

Getting Some Big Air

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Abstract

In this paper we address the problem of optimizing a ramp for a snowboarder to travel. Our approach is two sided. We first address the “forward problem” of modeling the motion of a snowboarder on a ramp assuming a given ramp and initial conditions for the snowboarder. We derive a second order ODE which we solve numerically. The second aspect of our approach, and arguably the more interesting and useful, is the “variational problem”, i.e. finding the optimal ramp for a snowboarder to travel on, given initial conditions for the snowboarder and a given jumping strategy for the snowboarder. To do this, we consider the space of all possible halfpipe curves and maximize final angular velocity by applying an adapted version of the Euler-Lagrange equations in several variables. We use Taylor series approximations to get a differential equation that is numerically tractable. The solution to this differential equation traces out a curve which represents an approximation to a candidate solution for a local extremum to the vertical velocity of a snowboarder on the halfpipe.

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1 The Forward Problem

1.1 Statement of the Problem

We define the “forward problem” of the snowboarding problem to be the problem of finding the position of the snowboarder as a function of time if we are given the shape of a halfpipe and the initial velocity and position of a snowboarder on the bottom of the halfpipe. To do this, we take the function $\phi : I \rightarrow \mathbb{R}^2$ (where I is some interval) to be a “sufficiently nice” function (infinitely differentiable, one-to-one, and with nonzero derivative) whose image represents the shape of the halfpipe. To find the position of the snowboarder on the ramp as a function of time, we essentially want to find some function $\psi : J \rightarrow I$ such that $\phi \circ \psi$ represents the position of a snowboarder on the halfpipe.

A technique skilled snowboarders use to increase the height of their jump on a halfpipe is known as “pumping”. To apply this technique, a snowboarder crouches as they approach the halfpipe, and then stand up as they travel up the halfpipe, thus changing their moment of inertia [3]. To model the snowboarder pumping (or changing their height in any way), we will define a function h to represent the height of the snowboarder. We define h to be a function of the position of the snowboarder on the ramp, and we assume that a skilled snowboarder could control their body to stand up while snowboarding in a manner described by h . We define the problem below:

Given:

- A parametrization of the ramp, $\phi : I \rightarrow \mathbb{R}^2$,
- The initial velocity v_0 of the snowboarder.
- The manner in which a snowboarder will stand up on the ramp. Namely $h : \phi(I) \rightarrow \mathbb{R}$ representing the distance from the snowboarder's feet to their center of mass. Note that h is a function of the snowboarder's position on the halfpipe.

To Find:

1. A function $\psi : J \rightarrow I$ such that a snowboarder following Newtonian physics would traverse the ramp parametrization with position $\phi \circ \psi(t)$.

1.2 Outline of Solution of the Forward Problem

Our general strategy for solving the forward problem is to fix a point with respect to the track and compute a system of equations based on the conservation of angular momentum about that point. We use this approach to derive a second order ODE that approximately represents the position of a snowboarder on a given ramp. We assume two initial conditions, namely the initial position and velocity of the snowboarder, thus yielding a unique solution to the ODE. Our two major simplifying assumptions are that

- we treat the snowboarder as a pointmass,

- we ignore friction and air resistance.

We will not explicitly write out the ODE in its full glory, but will instead sequentially derive a sequence of functions such that our ODE can be written as a sequence of compositions of these functions with each other evaluated at $\psi(t)$, $\psi'(t)$, and $\psi''(t)$. Recall that ψ represents the function we are trying to solve for.

1.3 Derivation of the ODE

Suppose that at each point $x \in \phi(I)$, the snowboarder maintains their center of mass at distance $h(x)$ from their feet, perpendicular to the path of travel. If we fix a point $\mathcal{O} \in \mathbb{R}^2$, then with respect to \mathcal{O} , we have by conservation of angular momentum that

$$I(0)\vec{\omega}(0) + \sum_i \int_0^t \vec{\tau}_i dT = I(t)\vec{\omega}(t),$$

where the sum is taken over all torques acting on the the snowboarder with respect to \mathcal{O} . Differentiating this with respect to t , we get that

$$\sum_i \vec{\tau}_i(t) = \frac{d}{dt}(I(t)\vec{\omega}(t)). \quad (1)$$

We note that $\vec{\tau}$, $I(t)$, and $\vec{\omega}(t)$ can all be written as functions of $(\phi \circ \psi)$, $(h \circ \phi \circ \psi)$ and their derivatives of various orders.

