Team #4094

Round and Round We Go

February 9, 2009

Abstract

The study of traffic flow and control has been a fruitful area of mathematical research for decades. Here we attempt to analyze and model the traffic flow that occurs in a traffic circle. We use a powerful macroscopic approach developed by B. Piccoli that uses network analysis and a wave-front tracking algorithm to produce powerful theoretical results with regards to right of way parameters in arbitrarily large networks of traffic circles. We then follow the classical approach of Lighthill and Whitham in order to model the effects of our proposed control system on the dynamics of a traffic circle, employing the Runge-Kutta algorithm to process multiple solutions to the famous conservation PDE with high-accuracy and speed. We found that prioritizing the right of way of the cars inside the circle optimizes the efficiency of the circle, and developed a control system that responds to incoming traffic density in real time and keeps the outgoing flux at a maximum regardless of the number of roads leading to the junction. Next we developed a far more descriptive discrete model, revising the older, standard car-following model, and reaffirmed our original control. We conclude with simulations of our models on specific examples, a reflection of our methods, and a technical description to a Traffic Engineer outlining how and when to use our methods in traffic control development for traffic circles.

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1 Introduction

This paper is concerned with modeling and optimizing traffic flow in a traffic circle. As in past studies of traffic flow (i.e. [1], [3], [4], [7]), we rely heavily on the following conservation law for cars:

$$\rho_t + f(\rho)_x = 0, \tag{1}$$

Where $\rho : \mathbb{R}_+ \times [a, b] \to [0, \rho_{max}]$ is the density of cars (number of cars per unit distance), and $f : [0, \rho_{max}] \to \mathbb{R}$ is the traffic flow or flux across a boundary (number of cars per unit time). The flux and density are related by the following rule: $f(\rho) = \rho v(\rho)$. This conservation assumption has been used in almost every macroscopic model for traffic flow so far, and it makes for a very useful description of the dynamics involved. Here it is applied to the idea of a traffic circle, which is an alternative to a traditional traffic light junction. We note here that no distinction is made in the paper between a traffic circle and a roundabout, even though there are slight technical differences. The rest of our assumptions we list here:

- No cars are created or destroyed, as described by (1)
- Traffic density within the roundabout is uniform.
- Traffic flux into the circle is constant **within a sufficiently small time interval**.
- All drivers that enter the traffic circle will exit without going around the loop multiple times.
- The velocity is a function of ρ and it is zero at ρ_{max}.
- The traffic from incoming roads is distributed on outgoing roads according to time-varying coefficients.
- Drivers attempt to maximize flux when possible.

Assuming traffic flux to be constant during small time intervals may seem like a bit of a stretch, but really it is just a condition that ensures rough continuity of incoming traffic density. In fact, our model allows for a very large number of jump discontinuities, so that the traffic control device can handle a piecewise constant wave of incoming traffic density. The condition on the velocity is one given in most traffic models, and it tests out empirically. We will be employing a velocity equation derived empirically from [8] [3]:

$$v(\rho) = \begin{cases} v_s & , \quad 0 \leq \rho \leq \sigma \\ \beta(\frac{1}{\rho} - \frac{1}{\rho_{max}}) & , \quad \rho > \sigma \end{cases} \tag{2}$$

Here, v_s is the velocity a car would go if it was by itself, in other words it represents the speed limit. The σ is what's known as the "capacity" of a system; it is the density at which the flux is maximum, and the β is a constant chosen to make the function continuous. Note that the function for velocity makes intuitive sense as well; as the density increases, the velocity decreases, until it hits the maximum density, where it becomes zero.

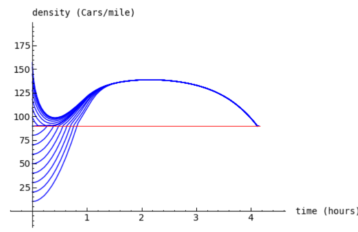


Figure 1: No Control Response to a One Peak Input

The final two assumptions are the basis for the most recent developments in traffic science that examines flows on a network; see [1] and [4] for details. Now, after modeling the system properly, we needed to choose something to optimize. In the past, researchers have looked at outgoing flux, traffic density, queuing time, and decreased possibility of congestion. Given the fact that the only way to reduce queuing times in this case is to increase the outgoing flux, we chose to control traffic density to **maximize the outgoing traffic flow**.

