

In this paper we present a method for determining the leaf mass of a tree from the height of the tree. Our model takes advantage of the field of allometry, the study of physiological size and shape. In particular, Kleiber's law, which states that metabolic rates in living things scale to mass to the power of three-quarters, is the basis for the theoretical component of this method. With some additional assumptions to those required by Kleiber, we were able to provide a rationale for the relationship  $M \propto H^2$  where  $M$  is the leaf mass and  $H$  is the tree height, which we obtained through analysis of data compiled in a previous study.

The assumptions we made were rational; the main ones are that the structure of trees is fractal-like or that they branch in a consistent manner (this is the same assumption made in the derivation of Kleiber's law), the trees in question were regular (which we defined to be adult, non-pruned, and in a non-extreme environment), and that photosynthesis is directly proportional to the area of the leaves.

The overall model is a synthesis of smaller components. The Leaf Distribution model roughly describes the distribution of leaves within the crown of a tree, primarily that they are not distributed uniformly with respect to mass. The Vascular Networks model replicates the argument for Kleiber's law to attain a relationship between the height of the tree  $H$  and the number of branches,  $H \propto n^{N/3}$  where  $n$  is the number of branching levels and  $N$  is the last branching level. Lastly, the Branch Distribution model describes the relationship between the volume of the crown and the mass of the leaves, which in turn again relates to the number of branches:  $M \propto n^{2N/3}$ . The synthesis of these models yields the above stated equation.

Since our model springs from data, analysis involved applying the formula to a specific subset of data. We choose to analyze the data for the species *Cryptomeria japonica* because it had a larger number of valid statistics. The model held up well, with a standard deviation of 5.416. This is better than estimations obtained in other studies using a volume/density approach. From a theoretical standpoint, the history of allometry attests to the validity of our model. And even though we are applying a general biological rule to individuals, the ecological fallacy does not apply as fully to this situation because of our relatively weak assumptions in establishing the theory. However, a weakness could be that a larger data sampling yields a different number, which would make our model obsolete. Additionally, our assumptions are not proven standards in dendrology. But for now, the model remains accurate and reasonable.

# Finding the Leaf Mass of a Tree

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## **Abstract**

In this paper we present a model for obtaining the leaf mass of a specific tree. We use allometric scaling to establish relations between the more readily measured height of the tree and the leaf mass without damaging the tree. The model is an explanation of results obtained from data analysis based upon established allometric relations and assumptions concerning the process of photosynthesis.

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# 1 Introduction

The leaf mass of a tree has several consequences in the field of dendrology and ecosystems study as a whole, as it corresponds to the amount of photosynthesis that takes place. From this can be obtained carbon dioxide consumption, an important factor in the study of global warming. Methods for obtaining the leaf mass of a region of forest have been established using satellite imagery or other advanced means, but they are not effective in measuring the leaf mass of a single tree [5]. This is an issue, for example, when a city is determining the ecological benefits of adding a specific type of tree to an area as a standalone plant. The only experimental way of obtaining the leaf mass is to remove the leaves and weigh them, a process both tedious and destructive. Instead, we have established a relationship between the mass of the leaves and the height of the tree using allometric scaling and regression on data compiled in previous studies.

## 1.1 Introduction to Allometry

Allometry is the study of relationships between physical features. One branch of plant allometry is based upon the tube structure for transporting nutrients within the plant. An explicit description for the derivation of the scaling relationship is described in multiple sources [1, 3]. The most famous of these relationships is the three-quarters relationship, sometimes called Kleiber's law, which is derived from assuming that the cross-sectional branch area of each branching level are equal. This law states that

$$B \propto M^{3/4}$$

where  $B$  is the metabolic rate and  $M$  is the mass of the animal (in our case, a tree). Since the metabolic rate of a tree is almost entirely dependent upon the leaves, the leaf mass is similarly proportional to the total tree mass [2]:

$$M_L \propto M^{3/4}$$

This serves as the necessary background for our model. While it is not easy to measure the mass of a tree, it is possible to find another relationship between the mass of the leaves and the height of the tree.

