Symmetry and chirality in discrete structures

Marston Conder
University of Auckland, New Zealand

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Where is New Zealand?
What is New Zealand famous for?

- Great place to live (South Pacific)
- Multi-cultural (European, Maori/Polynesian, Asian ...)
- Beautiful scenery (beaches, volcanoes, lakes, mountains)
- Kiwis (strange little birds), kiwi fruit (strange little fruit)
- Dairy produce (milk, butter, cheese, etc.)
- Film sites (Lord of the Rings, The Hobbit, Narnia, etc.)
- Rugby football
- Extreme sports (bungy-jumping, white-water rafting, etc.)
What is symmetry?

Symmetry can mean many different things, such as balance, uniform proportion, harmony, or congruence.

Generally, an object has symmetry if it can be transformed in a way that leaves it looking the same as it did originally.
Symmetry can be **reflective:**
... or rotational:
... or translational:

... or a combination of these types
Examples of these kinds of symmetry abound in nature

... but have also been manufactured by human fascination and enterprise

e.g. the **Platonic solids** (c. 360BC)
or earlier ... the ‘Neolithic Scots’ (c. 2000BC)

... as publicised by Atiyah and Sutcliffe (2003)
... but unfortunately a hoax!

The claim that the Scots knew about these five regular solids over 1000 years before Plato was based on the above five ‘Scottish stones’ at the Ashmolean Museum in Oxford — but **one has 14 faces, and none of them is an icosahedron**

[See John Baez’s website for the full story]
Tilings at the Alhambra Palace — on its walls, floors, ceilings, and even some of the furniture — amazingly exhibit all of the 17 “wallpaper symmetries” (in two dimensions)

[Rafael Pérez Gómez and José Mara Montesinos, 1980s]
Symmetry can induce strength and stability:
... or its more contemporary version, the $C_{60}$ molecule Buckminsterfullerene ("buckyball"): 
Symmetry can also arise unexpectedly …

Consider a network in which

- each node is directly connected to (at most) 3 others
- any two nodes are connected by a path of length \( \leq 2 \)

We call this a graph of degree 3 and diameter 2

**Question:** What’s the largest possible number of nodes?
This is the **Petersen graph**: 

![Petersen Graph Diagram](image-url)
The largest 7-valent graph of diameter 2

... is also highly symmetric: the Hoffman-Singleton graph
Symmetric graphs

The symmetries of a graph form an example of an abstract algebraic structure known as a group.

When the symmetry group has a single orbit on vertices, i.e. when the graph looks the same from every vertex/node, the graph is called vertex-transitive.

When the symmetry group has a single orbit on edges, i.e. when the graph looks the same from every edge, the graph is called edge-transitive.

When the symmetry group has a single orbit on arcs, i.e. when the graph looks the same along every ordered edge, the graph is called arc-transitive, or symmetric.
Example:

This is \textit{edge-transitive} but \textit{not vertex-transitive}
Example:

This is **vertex-transitive** but **not edge-transitive**
The Petersen graph is symmetric (in fact 3-arc-transitive).

Every 3-arc has form

$$ab \rightarrow cd \rightarrow ae \rightarrow bc$$
Tutte’s 8-cage
(5-arc-transitive)
Construction of symmetric graphs

The symmetry group of a symmetric graph has particular properties that can be modelled. All examples of a given type can then often be constructed from a ‘universal model’.

Using group theory and computational methods, various mathematicians have constructed families of examples, along with some ‘pathological’ cases (with exotic properties).

A recent example is the largest known 3-valent graph of diameter 10 (discovered almost by accident).
Recent work by Auckland student Eyal Loz

Tables of the largest known graphs of given degree \( d \) and diameter \( k \) have been built up and occasionally adjusted over the last 50 years (by computer scientists, engineers and mathematicians). Finding the largest possible is known as the degree-diameter problem.

In his PhD thesis project (2005-2008), Eyal used voltage graph methods to construct new examples as ‘covers’ of old ones. Roughly speaking, this involves linking together a chain of copies of a suitably-chosen small graph, with a ‘voltage group’ determining the linkages.

The result?
Degree-Diameter Table (as at December 2012)

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Regular maps

The five Platonic solids can be viewed as embeddings of symmetric graphs on the sphere

... e.g. the cube

The symmetry group of this object (called a 'map') has a single orbit on incident vertex-edge-face triples (or ‘flags’).

Any such graph embedding is called a regular map.

The face-size $k$ and the vertex-degree $m$ give its type $\{k, m\}$. 
Regular maps on the sphere $\sim$ Platonic solids

Regular maps on the torus:

Type $\{4, 4\}$

Type $\{3, 6\}$ (and duals)
Regular maps can also be constructed on surfaces of higher genus, and on non-orientable surfaces (like the Klein bottle).

One important family of these maps can be obtained from the universal $[3, 7]$ tessellation of hyperbolic space:
Analogues in higher dimensions: polytopes

We can also think of a map as a partially ordered set:

Allowing extra ‘layers’ but requiring certain extra conditions, we get the notion of a regular $n$-dimensional polytope, with symmetry group always an image of a Coxeter group.
Chirality

An object is called chiral if it differs from its mirror images.
The term 'chiral' means handedness, derived from the Greek word $\chi\epsilon\iota\rho$ (or ‘kheir’) for ‘hand’. It is usually attributed to the scientist William Thomson (Lord Kelvin) in 1884, although the philosopher Kant had earlier observed that left and right hands are inequivalent except under mirror image.
Chirality in mathematics

The right and left trefoil knots are inequivalent ... with Jones polynomials $t + t^3 - t^4$ and $t^{-1} + t^{-3} - t^{-4}$ respectively.

Many of the other invariants of these knots (including their Alexander polynomials) are exactly the same for both, some because they are mirror images of each other, and in purely mathematical terms they have equal importance, but ...
Chirality in biology/chemistry/medicine

The two enantiomorphs of thalidomide have vastly different effects ... one is a sedative, but its mirror image causes birth defects ... making the context important.

Similarly, differences between aspartame (sweetener) and its mirror image (bitter), and (S)-carvone (like caraway) and its mirror image (R)-carvone (like spearmint).
Chiral or reflexible?

If an object is equivalent to its mirror image (with respect to some axis/hyperplane) then it has reflectional symmetry. In biological/chemical/medical/physical contexts we have no reason to expect mirror symmetry — so objects tend to be chiral — but the following is a remarkable phenomenon:

When a discrete object has a large degree of rotational symmetry, it often happens that it has also reflectional symmetry, so that chirality is not necessarily the norm!

e.g. the Platonic maps are all reflexible
Maps of type \( \{6, 3\} \) and \( \{3, 6\} \) on the torus

These regular maps are chiral, and are dual to each other. (The one on the right is a triangulation of the torus using the complete graph \( K_7 \).)
Open question: How prevalent is chiraility?

- for regular maps?
- for Riemann surfaces?
- for abstract polytopes?
- for other orientable discrete structures like these?
THANK YOU