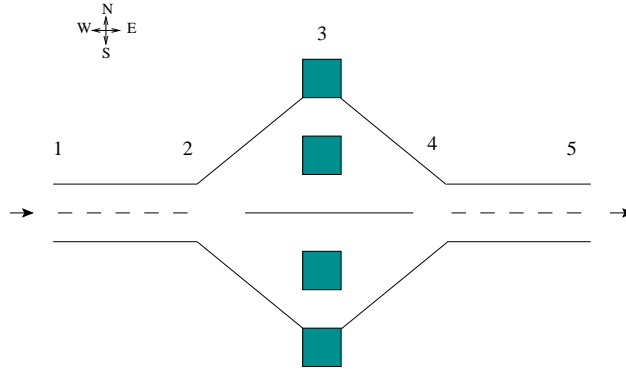


Figure 2: The five zones of a toll plaza, where $L = 2$ and $n = 4$.

Zone 3: *The toll booths.* Here we have cars delayed for a specified amount of time while the drivers pay the appropriate tolls.

Zone 4: *The merging area.* In this zone cars are required to merge from the n booths to one of the L lanes. Due to the difficulty and disorder of merging this zone becomes congested when $n > L$.

Zone 5: *The stretch of highway East of the toll plaza.* It is the sole intent of every driver to reach this point.

We refer to the average time it takes a motorist to proceed from Zone 1 to Zone 5 as the *mean wait time*.

3 Traffic Simulation

Suppose we monitor a fixed number of cars, say c of them. We would like to know how their mean wait time depends on the number of toll booths, the relative traffic density of these cars, and the approximate time it takes for each car to pay the toll. We simulate this as follows:

3.1 The Algorithm

1. Uniformly distribute the entry times $\{t_1, \dots, t_c\}$ for these c cars in some time interval. For simplicity we take this interval to be one minute, so that traffic may be measured in units of *cars per minute*.
2. Start the clock at $t = 0$. When the clock reaches the entry time of the j^{th} car, so $t = t_j$, we consider the car to have left Zone 1 and entered Zone 2. That is, the car has reached the plaza. When this occurs, the car looks for an available toll booth. If one is free, the car proceeds to it immediately,

Name	Description
L	Number of lanes on highway
n	Number of toll booths in plaza
c	Number of cars to be simulated
t	Time, which runs from 0 to 60s
t_1, \dots, t_c	Entry times of cars
μ, σ	Parameters for delay time at booth
$z_i(t)$	Number of cars in Zone i at time t
$M(t)$	Delay function for merging zone
α, β	Parameters for delay function
W	Mean wait time

Table 1: Variables used in the simulation.

and if not then the car idles until one becomes available and then proceeds to it. This is the process of progressing from Zone 2 to Zone 3.

3. The car is now sitting at the toll booth paying the fee. Since the time it takes different drivers to perform this task varies, each time a car enters this zone we choose a random delay factor from a *normal distribution* with parameters μ and σ and have the car idle for this amount of time. After this delay has ended, the car leaves Zone 3 and enters Zone 4.
4. Now the car is attempting to merge. If the total number of cars in Zone 4 is less than or equal to the number of lanes on the highway, then the time it takes to complete this zone is some constant, since each car simply has to drive straight through without changing lanes or merging. However, if there are more cars than lanes, the time it takes should grow exponentially with the number of cars. Formally, the merge time $M(t)$ is given by

$$M(t) = \begin{cases} \beta & \text{if } z_4(t) \leq L \\ \beta + e^{\alpha(z_4(t)-L)} & \text{if } z_4(t) > L \end{cases} \quad (1)$$

where $z_4(t)$ is the number of cars in Zone 4 at time t , and α and β are fixed parameters. At the moment a car enters Zone 4 it is assigned a delay factor $M(t)$ according to the above formula. The cars idles for this length of time and then proceeds to Zone 5.

5. The total time it takes the j^{th} car to pass through the plaza is given by the difference between the time when the car reached Zone 5 and the entry time t_j . The mean wait time W is then the average of these times for all c cars.

