

Norms on \mathbb{R}^n

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Theorem 1. *All norms on \mathbb{R}^n are equivalent (even norms you never heard of). In other words if $\|\cdot\|$ and $\|\cdot\|_2$ are norms then there are positive constants a, b such that*

$$a\|v\| \leq \|v\|_2 \leq b\|v\|, \forall v \in \mathbb{R}^n.$$

Proof. Let $\|\cdot\|$ be any norm. We will prove that there are $a > 0, b > 0$ so that

$$a\|v\| \leq \|v\|_2 \leq b\|v\|.$$

This will be good enough. First let $v = x_1e_1 + x_2e_2 + \dots + x_n e_n$, where $\{e_1, e_2, \dots, e_n\}$ is a basis for \mathbb{R}^n . By the triangle inequality

$$\|v\| \leq \sum_j |x_j| \|e_j\|.$$

By Cauchy's inequality, for the inner product $\sum_j |x_j| \|e_j\|$,

$$\sum_j |x_j| \|e_j\| \leq \left(\sum_j x_j^2\right)^{1/2} \left(\sum_j \|e_j\|^2\right)^{1/2},$$

so

$$a\|v\| \leq \|v\|_2, \text{ where } a = 1/\left(\sum_j \|e_j\|^2\right)^{1/2}.$$

Next consider the function $\|v\|$ on the set $\sum_j x_j^2 = 1$. Let $m > 0$ be its minimum. By what we have just proved the function $v \rightarrow \|v\|_2$ is continuous on \mathbb{R}^n , meaning that if $\|v_j\|_2 \rightarrow 0$ then $\|v_j\| \rightarrow 0$. Also the set $v : \|v\|_2 = 1$ is compact (closed and bounded). Now let v be any vector and let $u = v/\|v\|_2 \in \mathbb{R}^n$. Then $\|u\|_2 = 1$ so $\|u\| \geq m$. Hence

$$\|v\| \geq m\|v\|_2, \text{ or } \|v\|_2 \leq b\|v\|, \text{ where } b = 1/m.$$

□