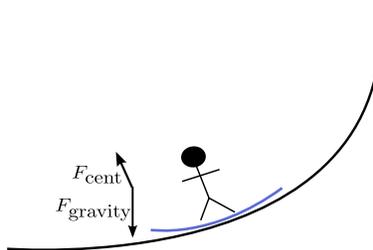


Figure 1: Snowboarder with forces

To compute the torques acting on the snowboarder, we consider the two most influential forces acting on the snowboarder, namely the normal force from the ramp and gravity. Thus

$$F_{\text{net}} = F_{\text{normal}} + F_{\text{gravity}}.$$

Calculating the normal force directly is difficult, however, so we instead calculate the centripetal force due to the curved motion of the path. This will comprise the entire component of F_{net} in the normal direction. Since the centripetal force and the normal force are parallel, we know that the remaining force will all be tangential, and hence will just be the component of the gravitational force

that is tangent of the motion of the center of mass. Since we assume that the snowboarder stays on the halfpipe until they reach the end of the halfpipe, we can compute the sum of these forces in the normal direction to be v^2/r where v^2 is the velocity of the center of mass and r is the radius of the osculating circle to the path of travel of the center of mass. On the other hand, the component of the net force parallel to the direction of travel is just the component of F_{gravity} which is parallel to the direction of travel. We compute these explicitly now using a sequence of explicit definitions and computations.

Define T_ϕ to be unit vector in the direction of travel of the feet of the snowboarder. We compute

$$T_\phi(t) = \frac{\phi'(\psi(t))}{|\phi'(\psi(t))|}.$$

We let M denote the matrix

$$M \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

i.e. rotation by $\pi/2$ counterclockwise. Thus the normal to the path of the feet of the snowboarder is given by

$$N_\phi(t) = M \cdot T_\phi(t).$$

We now define $p(t)$ to be the position of the snowboarder's center of mass at time t . By our assumptions on how the snowboarder stands, we compute this to be

$$p(t) \stackrel{\text{def}}{=} \phi(\psi(t)) + h(\phi(\psi)) \cdot N_\phi(\psi(t)).$$

The tangent to the path of the center of mass is thus

$$T_p(t) = \frac{p'(t)}{\|p'(t)\|}.$$

The normal to the direction of travel of the center of mass is

$$N_p(t) = M \cdot T_p(t).$$

The radius that we will need to compute the torques is simply the vector from \mathcal{O} to $p(t)$, i.e.

$$R(t) \stackrel{\text{def}}{=} p(t) - \mathcal{O}.$$

The force due to gravity is just

$$F_{\text{gravity}} = m\vec{g}$$

where m is the mass of the snowboarder and all of their equipment. The norm of centripetal force will be mv^2/r where r is the radius of curvature of the motion of the path of center of mass. If we compute r exactly, we will get an expression that involves the third derivative of ψ , but our initial conditions only give two conditions, so we need to make an approximation to make our differential equation have a unique solution. We phrase the following result below:

1.4 Approximating the Radius of Curvature

We take a short diversion to present a result about approximating the radius of curvature of p , as defined above.

Claim: Let $\rho_\phi(t)$ represent the radius of curvature of the curve $\phi(t)$ and suppose $|h(t)| \ll \rho_\phi(t)$. Then the radius of curvature of the curve $p(t) = \phi(t) + h(t)N_\phi(t)$ is approximately (as a second order approximation) equal to

$$\rho_p(t) \approx (\rho_\phi(t) - h(t)) - \frac{1}{2}(h'(t))^2 \frac{\rho_\phi(t)}{|\phi'(t)|^2}. \quad (2)$$

We omit the derivation of this approximation, but comment that it was derived through approximation of ϕ and p by osculating circles and then approximation of h as a power series. We also note that the above approximation is independent of reparametrization so the above approximation holds for p as written above.

