Optimization of the Brownie Pan

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Abstract

In this paper we address the effect that pan shape has on uniformity of heat distribution, as well as efficiency of spacial utilization. Specifically, we address the family of super ellipses known as Lamé Curves, analyzing the midway shapes between squares and circles. Regrettably, due to an unfortunate overlooking of an assumption which turned out false, our analysis is incomplete. So instead, we addressed some more creative answers that focus on solutions to these issues, rather than mere compromise.

Contents

1   Introduction

A brownie is a terrible thing to waste. It is a shame how so many corner pieces have lost the fight to remain soft and moist simply because of a non-optimal pan design. Of course, a cylindrical pan avoids these dangerous corners, but the space in the oven is no longer used efficiently, and again, a brownie is a terrible thing to waste.
It is with these issues in mind that we have developed a model to find the optimal baking solution. We focus on a compromise between nice, evenly-cooked brownies, and maximal brownies per batch.
In the following sections we outline how we parameterize this compromise in terms of superellipses and individually examine the effects on uniformity of heat transfer throughout the pan, as well as efficiency of packing in the oven. We then examine these two function simultaneously in order to maximize the benefits of both extremes.

2   Theory of Operation

We focus our analysis on the family of superellipses commonly known as Lamé Curves. These are curves of the form

$$\left| \frac{x}{a} \right|^n + \left| \frac{y}{b} \right|^n = 1, \tag{1}$$

where n is some integer greater than 2, a, b are real numbers, and $a = b$. For our optimizations, we hold the area of the pan fixed, regardless of pan shape. The area for a superellipse is given as follows:

$$A = 4ab \left(\frac{\Gamma \left(1 + \frac{1}{n} \right)^2}{\Gamma \left(1 + \frac{2}{n} \right)} \right), \tag{2}$$

where Γ is Euler's gamma function. So the constants a, b of our curve will be

$$a = b = \sqrt{\frac{A}{4} \cdot \left(\frac{\Gamma \left(1 + \frac{2}{n} \right)}{\Gamma \left(1 + \frac{1}{n} \right)^2} \right)}. \tag{3}$$

Note also that as n approaches infinity, the graph of the image tends to a square. A proof of this fact is given in the appendix.
In the following sections we examine how uniformity of heat distribution as well as efficiency of space usage is affected with changing variable n.

2.1 Heat Distribution Optimization

To analyze the heat distribution in various pans, we use a scheme similar to that of solving the heat equation in a cylinder. Notice that for Lamé curves of degree greater than two, although we know that the boundary $|x|^n + |y|^n = a^n$ is held at the temperature of the oven, parameterizing that boundary in a manner that is useful in solving the heat equation is difficult. To combat this, we impose a change of coordinates, sending

$$x = r \cos^{2/n} \theta; \quad y = r \sin^{2/n} \theta. \tag{4}$$

Furthermore, we can remove the absolute values from x and y, by only examining $x > 0$ and $y > 0$. Values in the other quadrants will be similar by symmetry.

After this change in coordinates, our Lamé curves will be mapped to a circle around the origin of radius a. Due to the symmetry of the image we can assume that the temperature T at any point in the brownie pan is independent of θ (NOTE: As it turns out, this is assumption does not hold). We proceed as in section 2.2 in Wilkinson, but instead considering the pan whose edge is defined by $x^n + y^n = a^n$, $0 \leq z \leq Z$, where a is a constant depending on the predefined area and the degree, as in the previous section. We examine the homogeneous equation

$$u(r, z, t) = T(r, z, t) - T_b, \tag{5}$$

where T_b is the oven temperature, and impose the boundary conditions

$$u(a, z, t) = u(r, 0, t) = u(r, Z, t) = 0. \tag{6}$$

We let T_i be the initial temperature of the brownie batter, so

$$u(a, z, 0) = T_i - T_b. \tag{7}$$

We apply these conditions to the following heat equation:

$$\frac{\delta u}{\delta t} = D \nabla^2 u = D \left(\frac{1}{r} \frac{\delta}{\delta r} \left(r \frac{\delta u}{\delta r} \right) + \frac{\delta^2 u}{\delta z^2} \right). \tag{8}$$

where D is a constant relating to thermal conductivity and other physical variables which we are holding constant. Then, we can employ separation of variables to find the general solution

$$u(r, z, t) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} A_{km} \sin \left(\frac{(2k-1)\pi z}{Z} \right) J_0 \left(\frac{r}{a} j_{0m} \right) e^{-\lambda_{km} D t} \tag{9}$$

where $\lambda_{km} = \left(\left(\frac{(2k-1)\pi}{Z} \right)^2 + \left(\frac{j_{0m}}{a} \right)^2 \right)$, for k, m positive integers, J_0 defines some Bessel function of the first kind, and j_{0m} is the mth root of J_0. Then applying our initial

and boundary values we solve for A_{km}, and arrive at the solution

$$T(r, z, t) = T_b + \frac{8(T_i - T_b)}{\pi} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin\left(\frac{(2k-1)\pi z}{Z}\right)}{(2k-1)} \frac{J_0(\frac{r}{a} j_{0m}) e^{-\lambda_{km} D t}}{j_{0m} J_1(j_{0m})}. \tag{10}$$