3.2 Data

We ran the simulation with the following information:

Parameters

$L = 4$: thus we are simulating a four-lane highway.

$\alpha = 0.2$: this was more or less an educated guess.

$\beta = 15$: we consider this to be approximatly the number of seconds it takes to accelerate through a merge zone when traffic is low.

$\sigma = 1$: this was estimated by calculating the standard deviation of time required to search our pockets for a specified amount of loose change upon repeated attempts.

Input

n : ranges from 4 to 16 booths.

c : ranges from 0 to 60 cars per minute.

μ : ranges from 0 to 30 seconds.

Output:

W : this is the mean wait time that we wish to minimize.

The results are pictured in Figures 3, 4, and 5.

3.3 Analysis

The first striking fact about these datasets is that the optimal number of booths is independent of traffic density. In fact, the choice of booths that minimizes mean wait time depends most strongly on μ , the average number of seconds to pay at the toll booth. We can most easily explain this phenomenon by considering extreme values of μ :

- $\mu \approx 0$: *This is equivalent to having a plaza without any toll booths. Cars may remain in their original lanes and any merging will only cause unnecessary complications, so the optimal choice is to have the same number of booths as lanes.*
- $\mu \rightarrow \infty$: *The delays induced by merging become small compared to the time required to pass each toll booth, so the optimal solution is to allow as many booths as possible.*

This means that in order for the model to predict the optimal number of booths correctly, it is essential to have an accurate estimate of μ .

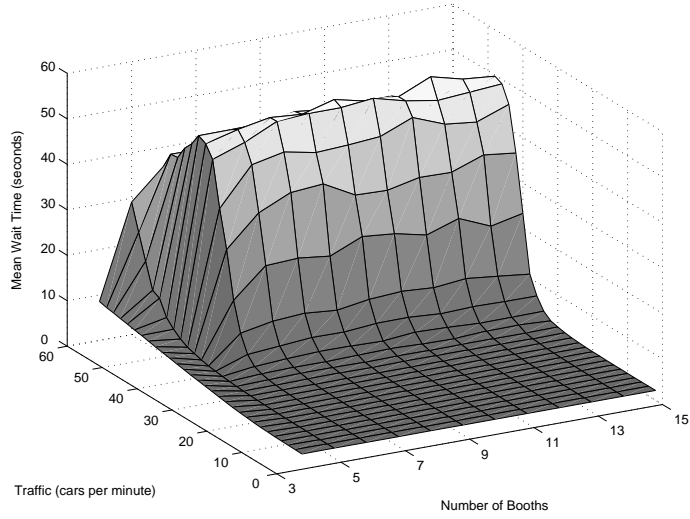


Figure 3: Mean wait time for $\mu = 5s$.

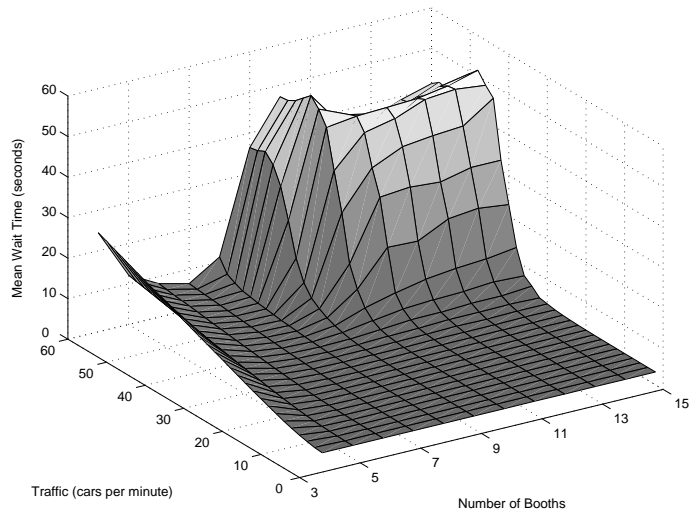


Figure 4: Mean wait time for $\mu = 15s$.