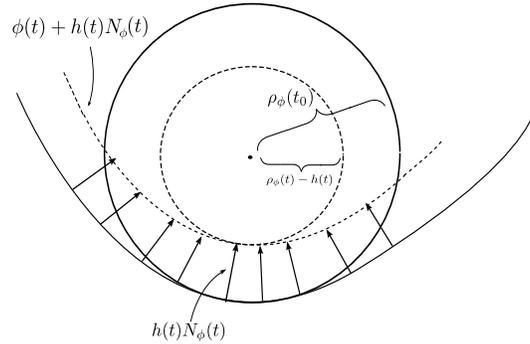


Figure 2: Case where h is constant

Remark: We note the interesting fact that the above approximation is actually an equality if h is constant (and $|h| < \rho_p$).

We omit the proof of the above remark so as to prevent cluttering of ideas, but we comment that it is a straightforward computation using standard formulas for the radius of curvature of a planar curve.

1.5 Deriving the ODE, continued...

We continue with the derivation of the ODE. Using the notation from before, the total force on the center of mass in the normal direction is

$$F_{\text{cent}}(t) = mv^2 / \rho_p(t) N_p(t).$$

Using the approximation result stated above, we see that

$$F_{\text{cent}}(t) \approx m \frac{\|p'\|^2}{(\rho_\phi - h) + \frac{1}{2}(h')^2 \frac{\rho_\phi}{|\phi'|^2}} N_p$$

where for readability, we omit the point at which each function is being evaluated in the above expression.

By definition, we have that

$$\vec{\tau}_i = R \times F_i.$$

Since R and F_i are all in the plane, we know that all of the $R \times F_i$ will be perpendicular to the plane, i.e., be parallel to the vector $R \times MR$. Using basic properties of the cross product, dotting our torques with the the vector $((M \cdot R) \times R)/(R, R)$ gives us a scalar representing the torque, which after some simple manipulations is just

$$\tau_{\text{total}} = \tau_{\text{gravity}} + \tau_{\text{cent}} = (M \cdot R, F_{\text{gravity}}) + (M \cdot R, F_{\text{cent}})$$

where (\cdot, \cdot) notates the inner product in \mathbb{R}^3 . Similarly, since $I(t)\omega(t)$ is a vector in \mathbb{R}^3 which is perpendicular to the plane in which ϕ travels, its derivative must be too. Hence we can take the dot product of $I(t)\omega(t)$ with the vector $(R \times M \cdot R)/(R, R)$ to transform $\vec{\omega}(t)$ into a scalar expression. By computing we have that

$$\omega = (\vec{\omega}, R \times (M \cdot R)/(R, R)) = \frac{\|p'(t)\|^2}{\|R\|}.$$

Since we are treating the snowboarder as a pointmass, we have that

$$I(t) = m\|R(t)\|^2$$

where m is the mass of the snowboarder. Combining the above results and terminology with our original differential equation (1), we have that

$$(M \cdot R, F_{\text{gravity}}) + (M \cdot R, F_{\text{cent}}) = \frac{d}{dt}(I(t)\omega(t)), \quad (3)$$

with all of the terms as defined as above. We note that both sides can be written as a function of ψ , ψ' , and ψ'' and that our initial conditions specify both $\psi(t_0)$ and $\psi'(t_0)$.

2 The Variational Problem

2.1 Introduction

The key idea of our approach is to use techniques from the calculus of variations to find the curve ϕ that maximizes the angular impulse the snowboarder exerts on the system. The calculus of variations was first systematized by Euler and Lagrange, who tackled the problem of finding the curve of fastest descent under gravity. In particular, they found necessary conditions for the functional

$$F(y) = \int_a^b f(x, y(x), y'(x)) dx$$

to have a local extremum on a subset of $\{y \in C^1[a, b] : y(a) = a_1, y(b) = b_1\}$. Our particular problem requires a considerable generalization of their techniques, which we will outline briefly as we state the relevant equations.