2 The Macroscopic Model without Control

We begin with the case of a single-lane traffic circle with n incoming roads and m outgoing roads. We let f_i represent the flux from the i^{th} incoming road, f^j represent the flux into the j^{th} outgoing road, and we let f be the total outgoing flux, which is what we are trying to maximize. Similarly, we let ρ_i represent the traffic densities of the incoming rows and ρ represent the uniform density inside the circle.
The input parameters are the ρ_i which can be made constants or can vary with time independently of each other. The outgoing flux is then modeled by the following set of equations:

$$f(\rho) = \begin{cases} \sum f_i(\rho_i) & , \quad 0 \leq \rho \leq \sigma \\ \sum f_i(\rho_i)(1 - \frac{\rho}{\rho_{max}}) & , \quad \rho > \sigma \end{cases} \tag{3}$$

Where ρ_{max} corresponds to the density in "bumper to bumper" traffic and $\sigma = \frac{\rho_{max}}{2}$ is the capacity as defined previously. In the model, following standard traffic laws currently in place regarding traffic circles, we assume that all entrances had yield signs placed on them, so that traffic entering yielded to traffic inside the circle. Fig(1) is an example of how an uncontrolled traffic circle with 4 incoming and outgoing roads responds to various initial traffic densities, as well as an incoming flux that varies with time. The red line shows the capacity, σ, and, as you can see, the density rises over capacity, which is not optimal, and this causes traffic congestion.

Apparently we need a control to keep the density from moving outside of some acceptable region of the capacity, and this will optimize the traffic circle. But first, we need to know what type of traffic control we should be looking for.

3 Network Considerations

Here we follow the development of [1] in order to narrow our search for a control procedure in the case of 2 incoming and 2 outgoing roads for a traffic circle. We then generalize these results to the case of n incoming and n outgoing roads. The purpose of this investigation is to show that it is better to give the cars in the circle priority over the cars entering. We will then design a control system that biases heavily towards the cars in the circle and produces a maximum result under a wide variety of incoming traffic densities.

Consider 2 incoming roads, I_1 and I_2, and 2 outgoing roads, I_3 and I_4 with 4 parts to the traffic circle filling each junction, I_{1R}, I_{2R}, I_{3R}, I_{4R}, so that I_{1R} connects I_1 to I_3, and I_{2R} connects road I_3 to I_2, and so on. Suppose that N cars enter at I_1 and αN exit off of I_3 while $(1-\alpha)N$ exit off of I_4. Define β as a similar coefficient for I_2 (i.e. βN cars go to I_4 and $(1-\beta)N$ go to I_3).

Now we define "right of way parameters," q_1 and q_2 so that if only a limited number of cars, C, can be on a given section of the traffic circle at a certain time, q_iC will come from inside the circle, and $(1-q_i)C$ will come from outside the circle. (The indices correspond to which incoming road we are dealing with). So, a high value of the right of way parameter would correspond to a strong bias to the cars already in the circle. With incoming densities constant, [1] gives us the following three results:

Proposition 1. The traffic on the circle never reaches maximum density (never "gets stuck") if the following holds:

$$q_1 \geq \frac{(1-\beta)f_2}{f(\sigma)}, \qquad q_2 \geq \frac{(1-\alpha)}{f(\sigma)}. \tag{4}$$

Proposition 2. If the right of way parameters satisfy

$$q_1 < 1 - \frac{f_1}{f(\sigma)}, \qquad q_2 < 1 - \frac{f_2}{f(\sigma)} \tag{5}$$

Then the circle does reach a maximum density ("gets stuck").

Combining these two results we see that, for low incoming traffic densities (i.e. $f_i < \frac{f(\sigma)}{2}$), whether or not there is a bias for traffic in or outside the circle does not hugely effect the results- as long as the bounds in (4) are kept. However, as soon as the incoming traffic density goes over $\frac{f(\sigma)}{2}$, **traffic outside the cirlce *must* yield the right of way to traffic inside the circle-** otherwise the circle will get stuck.

Indeed it is not hard to see that the same analysis used by B. Piccoli for the 2 by 2 case works for the n by n case, and we get a more general result.

Proposition 3. Given n incoming roads and n outgoing roads, define time-varying coefficients, ϕ_i^j so that

$$f(\rho) = \sum_{i,j} \phi_i^j f_i(\rho_i) \qquad \text{and} \qquad \forall i \quad \sum_j \phi_i^j \equiv 1$$

Where ϕ_i^j represents the percentage of cars from incoming road i that go out of road j. If the n right of way parameters satisfy

$$q_i < 1 - \frac{f_i}{f(\sigma)} \quad \forall i \tag{6}$$

then the circle gets stuck.