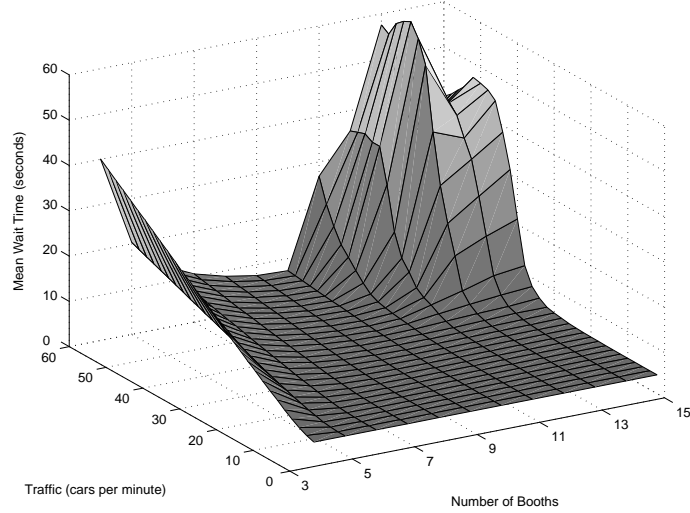


Figure 5: Mean wait time for $\mu = 25s$.

Another observation is that at some point, adding additional booths will have no effect. The reason for this is that eventually there will be plenty of booths available, so the predominant factor in the wait time will be due to merging.

However, we also note that the simulation does not seem to be accurate for high levels of traffic. The plots show a decrease in mean wait time as traffic increases from 50 to 60 cars per minute, which is totally unrealistic. Because of this, we find it necessary to build a model that predicts more accurate behavior at high traffic levels.

4 Ensemble Model

Our objective in this section is to produce a mathematical model to describe the above simulation. In doing so, it is crucial to understand that the mean waiting time is dependent on the number of booths, the traffic density, and the processing time of each toll booth. In other words,

$$W = W(n, c, \mu) \quad (2)$$

By inspecting the three datasets above, we note that the simulation exhibits two different types of behavior, namely one for high traffic and one for low traffic.

When traffic density is small, the model is given by

$$\frac{\partial W}{\partial n} = -\alpha_1 c \frac{1}{n^2} \quad (3)$$

$$\frac{\partial W}{\partial c} = \alpha_1 c \frac{1}{n} \quad (4)$$

Solving the system, we obtain

$$W = \alpha_1 \frac{1}{n} c \quad (5)$$

With this equation we see that...

For higher traffic density, a different model is necessary. Consider the logistic growth

$$\frac{\partial W}{\partial n} = lW(1 - \frac{W}{K})f(n) \quad (6)$$

where K is the carrying capacity of the curve, l is some constant, and $f(n)$ is the term to incorporate the fact that there is an optimal value for n .

It is evident from the simulation that only one minimum for W exists when traffic is fixed. Thus, at the optimal point n_0 we have

$$\frac{\partial W}{\partial n}(n_0) = lW(1 - \frac{W}{K})f(n_0) = 0 \quad (7)$$

Thus the relation between W and n is

$$\frac{\partial W}{\partial n} = lW(1 - \frac{W}{K})(\frac{n}{n_0} - 1) \quad (8)$$

For W versus c , the relation can be explained by

$$\frac{\partial W}{\partial c} = \alpha W \quad (9)$$

The system of partial differential equations can be analytically solved in the following manner. Consider first solving (6) by examining

$$\frac{dW}{dn} = lW(1 - \frac{W}{K})(\frac{n}{n_0} - 1) \quad (10)$$

Solving for W yields

$$W = \frac{K\theta e^{l(\frac{n^2}{2(n_0)} - n)}}{K - \theta e^{l(\frac{n^2}{2(n_0)} - n)}} \quad (11)$$

Since (6) involves a partial derivative with respect to n , a variation of parameters can be made to reach

$$W = \frac{K(c)\theta(c)e^{l(\frac{n^2}{2(n_0)} - n)}}{K(c) - \theta(c)e^{l(\frac{n^2}{2(n_0)} - n)}} \quad (12)$$

