Cholesky Factorization

An alternate to the LU factorization is possible for positive definite matrices A. The text's discussion of this method is skimpy. This is a more complete discussion of the method. A matrix is **symmetric positive definite** if for every $x \neq 0$

$$x^T A x > 0, \text{ and } A^T = A.$$

It follows the $\det(A) > 0$ and that all principal proper sub matrices have positive determinant. What follows is a description of Cholesky's method. We want to come up with a factorization of the form

$$A = LL^T,$$

Where L is lower triangular. We construct $L = [\ell_{ij}]$ inductively.

1. $$\ell_{11} = \sqrt{a_{11}},$$

   where we take the positive square root. Since A is positive definite, $a_{11} > 0$ and this gives a positive real number.

2. We want $\ell_{11}\ell_{21} = a_{21}$, so we take
   $$\ell_{21} = \frac{a_{21}}{\ell_{11}}.$$

3. We want $\ell_{21}^2 + \ell_{22}^2 = a_{22}^2$. So we set
   $$\ell_{22} = \sqrt{a_{22}^2 - \ell_{21}^2}.$$

   So now we have a 2×2 matrix
   $$L_2 = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix}$$

   that satisfies
   $$L_2 L_2^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A_2.$$

   Now take determinants of both sides to get

   $$l_{11}^2 l_{22}^2 = \det A_2 > 0.$$

   So the complex number ℓ_{22} has a *positive* square and must have been real, in other words

   $$a_{22}^2 - \ell_{21}^2 > 0,$$

   and we can take the (real) square root to be positive.

4. Now consider an $n \times n$ symmetric positive definite matrix A. Let's suppose we have already handled $(n-1) \times (n-1)$ matrices. We want to produce a partitioned matrix

$$L = \begin{bmatrix} L_1 & 0 \\ b & \ell \end{bmatrix}$$

so that

$$\begin{bmatrix} L_1 & 0 \\ b & \ell \end{bmatrix} \begin{bmatrix} L_1^T & b^T \\ 0 & \ell \end{bmatrix} = \begin{bmatrix} A_1 & c^T \\ c & a_{nn} \end{bmatrix}.$$

In this equation b and c are $n-1$ dimensional row vectors and A_1 is the upper left $(n-1) \times (n-1)$ part of A. One equation we can solve uniquely is

$$bL_1^T = c.$$

Now that we know b we have one more equation

$$\|b\|^2 + \ell^2 = a_{nn}^2.$$

Hence we let

$$\ell = \sqrt{a_{nn}^2 - \|b\|^2},$$

which is possibly a complex number, but it does give us a solution to

$$LL^T = A.$$

If we compute determinants we find

$$\det(L_1)^2 \ell^2 = \det(A) > 0.$$

Hence $\ell^2 > 0$ and ℓ must have been real and we choose it to be the positive square root.

Cholesky was a French soldier who was also a mathematician and geodesist. He was killed near the end of WWI. One of his aquaintances wrote a paper with Cholesky's method and credited him with it. It was published in 1924, after Cholesky's death.