## 1.2 Notation and Definitions

Here we define both biological terms and the notation used throughout the paper:

### 1.2.1 Ecological Terms

- Photosynthesis: the process in which plants turn  $CO_2$ , water, and sunlight into energy
- Crown: leaf bearing profile of a tree

### 1.2.2 Notation

- $M$ : the mass of the leaves
- $H$ : height of the tree
- $n$ : the number of daughter branches per parent
- $N$ : the number of branching levels in the tree
- $l_k$ : the average length of the  $k$ -level branches
- $V$ : the volume of the leaves

## 2 Method Descriptions

### 2.1 Assumptions

It is necessary for our model to make the following assumptions:

- The tree in question is mature, since young trees exhibit slightly varied scaling.
- The tree in question has not been pruned extensively, as this will affect the branching pattern heavily.
- The trees we studied are regular with respect to leaves, which we define to mean that the leaves are neither too large nor small in number because of strong sunlight, high temperature or other extreme environmental factors.
- The branches are, on average, distributed uniformly within the crown of the tree. This is valid since the number of branches becomes quite large with each successive branching level and because trees tend to grow roughly equally in all directions. Similarly, we assume that the number of leaves per branch on each branching level are equal. This also contains the hidden assumption that the tree has not been pruned.
- Since photosynthesis takes place primarily in leaves, the overall occurrence of photosynthesis within a tree is proportional to the mass of leaves.
- The production of photosynthesis is proportional to the effective area of the crown, which is defined by the projection of of the crown onto the ground.

### 2.2 Data

We used the data assembled by Brian Enquist in his *Biomass Allocation and Growth Data of Seeded Plants*. It had a wide range of data, even including some non-woody plants. We filtered out the non-trees and those trees that were younger than 3 years old to assemble a little over 500 individual trees to perform

data analysis on. Using a wide scale regression, we found that the relationship between height and leaf mass was given as follows:

$$\log(\text{Leaf Mass}) = 1.96 \log(\text{Height}) - 1.6 \quad (1)$$

with an  $R^2$  term of .78. We interpreted this as a squared relationship and found a theoretical model that matches this data.

### 2.3 Leaf Distribution Model

Leaves in the upper part of the crown are smaller and thicker, while leaves in the lower part of the crown are larger and thinner. This is because for any particular leaf, the upper part is mainly composed of cells responsible for taking in sunlight and the lower part is mainly composed of cells responsible for absorbing carbon dioxide. The thickness is related to photosynthesis. Therefore, as long as the sunshine is more stronger on the top of the tree, the leaves are thicker and smaller to avoid overheat and to minimize self-shading. Likewise, since the sunlight is weaker for lower part in the crown, the leaves are thinner and larger to increase the possible amount of photosynthesis [7]. In summary, the distribution of leaf mass within the crown is not assumed to be constant (although the number of leaves on each level is assumed to be constant).

### 2.4 Branching Vascular Networks Model

Here we introduce the formal aspects of the scaling law as it applies to the leaf mass model. We will introduce the relations between the height of the tree  $H$ , the number of leaves  $L_n$ , and the maximum level of branching  $N$ . From the assumptions of the allometric scaling law,

$$l_k = l_{k+1}n^{1/3} \quad (2)$$

where  $l_k$  is the average length of the branches at the  $k$ th level and  $n$  is the number of daughter branches each parent has. This series is multiplicatively telescoping; in other words,

$$l_{k+1} = l(n^{(N-k)/3}) \quad (3)$$

where  $l = l_N$ . The height of the tree is given by the sum of the  $l_k$ 's:

$$H = \sum_{i=0}^N l_{i+1} = \sum_{i=0}^N l(n^{(N-i)/3}) \quad (4)$$

which is a geometric series. Therefore

$$H = l \frac{1 - n^{N/3}}{1 - n^{1/3}} \quad (5)$$

This can be simplified to  $H = A + Bn^{N/3}$  where  $A$  and  $B$  are some constants, since for any given tree  $n$  is a constant and by the assumptions of tube based allometry, so is  $l$ . Therefore

$$H \propto n^{N/3} \quad (6)$$

## 2.5 Branch Distribution Model

Branches are assumed to have a uniform amount of daughters by layer in the crown of a tree. For this model, assume the distribution of leaves with respect to the crowding level is uniform in the crown (even though this contradicts the Leaf Distribution Model). Since we assume the leaves are uniformly distributed in the crown, we can conclude that the number of the leaves is proportional to the volume of the leaves, and since the number of branches is  $n^N$ ,