Now examine (9) in a similar way to see

$$\frac{dW}{dc} = \alpha W \quad (13)$$

This yields the solution in exponential terms and with the variation of parameter

$$W = \phi(n)e^{\alpha c} \quad (14)$$

To be able to separate the variables, we see that for (12), $K(c) = \theta(c)$, or

$$W = \frac{\theta(c)e^{l(\frac{n^2}{2(n_0)} - n)}}{1 - e^{l(\frac{n^2}{2(n_0)} - n)}} \quad (15)$$

Thus comparing (15) and (16), we obtain the relation

$$\theta(c) = e^{\alpha c} \quad (16)$$

$$W = \frac{e^{l(\frac{n^2}{2(n_0)} - n)}}{1 - e^{l(\frac{n^2}{2(n_0)} - n)}} \quad (17)$$

or

$$W = e^{\alpha c} \frac{e^{l(\frac{n^2}{2(n_0)} - n)}}{1 - e^{l(\frac{n^2}{2(n_0)} - n)}} \quad (18)$$

4.1 Analysis of the Model

In using any mathematical model, the justification for the terms being used is necessary. We will now proceed by examining the meaning behind the terms we have used.

For the low traffic model, the most crucial observation is that the simulation exhibits a very subtle linear growth for W with respect to c . However, the growth rate is more steep when the number of toll booths is lowered. Furthermore, as this number n is increased, the growth becomes very small. Thus, the relation in (4) was constructed.

For W versus n , we see a rapid damping of the curve going from high values to a limiting value. Thus, the relation

$$\frac{\partial W}{\partial n} \propto \frac{1}{n^h} \quad (19)$$

where h is a positive integer. To match n in (4), h was chosen to be 2. For higher traffic, the decrease rate of W with respect to n is sharper at the beginning. Thus, we were able to conclude that the relationship between W and n is given by (5).

For the situation in which traffic density is high, a logistic curve seemed the most appropriate fit to the output from the simulation. This can be explained

by the fact that after a certain number of booths are opened, the mean wait time will not be affected by any additional increases in n since the merging delay compensates for the quickened time it takes to enter the toll booths.

In modeling the curves relating W and c , a slightly different approach from the simulation was used. This difference concerns the idea of mean wait time actually dropping when enough traffic is introduced. For our model, we have decided that this occurrence is incorrect, meaning that as traffic increases sufficiently, the mean increases without bounds. This assumption can be justified by the fact that traffic jams are shown to be more likely with heavier traffic [5]!

Since W is allowed to grow without bounds as c does, the most common approach of relating the rate of change of W to the current W was made, namely the exponential growth.

5 Behavioral Model

In this section we develop a model to better understand the interplay between individual lanes rather than simply studying the overall flow of traffic. The model we arrive at is inspired by predator-prey systems, so we will refer to it as a *behavioral model*, meaning that cars driving through the various toll booth lanes must compete with each other.

5.1 Assumptions

In this model, the following assumptions will be made:

1. The traffic flow into the system is uniform. Call this rate $\frac{dC}{dt}$.
2. The time required to reach the toll booths and that required to merge can be combined into an overall delay referred to as *processing time*.
3. Each car entering the toll plaza has the ability to choose any of the toll booths.

In addition, as stated in [6], the lanes leading up to most toll plazas split in advance of the plaza itself, so it seems reasonable to assume that the difficulties of changing lanes occur only after passing the toll booths, not leading up to them.

5.2 Which Toll Booth to Choose?

Although motorists entering the plaza are given the option to choose any of the toll booths, several factors may influence their choice of lanes. Let us consider the probability distribution indicating the likeliness of the incoming traffic to choose each of the n booths.

Define p_i to be the probability that a given car chooses the i^{th} booth. In the simplest case, the distribution is uniform with respect to the n booths. That is,

$$p_i = \frac{1}{n}, i = 1, 2, \dots, n. \quad (20)$$

However, this assumes that the drivers do not consider how crowded each of the lanes are. The density of cars in each toll lane does indeed influence the drivers' choice to some degree. Specifically, drivers are more likely to choose a lane with the least number of cars. Denoting the number of cars in the i^{th} lane as y_i , we have that

$$\begin{aligned} p_j &\text{ decreases when } y_{i=j} \text{ increases} \\ p_j &\text{ increases when } y_{i \neq j} \text{ increases} \end{aligned}$$

This leads to the relation

$$p_i = \frac{1}{n} + (-\beta_i y_i + \sum_{j \neq i} \beta_j y_j) \quad (21)$$

Notice that $\sum_{i=1}^n p_i = 1$. Furthermore, in order to ensure that the probabilities satisfy the condition $0 < p_i < 1$ for all i , the β_i must be sufficiently small.

