Cholesky Factorization

An alternate to the LU factorization is possible for positive definite matrices $A$. The text’s discussion of this method is skimpy. This is a more complete discussion of the method. A matrix is **symmetric positive definite** if for every $x \neq 0$

$$x^T Ax > 0, \text{ and } A^T = A.$$ 

It follows the $\det(A) > 0$ and that all principal proper sub matrices have positive determinant. What follows is a description of Cholesky’s method. We want to come up with a factorization of the form

$$A = LL^T,$$

Where $L$ is lower triangular. We construct $L = [\ell_{ij}]$ inductively.

1. $$\ell_{11} = \sqrt{a_{11}},$$

   where we take the positive square root. Since $A$ is positive definite, $a_{11} > 0$ and this gives a positive real number.

2. We want $\ell_{11}\ell_{21} = a_{21}$, so we take

   $$\ell_{21} = \frac{a_{21}}{\ell_{11}}.$$

3. We want $\ell_{21}^2 + \ell_{22}^2 = a_{22}^2$. So we set

   $$\ell_{22} = \sqrt{a_{22}^2 - \ell_{21}^2}.$$

So now we have a $2 \times 2$ matrix

$$L_2 = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix}$$

that satisfies

$$L_2 L_2^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A_2.$$ 

Now take determinants of both sides to get

$$\ell_{11}^2 \ell_{22}^2 = \det A_2 > 0.$$

So the complex number $\ell_{22}$ has a **positive** square and must have been real, in other words

$$a_{22}^2 - \ell_{21}^2 > 0,$$

and we can take the (real) square root to be positive.
4. Now consider an $n \times n$ symmetric positive definite matrix $A$. Let’s suppose we have already handled $(n - 1) \times (n - 1)$ matrices. We want to produce a partitioned matrix

$$L = \begin{bmatrix} L_1 & 0 \\
 b & \ell 
\end{bmatrix}$$

so that

$$\begin{bmatrix} L_1 & 0 \\
 b & \ell 
\end{bmatrix} \begin{bmatrix} L_1^T & b^T \\
 0 & \ell 
\end{bmatrix} = \begin{bmatrix} A_1 & c^T \\
 c & a_{nn} 
\end{bmatrix}.$$ 

In this equation $b$ and $c$ are $n - 1$ dimensional row vectors and $A_1$ is the upper left $(n - 1) \times (n - 1)$ part of $A$. One equation we can solve uniquely is

$$bL_1^T = c.$$ 

Now that we know $b$ we have one more equation

$$\|b\|^2 + \ell^2 = a_{nn}^2.$$ 

Hence we let

$$\ell = \sqrt{a_{nn}^2 - \|b\|^2},$$

which is possibly a complex number, but it does give us a solution to

$$LL^T = A.$$ 

If we compute determinants we find

$$\det(L_1)^2\ell^2 = \det(A) > 0.$$ 

Hence $\ell^2 > 0$ and $\ell$ must have been real and we choose it to be the positive square root.

Cholesky was a French soldier who was also a mathematician and geodesist. He was killed near the end of WWI. One of his acquaintances wrote a paper with Cholesky’s method and credited him with it. It was published in 1924, after Cholesky’s death.