$$V \propto n^N. \quad (7)$$

Finally, we assume that the effective area of the leaves, defined as the projection of the crown onto the ground, is proportional to the photosynthesis that takes place in the leaves, and that this photosynthesis is proportional to the mass. Treating the crown as a solid ball, we can conclude that the mass of the leaves  $M$  is

$$M \propto n^{2N/3} \quad (8)$$

## 2.6 Unifying Model

We have already assumed that the distribution of leaf mass is not consistent in the crown. However, mass is related to the effective area for photosynthesis, so as a result of the Branching Vascular Networks Model and the Branch Distribution Model, we have equations 6 and 8. Combining the two gives the relationship between the leaf mass relative to the height of the tree:

$$M \propto H^2 \quad (9)$$

Taking the logs of these equations gives the final model:

$$\frac{\log M}{\log H} = 2 \quad (10)$$

# 3 Application

As a single species selection from our data we chose to analyze *Cryptomeria japonica* based on its larger number of valid trees [4]. Our theoretical formula for leaf mass is  $M = cH^2$ , where  $c$  is some constant. Thus, there are two things we need to do:

## 3.1 Finding the Coefficient

First we must find  $c$ . Using Matlab to plot and find the regression lines for the comparison of tree height to leaf mass yielded the results in figure 1. The theoretical equation had a standard of deviation from the data of 5.416. This establishes the relation between the height of the tree and the leaf mass. So the leaf mass problem has been reduced to finding the height of the tree. We suggest making use of the shadow to measure the height of the tree.

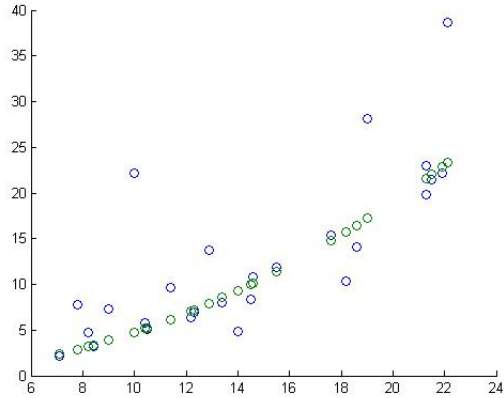


Figure 1: The blue points are real measured values for height vs. leaf mass of the *Cryptomeria japonica*, while the green points are the estimated values.

### 3.2 Measuring Height

Using similar triangles, the height of a tree can be measured relatively easily. Let  $Y$  denote the height of the tree, and  $X$  to be the length of its shadow at any set time. Measure 1 meter up the trunk of the tree (if no such place exists, the tree can probably be measured directly) and determine the length,  $A$ , of the shadow cast by this 1 meter segment of tree. With these measurements in place, the height of the entire tree is given by the equation

$$Y = X/A \quad (11)$$

## 4 Evaluation

### 4.1 Theoretical Analysis

The most distinct advantage of our model is that we have strong theoretical support for the validity of the scaling law. This law was first discovered 70 years ago, and has therefore been tested by many both experimentally and theoretically. It is widely accepted, and therefore highly valid.

Additionally, we avoided harsh assumptions that would constrain the possible types of trees. Most of the assumptions we did make could be accounted for as well in a more general model; for example, younger trees could be analyzed independently in a similar way to how we analyzed adult trees. Our model will tend to become less accurate if the leaf mass of a given type of tree is irregular for some reason that violates our assumptions. But generally speaking, our model should always perform relatively well.



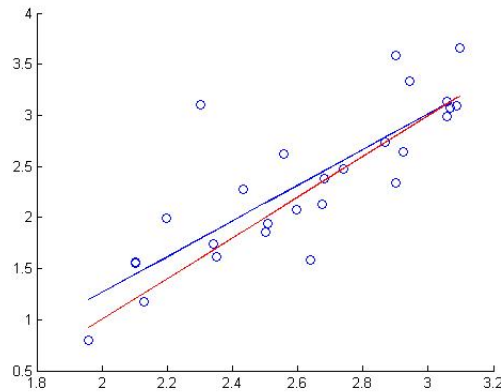


Figure 2: The black line is the regression line of the data, while the red line is that derived from the formula. Visually, these lines are close.