5.3 Some Rates of Traffic

With probabilities defined as above, we see that the probabilistic rate at which the cars enter each of the toll lanes is

$$(\text{rate into booth } i) = p_i \frac{dC}{dt}$$

Now define the number of cars leaving the toll lanes by

$$(\text{number of cars exiting toll lane } i) = r_i(t)$$

so that

$$(\text{rate of cars leaving lane } i) = \frac{dr_i}{dt}$$

Let us proceed in determining the number of cars in each of the toll lanes at a specific time. Assume that initially there are $y_i(0)$ cars in lane i . Then the number of cars in lane i at a later time can be written as

$$(\text{number of cars in lane } i) = (\text{initial number}) + (\text{cars entering}) - (\text{cars exiting})$$

or more formally,

$$y_i(t) = y_i(0) + (p_i \frac{dC}{dt} - \frac{dr_i}{dt}) t e^{-\frac{1-\nu^2}{\nu} t} \quad (22)$$

where ν is the time required to pay the toll and proceed through a toll booth. We refer to this as the *processing time*. This parameter is equivalent to $\frac{1}{n}$ with

the notation from the first model.

We now examine the specific way in which cars exit the toll booth lanes. Notice that if the processing time ν is zero, we would expect the flow into the system to equal the flow out of the system. That is,

$$\lim_{\nu \rightarrow 0} \frac{dr_i}{dt} = p_i \frac{dC}{dt}. \quad (23)$$

Furthermore, observe that as $\nu \rightarrow 0$, $y \rightarrow 0$ and when $\nu \rightarrow \infty$, $y \rightarrow \infty$. This can be explained by the fact that when the processing time is zero, no build up of cars in y occurs. In contrast, when ν is extremely high, no cars can exit the toll booths leading to a completely jammed traffic.

Also, notice that if processing time is infinite, the rate out of the system should be zero since no cars are able to exit the toll booths. Thus,

$$\lim_{\nu \rightarrow \infty} \frac{dr_i}{dt} = 0. \quad (24)$$

Now let us examine how the rate of exit is affected by the number of cars r_i that are leaving. Naively, we would expect the rate of exit in each lane to be given by a direct proportionality:

$$\frac{dr_i}{dt} \propto r_i(t) \quad (25)$$

However, this assumption would imply that $\frac{dr_i}{dt}$ could grow without bound. To reconcile this dilemma, a logistic approach can be made to the exit rate:

$$\frac{dr_i}{dt} \propto r_i \left(1 - \frac{r_i}{U}\right) \quad (26)$$

where U is the carrying capacity of the number of cars in each of the toll lanes. By examining this relation, we see that the equilibria are

$$r_i \left(1 - \frac{r_i}{U}\right) = 0 \quad (27)$$

or

$$r_i = 0 \text{ and } r_i = U$$

Stability analysis of these equations leads to the data in Figure 5.3. Thus, $r_i = U$ is the stable equilibrium while $r_i = 0$ is the unstable equilibrium.

Another factor that plays a key role in the behavior of the exit rates is the interdependence between each of the lanes. There is not enough room for all cars to merge into one lane simultaneously, so traffic flows in each lane become erratic and certain lanes are able to exit more quickly than others. This phenomenon has been studied previously (see [2], [4]), but the pre-existing models seem to vary greatly and depend heavily on heuristics and observation. We develop a more systematic model in the ensuing section.