And the same conclusion holds as above- **for an arbitrary number of roads coming to the junction, even a moderate level of traffic demands priority to cars inside the circle**. With this in mind we quote the last proposition regarding traffic circles by B. Piccoli.

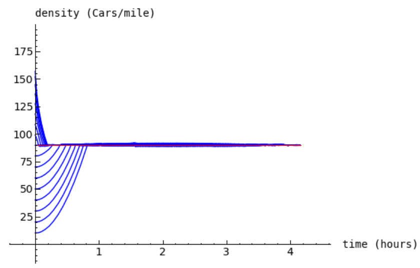
Proposition 4. A multi-lane traffic circle with inner lanes not interacting with outer lanes (i.e. cars can always enter on the outside lane without affecting traffic flow on the inner lane), is isomorphic to a traffic circle that yields to *entering* traffic.

This is certainly surprising- it means that, under heavy traffic, *a multi-lane traffic circle is more likely to get stuck than a single-lane traffic circle*. So, in our optimization schemes, we will only consider single-lane models to deal with control and suggest that traffic circles with multiple lanes should shut down inner lanes during peak traffic volume hours, or get rid of them all together.

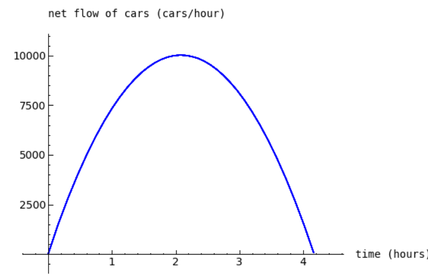
Armed with these theoretical results, which tell us that, under correct controlling of the right of way parameters, we can always avoid traffic getting stuck in a single-lane traffic circle, we now move on to a control system and model.

4 Macroscopic Model with Control

For our control system, we were inspired by a simple idea: the thermostat. We know we have to control incoming traffic, and we know we want the traffic density to be as close to capacity as possible, so we designed a control system that would ensure that- in every possible situation. We start with a normal traffic circle where the rule of thumb is to yield to the traffic inside the circle. But then we add one more level of control- red lights on each of the entrances to the traffic circle. In other words, the flow coming in from a road is either the same as it would be with a yield sign, or zero. Then we measure the traffic density every thirty seconds inside the circle, and the incoming fluxes in each lane: if it is too low, one of the red lights is turned off; if it is too high, one of the red lights is turned on; if it is exactly at capacity, nothing happens. The measuring device used for calculating the density in the traffic circle is described in our directions to the traffic engineer (it is, in fact, a real device that is currently on the market, and would require only one camera), and the measuring tool for incoming fluxes can be any number of simple stopwatch-like devices located on the incoming roads. The details of the algorithm for how to decide which lanes to open up are as follows:



(a) Traffic Denisty Control Response



(b) Traffic Flux Input

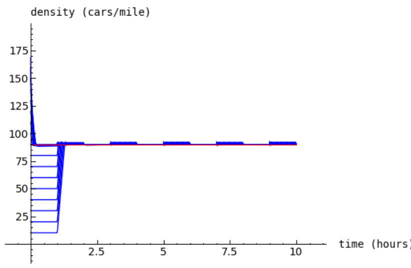
Figure 2: Response Denisty and Input Flux for One Peak

- We measure the traffic density inside the circle, call it ρ_0.
- If $\rho_0 = \sigma$ we do nothing.
- If $\rho_0 > \sigma$, then we list all of the incoming non-red-lit traffic fluxes in order of magnitude, and put a red light on the highest one.
- If $\rho_0 < \sigma$, then we list all of the incoming red-lit traffic fluxes in order of magnitude, and take off the red light on the lowest one.
- Re-measure and repeat every 30 seconds.

