Sample Problems

Math 464

The test (on Wednesday, October 24) will be closed book. One notebook-size page of notes will be allowed on the test. You should give any numerical answers in the form of a rational number or a simple expression involving radicals. Calculators will not be allowed.

1. In using Newton's method to find a root of an equation $f(x) = 0$, with initial guess $x_0 = 3$, we compute $f(x_0) = 1$, and find that $x_1 = 5$. What is $f'(x_0)$?

2. Let a be a given number and suppose that $0 < \epsilon < 1$ is also a given number, Show that the equation $a = x - \epsilon \sin x$ has a unique solution and it can be found by fixed point iteration of an appropriate function with any starting value.

3. The following formula is used to compute a function f:

$$f(x) = \sqrt{x-1} - \sqrt{x-2}$$

Suggest a more accurate way to compute the same function.

4. The reciprocal of a number R can be computed by the following iteration (no divisions!):

$$x_{n+1} = x_n(2 - x_n R).$$

This is Newton's method applied to find the zero of a certain function f. What is f?

5. Each of the following iterations may converge. If the iteration does converge it will be to a root of a polynomial $p(x)$. In all three cases the convergence will be to some root of the same polynomial.

$$x_{k+1} = x_k^2 - 2x_k + 2 \tag{1}$$

$$x_{k+1} = x_k - \frac{x_k^2 - 3x_k + 2}{2x_k - 3} \tag{2}$$

$$x_{k+1} = -x_k^2 + 4x_k - 2 \tag{3}$$

   a) What is $p(x)$? What are its roots?

   b) One of the iterations will converge to either of the two roots of $p(x)$ if the initial guess is sufficiently close. Each of the others will converge to one root, but can't possibly converge to the other unless the initial guess is exactly equal to the root. Which is which? Give a justification for your answer.

6. a) Find the $A = LU$ factorization of
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$$
by using Gaussian elimination. L should have 1's on the main diagonal.

   b) Use part a) to solve:
$$Ax = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

7. Find the best least squares straight line fit $y = ax + b$ to the data $(0, 1), (1, -1), (2, 4)$.

8. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 3 \end{bmatrix}$ $b = \begin{bmatrix} 5 \\ 0 \\ -4 \end{bmatrix}$. Find the least squares solution to the problem $Ax = b$ using the normal equations.

9. a) Let A be row diagonally dominant. Prove that A^{-1} exists.

   b) Let
$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix},\ b = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
Compute two steps, $x^{(1)}, x^{(2)}$ of the Jacobi method for solving $Ax = b$. Give an error estimate for $\|x^{(2)} - x_t\|$ in terms of $\|x^{(1)} - x^{(0)}\|$, where x_t is the true solution.

10. Let
$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

   a) Compute $\kappa_\infty(A)$.

   b) Let
$$b = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$
and suppose that
$$x_c = \begin{bmatrix} .9 \\ 1.1 \end{bmatrix}$$
is a computed solution of $Ax = b$. Give upper and lower estimates on $\dfrac{\|x_c - x_t\|}{\|x_t\|}$, using $\kappa_\infty(A)$, where x_t is the true solution.

11. Consider the fixed point iteration
$$x_{n+1} = \frac{1}{3 + x_n}, x_0 = 0$$

a) Prove that

$$|x_{n+1} - x_n| < \frac{1}{9}|x_n - x_{n-1}|$$

b) Prove that x_n converges to a limit, s.

c) Prove that $x_n - s$ alternates in sign.

d) Find s.