Inverting our change of coordinates, we find that

$$T(x, y, z, t) = T_b + \frac{8(T_i - T_b)}{\pi} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin\left(\frac{(2k-1)\pi z}{Z}\right)}{(2k-1)} \frac{J_0(\frac{\sqrt[n]{x^n + y^n}}{a} j_{0m}) e^{-\lambda_{km} D t}}{j_{0m} J_1(j_{0m})}. \tag{11}$$

Using this solution, we can plot how variable n affects temperature at various points and times throughout the brownies.

2.2 Spacial Optimization

Under the assumption that the oven has dimensions L, W where L is the length and W is the width, we map the rectangle into a square using

$$\begin{bmatrix} \frac{L}{W} \\ \\ \frac{W}{L} \end{bmatrix}.$$

Thus both W, L map to 1 and we are now dealing with a unit square.
Each Lamè curve can be superimposed on top of a square with the same width and height. Using the equation for the area of a Lamè curve, we can write

$$A = 4a_n^2 \left(\frac{\Gamma\left(1 + \frac{1}{n}\right)^2}{\Gamma\left(1 + \frac{2}{n}\right)} \right), \tag{12}$$

where $a_n = \frac{1}{2} s_n$ and s_n is the side length of the superimposed square. Define c_n as

$$c_n = 4 \left(\frac{\Gamma\left(1 + \frac{1}{n}\right)^2}{\Gamma\left(1 + \frac{2}{n}\right)} \right). \tag{13}$$

Analyzing extra area, w_n, of the superimposed square, we arrive at the function

$$w_n = \frac{1}{4} s_n^2 (4 - c_n). \tag{14}$$

Using this equation, we find that $w_n > w_{n+1}$ thus we know that, per superimposed square, Lamè curves with higher degree n will have less extra area per square and thus are more space efficient individually.
In optimizing the space efficiency, we will need square packing. As of now, there is no algorithm for this, thus we will need to proceed on a case-by-case basis. We will operate under the assumption that an oven tray cannot hold more than 50 pans.

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3   Methodology

It is apparent that the most uniform heat distribution will be from using cylindrical pans, whereas the best use of space will be from square pans. Our goal in this section is to analyze just how quickly we lose efficiency when straying from these extremes. In order to do this, we assign to each n an efficiency rating for uniform heat distribution, H_n, and an efficiency rating for spacial utilization, S_n.

We set H_2, (the circle) to be 1, and H_∞ (the square), to be 0. Similarly, we set $S_2 = 0$, and $S_\infty = 1$.

We can then consider these efficiency ratings simultaneously and decide on the brownie pan that captures the benefits of squares and circles most effectively.

3.1   Numerical results of heat distribution optimization

To assign H_n, we take measurements of temperature from the heat equation derived in section 2.1 at points a small, fixed distance from the boundary. This will be constant for $n = 2$. We call this constant α_2. For all other n, we measure the least squares distance of the temperatures from the heat equation and the horizontal line at α_2. Let ϵ_n be this least squares measurement. Then we can map each n to a point in $[0, 1]$ as follows:

$$H_n = 1 - \frac{\epsilon_n}{\epsilon_\infty}. \tag{15}$$

3.2   Numerical results for optimal spacial usage

Using square packing data, we can find the side lengths of the pan when given the number, m, of pans to fit in the oven. In the data, we found the optimal side length for packing m unit squares inside a square was given as size t, thus the side length $\frac{1}{t}$ is the maximum side length to fit m pans inside the unit square.

4   Simulations

As for simulations of the optimization, we have none. But we did make brownies and concluded that the corners of brownies are indeed crispier. Furthermore, we noticed that as we increase cooking time, the entire brownie gets hard and relatively more uniform regardless of the shape of the pan.

5   Conclusion

5.1   Limitations

Our most obvious limitation is the restricted geometry of the pan. We can see that Lamé curves are non-optimal when the square root of the area does not divide both the width and the length of the pan. To provide a more complete picture for arbitrary oven sizes, general superellipses (as well as other shapes) should be considered.

In this study we assumed the the diffusion rate remains constant throughout the baking process, whereas previous studies have suggested otherwise (Wilkinson 2008, Olszewski 2006). Extensions of this study could take into account this non-linearity. Finally, we assume that the boundary values of the pan in the oven are constant (at oven temperature), ignoring any temperature fluctuation or any internal heat generation.