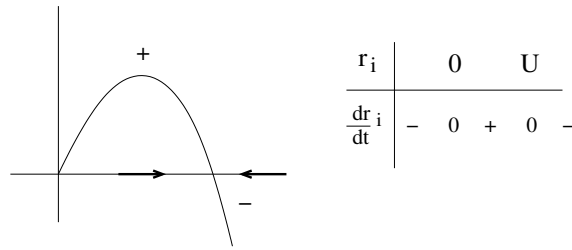


Figure 6: Stability analysis for r_i .

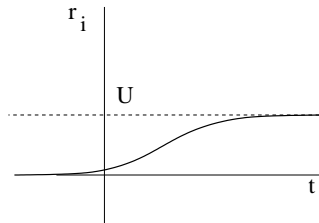


Figure 7: The logistic growth of r_i .

5.4 The Behavior of Merging

There are two possible types of dependences that can occur between lanes: one in which competition between lanes directly affects the other lanes, and one in which this effect is induct. In designing our model, we have decided to make use of the latter dependence.

Greater competition among the lanes should decrease the rate of cars leaving the plaza. That is, inter-lane competition negatively affects the exit rates of the traffic since there are only limited space available for the cars to exit. Thus, with the incorporation of competition, we see that

$$\frac{dr_i}{dt} \propto r_i \left(1 - \frac{r_i}{U}\right) - r_i \sum_{k \neq i} q_k r_k \tag{28}$$

for some coefficients q_i . Applying the conditions (24 and (25),

We now arrive at the following system:

$$\frac{dr_1}{dt} = e^{-\nu} \left(p_1 \frac{dC}{dt} + \nu \left[r_1 \left(1 - \frac{r_1}{U}\right) - r_1 \sum_{k \neq 1} q_k r_k \right] \right)$$

⋮

$$\frac{dr_n}{dt} = e^{-\nu} \left(p_n \frac{dC}{dt} + \nu \left[r_n \left(1 - \frac{r_n}{U} \right) - r_n \sum_{k \neq n} q_k r_k \right] \right)$$

Observe that when processing time is small, the term

$$e^{-\nu} p_i \frac{dC}{dt} \quad (29)$$

dominates the growth of r_i . In contrast, when ν is large,

$$e^{-\nu} \nu \left[r_i \left(1 - \frac{r_i}{U} \right) - r_i \sum_{n \neq i} q_n r_n \right] \quad (30)$$

will dominate the rate at which r_i changes.

5.5 Solving the System of Equations

In solving the system of differential and algebraic equations above, some modifications of the parameters were made:

- The initial number of cars in the system is zero.
- The initial number of cars leaving the system is zero.

Furthermore, the probability relation (21) was replaced by the following:

$$p_i = \frac{1}{n} + \beta(-2y_i + \sum_{j \neq i} y_j) \quad (31)$$

The substitution $\beta_j = \beta$ for $j \neq i$ and $\beta_i = 2\beta$ was made to indicate that the probability of choosing lane i is most affected by the number of cars in lane i .

Another important modification was to make the inter-lane dependence the same between all lanes. This can be justified by the fact that each lane experiences the same degree of competition as any other lane.

5.6 Numerics

Finding an exact solution to the system described above is a daunting task. In fact, from the early stages of our modeling process, we expected to be confronted with such a problem. For this reason we made use of several different numerical methods.

The first step in solving this system was to solve for each of the probabilities, p_i . For the purposes of later computation, we will denote \hat{P} to be the vector having the probabilities p_i for its components. Or

$$P = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

In computing the \hat{P} , MATLAB's symbolic algebra functionality was utilized.

We see that substitution of (22) into (21) results in to the following matrix expression

$$W\hat{P} = \hat{V} \quad (32)$$

where

$$W = \begin{bmatrix} 1 - 2K & K & \cdots & K \\ K & 1 - 2K & \cdots & \vdots \\ \vdots & \vdots & \ddots & K \\ K & \cdots & K & 1 - 2K \end{bmatrix} \quad (33)$$