2.2 Problem Statement

We now give a precise statement of the variational problem we are trying to solve. To maximize the height the snowboarder can attain above the end of the halfpipe, we want to maximize his or her linear velocity at this end. Based on our choice of \mathcal{O} , this becomes equivalent to maximizing the angular velocity $\omega(t)$ about \mathcal{O} at this endpoint. Let $\tau(t) = (M \cdot R, F_{\text{gravity}} + F_{\text{cent}})(t)$ be the (signed) magnitude of the torque exerted about point \mathcal{O} at time t . Then recall that by the conservation of angular momentum,

$$\int_0^{t_{\text{final}}} \tau(s) ds = I(t_{\text{final}})\omega(t_{\text{final}}) - I(0)\omega(0).$$

Since we assume that the snowboarder is standing at full height when they reach the top of the ramp, as is common among skilled snowboarders, the quantities $I(0), \omega(0)$ and $I(t_{\text{final}})$ are constant, and we want maximize $\omega(t)$ when we maximize the integral of the torque. Our objective, then, is to find the sufficiently smooth plane curve $\alpha(t) = (\phi \circ \psi)(t) : [0, t_\alpha] \rightarrow \mathbb{R}^2$ that maximizes the functional

$$J[\alpha] = \int_0^{t_\alpha} \tau(\alpha(s), \alpha'(s), \alpha''(s)) ds$$

subject to physical constraints $p(0) = 0, p'(0) = v_0$ and the requirements that $\alpha(t)$ be at a fixed predetermined height H above the base of the ramp and have a vertical tangent at this point. Here we have written out explicitly the dependency of τ on the various derivatives of α that we derived in the formulation of the forward problem, but incidentally not on s itself. This type of problem is a generalization of the kind Euler and Lagrange initially studied in several regards. Namely, we are solving for an optimal vector-valued instead of scalar-valued function, the integrand depends on higher order derivatives, and the end limit of integration is variable.

2.3 The Euler-Lagrange Equations

Here we present the system of differential equations that α must necessarily satisfy in order to be a solution to the variational problem. These conditions are analogous to the requirement that for a smooth function to have a local extremum at a point, its derivative must be zero. By an obvious generalization of an argument in [1], we can show that $\alpha(t) = (\alpha_x, \alpha_y)$ must satisfy the **Euler-Lagrange equations**

$$\tau_{\alpha_x} - \frac{d}{dt} \tau_{\alpha'_x} + \frac{d^2}{dt^2} \tau_{\alpha''_x} = 0 \quad (4)$$

$$\tau_{\alpha_y} - \frac{d}{dt} \tau_{\alpha'_y} + \frac{d^2}{dt^2} \tau_{\alpha''_y} = 0. \quad (5)$$

The proof involves assuming that α is a solution, and considering variations of the form $\alpha + \epsilon \cdot \nu$ where ν is an arbitrary perturbation function whose value and

derivatives at 0 and t_α are zero. For α to be a local solution, we must have that

$$\delta J = \epsilon \frac{d}{d\epsilon} J[\alpha + \epsilon\nu] \Big|_{\epsilon=0} = 0$$

for all such ν . Repeated integration by parts of δJ gives the desired result.

Now we handle the “problem” that the the end limit of integration depends on α . Recall that we fix the final height of α at H , so we must have

$$\alpha(t_\alpha) = (p_x(t_\alpha) + h(\alpha(t_\alpha)), H).$$

Since p depends only on α and its first derivative, we can express this condition as

$$\beta(\alpha(t), \alpha'(t))|_{t=t_\alpha} = \alpha(t_\alpha) - (p_x(t_\alpha) + h(\alpha(t_\alpha)), H) = 0.$$