However there is a problem with the fact that our model is based of data. While the model is reasonable in its assumptions, if more data was compiled and analyzed to different results then our model would be defunct. We admit to a lack of irrefutable theoretical reasoning.

Consideration has also been given to the problem of ecological fallacy, since we are applying a statistical derivation to the analysis of single trees. We have concluded, however, that this particular inaccuracy is lessened because of the fact that the scaling law should apply to every tree (except for perhaps those listed above). Theoretically, error due to ecological fallacy should not play a tremendous role under our model.

## 4.2 Empirical Analysis

We again use the 27 data points of the *Cryptomeria japonica* from before. Now we use them to plot a log log diagram. The Y-axis is the log of the mass and the X-axis is the log of height. We can easily see that the error between the different lines is quite small, and that our model holds up to the data here.

However, it is almost circular to reason that our data-based model is valid because it matches the data. Realistically, more data needs to be compiled and compared to the model, as this is the strongest means of validation.

## 4.3 Alternative Models

We did consider a few other methods, and a few of them are listed here.

### 4.3.1 A Volume/Density Approach

Research has been done into approximating the volume of a the crown and multiplying this by a measured leaf density within the crown. However, this method seems to be less accurate and impractical for multiple reasons. Diversity in leaves could lead to errors, since the distribution of leaves within the crown varies. While we assumed that the branches are relatively evenly distributed, we did not assume a similar property for leaves due to the heteroblasty in trees, or tendency to have different shaped leaves within the same tree. Leaves in the outer layer of the crown are smaller and thicker in order to minimize self-shading and maximize the amount of photosynthesis[7]. Likewise, leaves in the lower part of the crown are larger and thinner. This makes the method of finding the mass through an estimated crown volume and leaf density inaccurate. On the extreme end, the differences between the leaf spacing of conifers and angiosperms show the disparity. Complex computer imaging may advance this method, but this is beyond our scope.

Additionally, this method would have required measuring leaf density within trees, and having good data to compare it to, which would have been impractical.

### 4.3.2 Measuring Photosynthesis Directly

For our particular model, a major assumption we made is that the shape, size, and arrangement of the leaves tend to maximize sun exposure. This is because natural selection tends to maximize tree photosynthesis at minimal metabolic cost while at the same time adapting to environmental pressures. If it were possible to accurately measure the photosynthesis that occurs within a tree, a good estimate could be obtained on the leaf mass.

However, technological constraints arise when pursuing this methodology. For example, trying to measure the  $CO_2$  consumption of a tree involves surrounding it in a controlled mylar bubble [6], which is impractical for large trees. Another method is using radiation measures to attempt to quantify the sunlight absorbed. This is beyond the capabilities for most people however.

## 5 Conclusion

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## A Letter to the Editor

In this paper, we propose a model for finding the leaf mass of a single based on the height of the tree. Most studies on this topic either focus on finding the leaf mass of whole forest canopies rather than individual trees, or on a volume/density approach that seems to be statistically inaccurate. While advancements in these methods could eventually lead to a more stable model, currently the theory of allometry provides the best means of approximating the leaf mass.

We start off by making some basic assumptions about the profile of the tree, such as a uniform branching structure and a proportional relationship between leaf mass and photosynthesis. Based on these assumptions and combined with the allometric scaling theory, we make three models which together lead to estimation of the leaf mass. In theory, we derived a relationship between the leaf mass and the height of the tree, which we found by data analysis to be, to be given by  $M = cH^2$  where  $c$  is some constant. The strengths of this method are support from the data, historical acceptance of allometric principles, and a reasonable theory behind the model. Some weaknesses are that the theory is dependent upon the data, and if a larger collection of statistics could be collected that change the numbers, the theory would have to be labeled as inaccurate.

Reasons for attaining the leaf mass of a specific tree mostly focus on the environmental influence a single tree can make. Since the leaf mass is directly related to the amount of photosynthesis and therefore the amount of carbon dioxide consumption of a tree, ecologists and policymakers can use the leaf mass of a single tree in determining what type and how many standalone trees to plant.

Thank you for your consideration.