$$K = B \frac{dc}{dt} e^{-\frac{1-\nu^2}{\nu} t} \quad (34)$$

$$\hat{V} = \begin{bmatrix} 1/n + Be^{-\frac{1-\nu^2}{\nu} t} [2 \frac{dr_1}{dt} - (\prod_{(j \neq 1) \neq (m \neq 1)} \frac{dr_j}{dt} \frac{dr_m}{dt})] \\ 1/n + Be^{-\frac{1-\nu^2}{\nu} t} [2 \frac{dr_2}{dt} - (\prod_{(j \neq 2) \neq (m \neq 2)} \frac{dr_j}{dt} \frac{dr_m}{dt})] \\ \vdots \\ 1/n + Be^{-\frac{1-\nu^2}{\nu} t} [2 \frac{dr_n}{dt} - (\prod_{(j \neq n) \neq (m \neq n)} \frac{dr_j}{dt} \frac{dr_m}{dt})] \end{bmatrix} \quad (35)$$

Here \hat{P} was solved by inverting W and multiplying through both sides of the equation. Once \hat{P} was obtained, probabilities were plugged into equation().

We numerically solved the equation using many of the methods available to MATLAB. Of main interest was the fact that MATLAB's ODE45 and ODE23 methods produced obvious numerical errors. This led us to believe the system of ODEs to be *stiff*. Indeed, invoking ODE23S (a standard algorithm for solving stiff systems of differential equations), our results became much more clear (see 5.6 for one such solution).

From this solution we obtain the number of cars leaving per lane at a given time, $r_i(t)$. Furthermore, the total number of cars leaving the system at time t is given by the sum:

$$R(i) = r_1(t) + r_2(t) + \dots + r_n(t)$$

By using centered differences we are able to compute the average exiting rate. The following table summerizes our results for different number of toll booths.

6 Conclusion

Both of our models predict interesting solutions to the optimal number of toll booths for a given plaza.

The simulation shows that the optimal number of booths to minimize mean wait time depends strongly on the processing time at each booth, and is independent of traffic density. Specifically, for a highway with four lanes it predicts

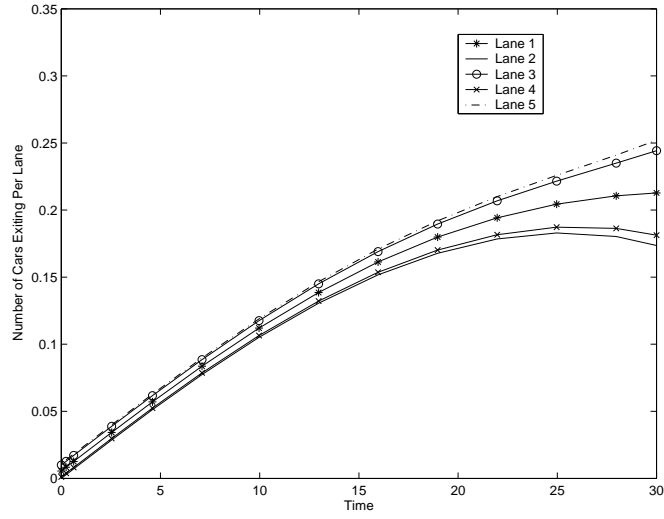


Figure 8: ODE23s' solution to the system.

Number of Gates	Average Rate Out
1	3.81
2	4.08
3	2.46
4	2.91
5	3.80
6	3.30
7	1.02
8	1.01
9	3.20
10	3.18

htb

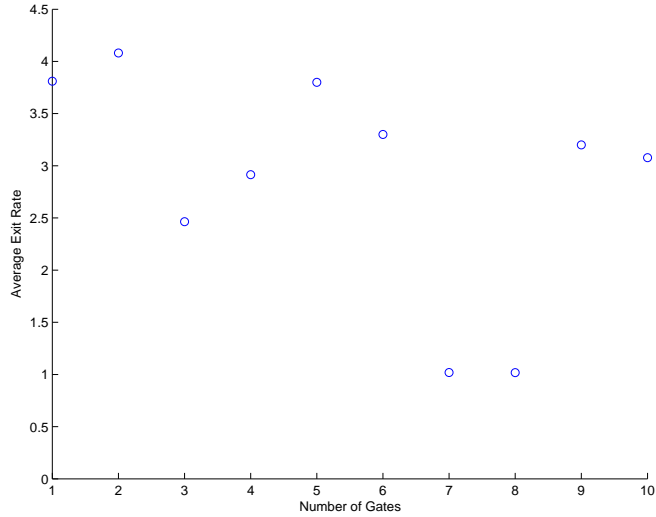


Figure 9: Plot of data in table 5.6

Name	Description
Average number of lanes	4 in each direction
Average traffic density	70 cars per minute
Standard processing time	12 seconds
EZ-Pass processing time	3 seconds

