$Ax = b$ and $Ax = 0$

Theorem 1. *Let A be a square $n \times n$ matrix. Then $Ax = b$ has a unique solution if and only if the only solution of $Ax = 0$ is $x = 0$. Let $A = [A_1, A_2, \ldots, A_n]$. A rephrasing of this is (in the square case) $Ax = b$ has a unique solution exactly when $\{A_1, A_2, \ldots, A_n\}$ is a linearly independent set.*

Proof. First, if $Ax = b$ has a unique solution (call it x_1), then $Ay = 0$ can't have nonero solution. For if we have $Ay = 0$ with $y \neq 0$ then $x_1 + y$ would give a new solution of $Ax = b$.

So assume the only solution of $Ax = 0$ is $x = 0$. Consider the equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \qquad\qquad &= 0 \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0
\end{aligned}$$

The coefficients of x_1 cannot all be 0 or else $x_1 = 1, x_2 = 0, \ldots, x_n = 0$ would be a non zero solution of $Ax = 0$. By rearranging the equations we may assume $a_{11} \neq 0$ and subtract multiples of the first equation from the rest to produce a new set of equivalent equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
+ a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \qquad\qquad &= 0 \\
+ a_{n2}x_2 + \cdots + a_{nn}x_n &= 0,
\end{aligned}$$

where I have used the same letters a_{ij} to represent the new equivalent equations (which still only have $x = 0$ as solution). Proceeding in a similar manner (perhaps by interchanging some rows) we get a set of equivalent equations (new notation) of the form

$$Ux = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \ldots & u_{1n} \\ 0 & u_{22} & u_{23} & \ldots & u_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where every $u_{kk} \neq 0$. Now if we perform the identical steps on the system $Ax = b$ we find an equivalent set of equations of the form

$$Ux = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \ldots & u_{1n} \\ 0 & u_{22} & u_{23} & \ldots & u_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Where the c_k are the result of applying the same operations on the b_k. This is a summary of Gauss elimination. The final set of equations $Ux = c$ has a unique solution and this solution is the unique solution of $Ax = b$. □