Or in other words, α and α' lie on the zero level set of the plane curve β . From this boundary condition, we can derive a set of equations that form what is known as a **transversal condition**. By generalizing the Lagrange multiplier approach in [2] to integrands depending on higher derivatives, we obtain an additional set of equations that α must satisfy at $t = t_\alpha$:

$$\left(-\frac{d}{dt} \tau_{\alpha''} + \tau_{\alpha'} \right) \cdot d_\alpha + \tau_{\alpha''} \cdot d_{\alpha'} \Big|_{t=t_\alpha} = 0 \quad (6)$$

$$\text{for all } \begin{pmatrix} d_\alpha \\ d_{\alpha'} \end{pmatrix} \in \ker \begin{pmatrix} B_\alpha & B_{\alpha'} \end{pmatrix}$$

where $T_{\alpha'}$ is an abbreviation for the vector $(T_{\alpha_x} \ T_{\alpha_y})^T$, etc. Note that if we assume $(B_\alpha \ B_{\alpha'})$ has full rank, then by linearity, we only need to check two linearly independent values of $(d_\alpha \ d_{\alpha'})^T$.

As a consequence of this analysis, we start our search for the optimal α by solving the system of equations consisting of (4), (5) and (6), along with their appropriate boundary values.

2.4 Approximating the Euler-Lagrange Equations

Given the extremely involved derivation we underwent to produce τ , one reasonably expects that the Euler-Lagrange equations (4), (5) are too complicated to solve even numerically. Our initial experiments solving these equations in Mathematica confirmed these expectations. To have any hope of producing an approximate solution α , we decided to use a Taylor series to approximate the torque as a function of α and its derivatives. That is, we can write τ as a function of six scalar variables

$$\tau(x_1, x_2, \dots, x_6) = \tau(\alpha_x, \alpha_y, \alpha'_x, \alpha'_y, \alpha''_x, \alpha''_y)$$

and approximate it with a Taylor polynomial

$$\tau(x_1, x_2, \dots, x_6) \approx \sum_{|\gamma|=0}^n \frac{1}{\gamma!} \frac{\partial^\gamma \tau(\vec{a})}{\partial \vec{x}^\gamma} (\vec{x} - \vec{a})^\gamma.$$

Due to the computational intensity of even this approximation, we decided to take $n = 2$. With this approximation in hand, we were able to substitute α and its derivatives back into τ . This greatly improved our ability to take derivatives with respect to t as mandated by the Euler-Lagrange equations. By explicitly computing the derivatives of $\tau(x_1, x_2, x_3, x_4, x_5, x_6)$, we computed for our fixed parameters that the Taylor approximation to τ up to second order was

$$\begin{aligned} \tau(x_1, x_2, x_3, x_4, x_5, x_6) \\ \approx -.686x_1 + 1.426x_4 \\ + 1.005x_2x_6 + .62292x_4x_5 + 21.716x_1x_6 + .290x_2x_4 + .0025x_6^2 \end{aligned}$$

3 Results

We used Mathematica to numerically solve the approximate Euler-Lagrange equations under several simplifying assumptions. First, we fixed the height function as

$$h(\alpha(t)) = 1 + \frac{2 \arctan(5(\alpha_y(t) - 5))}{\pi}$$

which we determined to be empirically reasonable by studying the behavior of skilled snowboarders on film. We also fixed the end time t_α at a value slightly smaller than the time it would take a snowboarder to reach the top of a halfpipe without pumping. This was because the presence of this additional t_α variable prevented Mathematica from solving through the transversal conditions (6).

Below is a figure of containing the parametrization of the approximate solution:

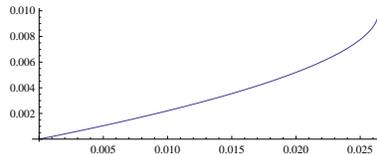


Figure 3: A reproduction of an approximation to an optimal snowboard ramp

As one can easily see, the figure is not perfect, but with more computing power and more sophisticated algorithms in computing the solutions of ODEs, one could expect a picture resembling a more typical snowboard ramp.

References

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- [2] B. Chachuat, *Nonlinear and Dynamic Optimization: From Theory to Practice*, Automatic Control Laboratory, EPFL, 2007.
- [3] *Physics of Snowboarding* <http://www.real-world-physics-problems.com/physics-of-snowboarding.html> Accessed 2/12/2011.