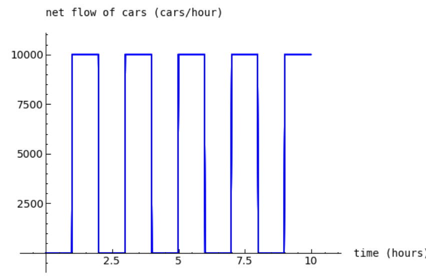
The time step was chosen out of an appeal to reality; changing the lights any faster than thirty seconds would not be feasible for a working traffic system, so we needed a lower bound of 30 seconds. We ran our control model simulation for many different types of input fluxes from the incoming road. First, just to demonstrate that our control does better than just a yield sign, we ran the simulation against the same traffic flux given to produce Fig1. The result is Fig 2, which shows the input flux, Fig 2(b), and the corresponding traffic density in the circle, Fig2(a), with control. Clearly, the control keeps the density within a very close margin of the capacity.

In order to test our control to ensure it could handle the toughest of situations, we tested it against a non-physical incoming flux: a square wave that oscillates from 0 to 10,000 cars/hour every hour for a traffic circle that has 4 incoming and 4 outgoing roads. Even with these instantaneous discontinuities in traffic densities, to epic proportions, our control was able to keep the traffic density very close to capacity, as evidenced in Fig3.

This wasn't enough. We now ran a simulation to test our control on an absolutely ludicrous level of traffic coming from 10 different incoming roads and leaving from 10 different outgoing roads. We let each incoming road have a flux that varied as a square wave. Just to understand how non-physical and ridiculous of a test this is; imagine an

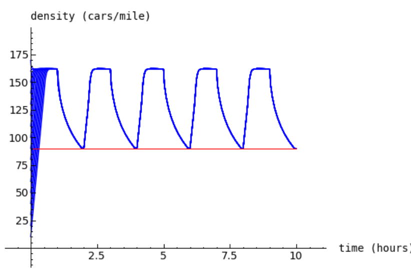


(a) Traffic Density Control Response

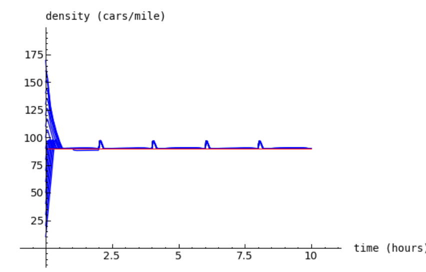


(b) Traffic Flux Input

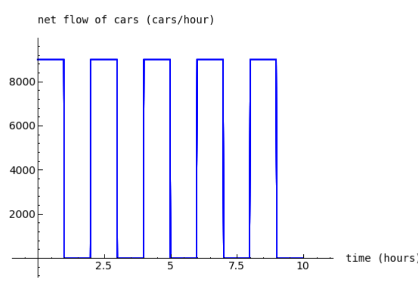
Figure 3: Response Density and Input Flux for Square Wave



(a) Uncontrolled Density

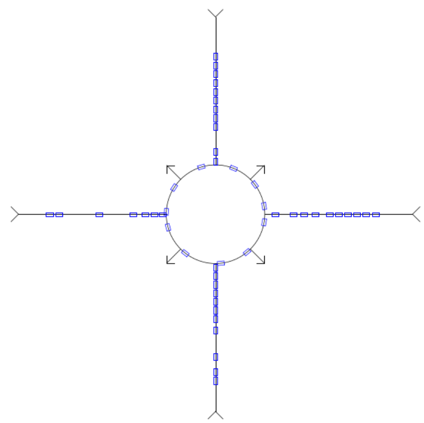


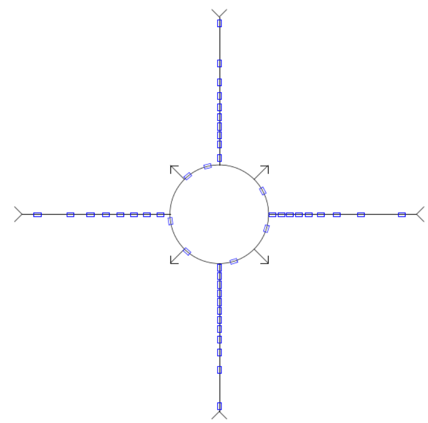
(b) Controlled Density



(c) Square Wave Incoming Flux

Figure 4: Monstrous Incoming Flux With and Without Control- 12 roads





(a) Low Speed Limit (b) High Speed Limit

Figure 5: Microscopic Model Still Shots

empty traffic circle that periodically deals with no traffic, and then a sudden hour-long onslaught of 10,000 cars, all coming in from different directions. Certainly any system that can handle such an overload can do better than just a yield sign. For comparison's sake, we show the controlled and uncontrolled response to the square wave input flux in Fig 4. Notice in Fig 4(b) that our control performs phenomenally against the monstrous wave of cars.

