To obtain A_0, we notice that $P(-1) = 0$; and since $T_k(-1) = (-1)^k$, we find

$$A_0 = A_1 - A_2 + A_3 + \cdots + (-1)^n A_{n+1}. \tag{6.51b}$$

This technique is valid for all x in $[-1, 1]$; and moreover $P(x)$ can be efficiently evaluated at any x by (6.45c). If $x = 1$ and $p(x)$ in (6.50) is given by either the interpolating polynomials (6.45a) or (6.45b), then $P(1)$ is an interpolatory quadrature for $I(f) \equiv \int_{-1}^{1} f(x)\,dx$. Thus, (6.51) can be written in the form $Q_n(f) = \sum_{j=0}^{n} A_j f(z_j)$. Although this form is not computationally efficient, it can be shown in either case that the weights are positive; so the convergence result of Theorem 6.1 is applicable. Formulas of this type are called *Clenshaw-Curtis* quadratures.

Finally, we remark that we seem to have placed a great deal of emphasis on Chebyshev-type approximations in these sections. However, both from a theoretical background and from practical experience, Chebyshev methods have proven to yield excellent procedures in terms of truncation errors and round-off propagation.

EXAMPLE 6.12. As an illustration of some of these ideas, let $f(x) = \sin(x)/x$ for $-1 \leq x \leq 1$. First, using (6.45b), we construct the interpolating polynomial for $f(x)$ with $n = 4$. Since $f(x)$ is an even function on $[-1, 1]$ and $T_k(x)$ is odd when k is odd, we see that γ_1 and γ_3 in (6.45b) are zero. For even k, since $f(x)$ and $T_k(x)$ are even, we have (by symmetry between t_1 and t_3 and between t_0 and t_4)

$$\gamma_k = \frac{2}{4}\,[f(t_0)T_k(t_0) + 2f(t_1)T_k(t_1) + f(t_2)T_k(t_2)].$$

Now $T_k(t_0) = 1$ for $k = 0, 2, 4$; $T_k(t_1)$ has the value 1 for $k = 0$, 0 for $k = 2$, and -1 for $k = 4$; $T_k(t_2)$ has the value 1 for $k = 0$, -1 for $k = 2$, and 1 for $k = 4$. Thus with $f(0) = 1$,

$$\gamma_0 = \frac{1}{2}\left[f(1) + 2f\left(\cos\frac{\pi}{4}\right) + f(0)\right] = 1.839461$$

$$\gamma_2 = \frac{1}{2}[f(1) - f(0)] = -0.079265$$

$$\gamma_4 = \frac{1}{2}\left[f(1) - 2f\left(\cos\frac{\pi}{4}\right) + f(0)\right] = 0.002010.$$

Thus $p(x) = 0.919731 - 0.079265T_2(x) + 0.001005T_4(x)$ where the rapid decrease of the coefficients is typical of a well-behaved function $f(x)$.

To demonstrate Clenshaw-Curtis quadrature, we derive a polynomial $P(x)$ that approximates

$$Si(x) = \int_0^x \frac{\sin(t)}{t}\,dt.$$

Writing the interpolating polynomial $P(x)$ above in the form of (6.50), we have $b_0 = 0.919731$, $b_1 = 0$, $b_2 = -0.079265$, $b_3 = 0$, and $b_4 = 0.001005$. Thus from (6.51a) and (6.51b), we obtain $A_5 = 0.000101$, $A_4 = 0$, $A_3 = -0.013378$, $A_2 = 0$, $A_1 = 0.959364$, and $A_0 = 0.946087$. Thus

$$P(x) = 0.946087 + 0.959364T_1(x) - 0.013378T_3(x) + 0.000101T_5(x)$$

is an approximation for $\int_{-1}^{x} (\sin(t)/t)\, dt$. Since $\sin(t)/t$ is even, we have

$$\int_{-1}^{0} \frac{\sin(t)}{t}\, dt = \int_{0}^{1} \frac{\sin(t)}{t}\, dt = Si(1).$$

Thus $P(0) = 0.946807$ is an estimate to $Si(1) = 0.946083$. Moreover, we can use $P(x)$ to provide an estimate $Si(x)$, $0 < x < 1$, by observing (again since the integrand is even) that

$$\int_{-1}^{-1+x} \frac{\sin(t)}{t}\, dt = \int_{1-x}^{1} \frac{\sin(t)}{t}\, dt = Si(1) - Si(1 - x).$$

Therefore for $0 < x < 1$, we have the approximation

$$Si(1 - x) \approx P(0) - P(-1 + x),$$

which is an easily computed and accurate approximation. For example, with $x = 0.5$, $Si(0.5) = 0.493107$ and $P(0) - P(-0.5) = 0.493060$, which is in error by 0.000047.

PROBLEMS, SECTION 6.5.3.

1. Using the result of Theorem 6.4, find the weights and the nodes of the two- and three-point Gauss-Legendre quadrature formulas. Find the weights by undetermined coefficients. [The second- and third-degree monic Legendre polynomials are respectively $P_2(x) = x^2 - (1/3)$ and $P_3(x) = x^3 - (3/5)x$. The weights can be verified by checking precision in (6.34).]

2. Use the three-point Gauss-Legendre formula of Problem 1 and the five-point formula given in Example 6.10 to estimate

a) $\int_{-1}^{1} \sin(3x)\, dx$ b) $\int_{1}^{3} \ln(x)\, dx$ c) $\int_{1}^{2} e^{x^2}\, dx$.

3. Write a computer program to generate the nth Legendre polynomial from the three-term recurrence relation and find the zeros by Newton's method. Next, use (6.36) to find the Gauss-Legendre quadrature weights. Check your results for various values of n against tabulated formulas.

4. Let $q_k(x) = (1/2^{k-1})T_k(x)$ denote the monic Chebyshev polynomial of the first kind. Verify that $q_k(x) = xq_{k-1}(x) - b_k q_{k-2}(x)$, $k \geq 2$ where $b_2 = 1/2$ and $b_k = 1/4$, $k \geq 3$.

5. Use (6.40a) and (6.40b) to bound the error made in estimating the integral in Problem 2(a) by the three-point Gauss-Legendre formula [that is, $n = 2$ in (6.40a) and (6.40b)]. Use (6.40b) and an appropriate Jackson theorem to bound the error for five-point Gauss-Legendre formula applied to the integral in 2(a).

6. Show that no matter how the nodes and weights of a quadrature formula

$$Q_n(f) = \sum_{j=0}^{n} A_j f(x_j)$$

are chosen, the formula cannot have precision greater than $2n + 1$. [Hint: Assume that $Q_n(f)$ is designed to approximate $\int_a^b f(x)w(x)\, dx$. Use Theorem 6.4 and find a polynomial $p(x) \in \mathcal{P}_{2n+2}$ for which $Q_n(p) \neq \int_a^b p(x)w(x)\, dx$.]