Table 2: Facts about the Garden State Parkway.

the optimal number of booths to be four when the EZ-Pass system is being used. This is because the processing time is small. When processing time is around 12 seconds, our estimate for the number of booths when they are manual is eight.

In our behavioral model, a more intriguing result can be obtained since the model incorporates stochastic and dynamic aspects. In particular we find that once a traffic rate has been set, the processing time determines mean wait time. Our results are summarized in the following figure 6.

6.1 The Garden State Parkway

According to [7], the Garden State Parkway has between two and six lanes in each direction, depending on the location. Because of this variation, we consider the average case of four lanes. Approximately one hundred thousand motorists traverse this highway each day, which means traffic may be estimated at an

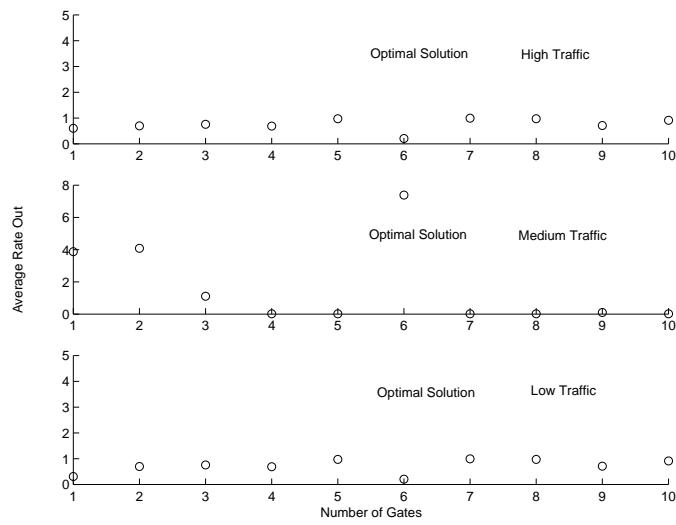


Figure 10: Optimal solution rates.

average of seventy cars per minute. In [8], it is calculated that the processing time of a manual toll booth is twelve seconds. The Garden State Parkway, like many other toll roads these days, has some toll booths devoted to customers with an electronic payment system called EZ-Pass. Our guess is that booths operating under this system have a processing time of about three seconds. This information is summarized in Table 2. Applying this data to our two models, we predict the optimal number of toll booths to be eight.

References

- [1] <http://www.dsafinearts.com/ArtAXS/Painting1T/2TBurnettJ/BurnettHome.html>, Accessed Feb. 6th, 2005.
- [2] Cowan, R. The uncontrolled traffic merge. *J. Applied Probability* **16** 384-392 (1979).
- [3] Haight, F. Mathematical Theories of Road Traffic. Institute of Transportation and Traffic Engineering, Univeristy of California, Special Report (1960).
- [4] Wynn, H., Gourlay, S., and Strickland, R. Studies of Weaving and Merging Traffic. Technical Report No. 4, Bureau of Highway Traffic (1948).
- [5] Lesort, J. Transportation and Traffic Theory. Proceedings of the 13th International Symposium on Transportation and Traffic Theory (1996).
- [6] Mobilizing the Region, Issue 277. Available at <http://www.tstc.org/> (2000).
- [7] <http://www.nycroads.com/roads/garden-state/>, Accessed Feb. 7th, 2005.
- [8] [http://ntj.tax.org/wwtax/ntjrec.nsf/0/129575aea524d7b5852567ef0057a8bd/\\$FILE/v48n2217.pdf](http://ntj.tax.org/wwtax/ntjrec.nsf/0/129575aea524d7b5852567ef0057a8bd/$FILE/v48n2217.pdf), Accessed Feb. 7th, 2005.