These limitation are all due to the difficulty faced in solving non-linear partial differential equations on arbitrary geometries.

5.2   Results

Unfortunately, we were not able to compute any results (see abstract).

6   Other solutions

Due to the failure of our previous analysis, we now present two examples of more creative answers:

6.1   Hilbert's Space Filling Curve

Although less suited for the real world (and thus more difficult to market), we turn to fractals for a solution. We noticed that the brownies in square pans were burnt as a result of the large curvature in the corners. We hypothesize that uniformity of heat distribution is related to change in curvature. With this in mind, instead of minimizing this change and dealing with space utilization as a separate issue, we thought of maximizing the change in curvature and as a result the analysis becomes trivial. Because of Hilbert's Curve's space filling nature, a square of area A with this curve cut out will have A brownie area. Thus, we choose a pan whose side lengths are aW, aL, such that $A = a^2WL$. Then we can achieve maximal space utilization. Furthermore, because of the fractal patterns, the distance between corners goes to zero, and effectively every point within the pan becomes a corner. Thus, the heat distribution is also uniform, although the brownie will end up uniformly burnt. Thus, this space filling fractal is apparently the most efficient design, as it maximizes both variables.

6.2   Wire mesh

This solution is actually implementable. We propose that we should not worry about any of this silly compromise. Uniformity of heat distribution and space utilization, represented as percentages, are both bounded values and so have maximums. So why not maximize both? Again it is apparent that the pan shape from the previous section is optimal in spacial utilization, while non optimal in uniformity of heat distribution. So we propose a solution that maintains shape while fixing this heat issue. We propose fixing a wire mesh to the inside of the brownie pan. The diffusion from metal to metal is much larger than the diffusion of heat into the brownie. We must then design this mesh to provide internal heat generation in a manner which counteracts the hotter corners, creating a uniform distribution of heat throughout the brownie.

A   Proof that Lamé Curves approach a square

The graph of a Lamé Curve approaches a square as $n \to \infty$.

The equation for a Lamé Curve is

$$\left|\frac{x}{a}\right|^n + \left|\frac{y}{a}\right|^n = 1, \tag{16}$$

where n is some positive integer and a is a real number. By raising both sides to the power $1/n$ and pulling out the constant a, we get the equation

$$(|x|^n + |y|^n)^{1/n} = |a|. \tag{17}$$

If we let $\mathbf{x}$ be the vector $\mathbf{x} = (x, y)$, then equation (17) can be interpreted as the set of all points such that the n-norm of $\mathbf{x}$ is equal to $|a|$:

$$\|\mathbf{x}\|_n = |a|. \tag{18}$$

We shall take the definition of the infinity-norm of $\mathbf{x}$ to be the maximum of its components, that is,

$$\|\mathbf{x}\|_\infty = \max\{x, y\}. \tag{19}$$

We will now show that the limit of the n-norm as n approaches infinity is in fact the infinity-norm.

Without loss of generality, take $\|\mathbf{x}\|_\infty$ to be $|x|$. The value of $|x|^n$ is clearly less than or equal to that of $|x|^n + |y|^n$ for all positive integers n. If we raise both sides of this inequality to the power $1/n$, we find that $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_n$ for all n. Thus, we find that

$$\|\mathbf{x}\|_\infty \leq \lim_{n \to \infty} \|\mathbf{x}\|_n. \tag{20}$$

Again, without loss of generality, assume $\|\mathbf{x}\|_\infty$ to be $|x|$. Pulling $|x|$ out of the equation for the n-norm, we have

$$|x| \left(1 + \left| \frac{y}{x} \right|^n \right)^{1/n}. \tag{21}$$

for all positive integers n. Since $|x|$ is the infinity-norm of $\mathbf{x}$, $|y|$ must be less than or equal to $|x|$, and so $|y/x|$ must be less than or equal to 1. Thus, we have

$$\|\mathbf{x}\|_n = |x| \left(1 + \left| \frac{y}{x} \right|^n \right)^{1/n} \leq |x| \cdot 2^{1/n} = \|\mathbf{x}\|_\infty \cdot 2^{1/n}. \tag{22}$$

The limit of $2^{1/n}$ as n approaches infinity is simply 1, so the limit of equation (22) as n approaches infinity is

$$\lim_{n \to \infty} \|\mathbf{x}\|_n \leq \|\mathbf{x}\|_\infty. \tag{23}$$

Looking at equations (20) and (23) together, we find that

$$\lim_{n \to \infty} \|\mathbf{x}\|_n = \|\mathbf{x}\|_\infty, \tag{24}$$

and thus equation (18), as n approaches infinity, represents the set of all points (x, y) in $\mathbb{R}^2$ such that the maximum of x and y is equal to $|a|$; in other words, a square centered on the origin with edges parallel to the axes located $|a|$ units from the origin. $\square$

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