5 The Microscopic Model

In addition to the macroscopic model, we have also formulated a microscopic, discrete model, where each car is modeled individually. It is a variation on a classical "car-following model." Here, however, we take a different approach from previous standard models and focus mainly on the following-distance and acceleration as opposed to the velocity of each individual car. From our animations, this approach deals far better with the stop and go traffic in and around a traffic circle.

The differential equation that we created for this model was:

$$x''(t+T_r) = \beta[(x_{front} - x) - g(x'(t))] \tag{7}$$

Where $x(t)$ is the position function for some car on a road, and $x_{front}(t)$ is the position function for the car directly in front of it on the road. T_r is the reaction time, $g(x'(t)) = \alpha x'(t)$ (where $\alpha > 0$ is some constant) is the desired following distance, and $\beta > 0$ is a scaling factor. Notice that this makes perfect sense: when the car is too close to the car in front, the negative term dominates the difference, and the car reacts after some delay time

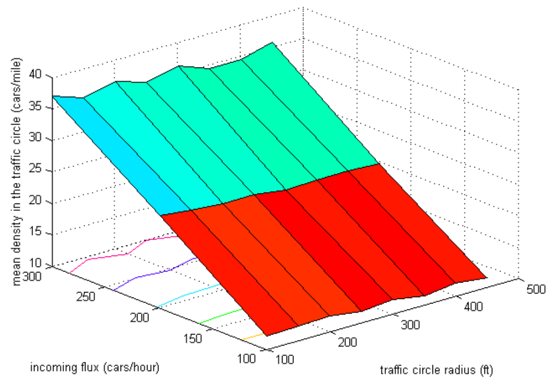


Figure 6: Density for different incoming fluxes and Radii

by decelerating. When the car is too far away, the positive term dominates the difference and so the car reacts by accelerating after some reaction time. Compare (7) to the old standard car-following model (seen in [3] and others):

$$x''(t + T_r) = -\lambda(x'_{front}(t) - x'(t)) \tag{8}$$

Clearly this model is flawed; when people drive, they don't gauge relative *velocities*, they gauge relative *distances*, and they model their behavior to control for that. We wrote up a program that models two incoming and two outgoing roads with an unlimited amount of cars coming in, and then driving around the circle and exiting. We used a random number generator to determine a choice distribution on the entering cars: a certain percentage would take one exit and a certain percentage would take the other. In addition, each car went through a decision-making procedure as it neared the entrance about whether to accelerate on through or to decelerate to a stop, based on its proximity to other cars already in the circle. Combining this with our control system of red lights and yielding produced wonderful results- the control system is equally as effective in this more descriptive, discrete model. This is surprising because even though the results were the same, there are some clear differences with this discrete model. For example, the density inside the ring depends highly on the speed limit in the traffic circle, as evidenced by Fig 5.

We ran over 500 different simulations, changing the radius of the circle and other parameters across a wide range (Notice that the radius does not affect the traffic density in the circle too dramatically as in 6). The data sample is far too large to include here, but a sample of the results can be seen in 7. The straight line is the capacity, the line that oscillates closer to that line is the density results with our control, and the highest line represents the density results without control. **The traffic control system works in the more descriptive model as well**.

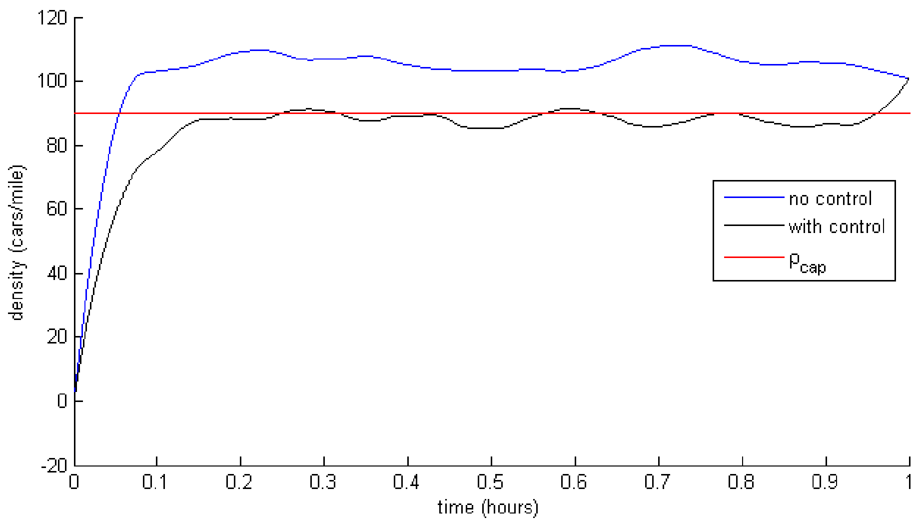


Figure 7: Typical Example of Micro-Model Density Results

6 Directions for a Traffic Engineer

Our control system works for any single-lane traffic circle. Multi-lane traffic circles are dealt with at the end of the section.

6.1 Measuring Traffic Density

In order to implement our control system, it is necessary to have the ability to analyze average traffic density inside the circle every thirty seconds. This is easily achieved with the installation of a camera that can view the entire traffic circle. Once this camera is installed, it must be programmed to count the number of cars it sees in the circle and divide by the pre-determined length of the circle. Many programs and camera systems like this already exist; one example can be found in [9]. The capacity can then be computed as $\frac{\rho_{max}}{2}$, where ρ_{max} is just the density that corresponds to the traffic circle filled with average-sized cars bumper to bumper.

6.2 Measuring Traffic Flow

This topic has been studied extensively, and is already employed in traffic control currently. Traffic flow must be measured on each of the incoming roads *away from the entrances.* It is important that the incoming flux is measured on the road away from the entrance, otherwise the traffic flow would be zero on all of the roads when there is high enough traffic congestion, and this would confuse the control system. The flow measuring device should be placed as far away as possible from the entrance without running into previous junctions- this will ensure the best results for the control system.

6.3 Programming the Control

Install red lights on each of the entrances to the traffic circle. Let the rule for incoming cars be: "Yield to traffic in the circle unless there is a red light. if there is a red light, stop." Set the measuring devices to send in data every thirty seconds to a traffic control box. The traffic control box then separates the incoming road data into two sets: roads with a red light currently on and roads with a red light currently off. The data is ordered by magnitude within each set. If the density inside the circle is higher than the capacity, then the traffic control box sends a signal to the greatest member in the set with red lights off to turn the red light on. If the density inside the circle is lower than capacity, then the traffic control box sends a signal to the least member in the set with red lights on to turn the red light off. If the density inside the circle is exactly at capacity, then the traffic control box sends a signal to tell all lights to remain in the current configuration. As stated before- this process repeats every 30 seconds.

6.4 Multiple-Lane Traffic Circles

Due to the results in Section 3, it would be perfectly reasonable to shut down the inner lanes of the traffic circle to avoid the almost inevitable situation of cars getting stuck inside during high traffic flow. If this option would not be feasible, then a survey of the traffic flow through the circle would be necessary. At peak traffic flow hours, the inner circle should be shut down, but during low traffic, it would be excusable (though not optimal), to allow it to remain open. During all these times the current control should be used as well.

7 Conclusion

We successfully modeled a single-lane traffic circle and designed a control system that keeps the traffic system running smoothly at very low operating cost. But we should note some of the limitations here. A full-fledged test of our first model on multi-lane traffic circles was not achieved. We also did not consider the possibility of pedestrian crossing delays in our model. Furthermore, it has been shown that the original model proposed by Lighthill and Whitham does not hold up to reality in certain occasions, so we cannot know if the control system truly works until it is tested in a real world situation. Much of the difficulty in traffic flow analysis arises for the same reasons that fluid dynamics is difficult: the differential equations governing the system are very difficult to deal with analytically. Indeed, most complete traffic flow analyses invoke Eulerian dynamics and the *Navier-Stokes equations*- for which some analytic descriptions merit a Millenium Prize.

Fortunately, the relatively new frameworks and models proposed by B. Piccoli and others provides a promising new field of inquiry in modeling traffic systems, and allows for arbitrarily complex configurations. Our strongest results came from the use of this model, and allowed us to focus our efforts on controlling the right-of-way parameters in the traffic circle. However, the correct definition and rigorous use of right of way parameters has yet to be hammered out and standardized, and the original use of these in [1] has been critiqued in [4], so the use of these results in our paper stands on shaky theoretical framework. Nonetheless, it is difficult to doubt the power and usefulness of our model with control given its exceptional ability to handle various onslaughts of traffic flows. Indeed, with the addition of our descriptive, albeit computationally costly, discrete model we managed to strengthen the prospect of our control as a valid traffic procedure.

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