Gamelin, Problem 11, VI.2

May 18, 2014

This note contains a solution of Problem 11, Section VI.2 of Gamelin, [1].

Problem. *Suppose $f(z) = \sum a_k z^k$ is analytic for $|z| < R$, and suppose that f extends to be meromorphic for $|z| < R + \epsilon$, with only one pole z_0 on the circle $|z| = R$. Show that $a_k/a_{k+1} \to z_0$ as $k \to \infty$.*

Solution. Let

$$f(z) - \sum_{j=1}^{m} \frac{b_j}{(z - z_0)^j} = \sum c_k z^k$$

be analytic in $|z| < R + \epsilon$. Then using the binomial theorem, or otherwise

$$a_k = c_k + \sum_{j=1}^{m} \frac{(-1)^j}{z_0^j} b_j \frac{(k+j-1)\dots(k+1)}{z_0^{k+j-1}}$$

$$a_{k+1} = c_{k+1} + \sum_{j=1}^{m} \frac{(-1)^j}{z_0^j} b_j \frac{(k+j)\dots(k+2)}{z_0^{k+j}}.$$

Dividing and multiplying top and bottom by z_0^{k+1} we get

$$\begin{aligned}\frac{a_k}{a_{k+1}} &= z_0 \frac{c_k z_0^k + \sum_{j=1}^{m} (-1)^j b_j (k+j-1)\dots(k+1) z_0^{1-2j}}{c_{k+1} z_0^{k+1} + \sum_{j=1}^{m} (-1)^j b_j (k+j)\dots(k+2) z_0^{1-2j}} \\ &= z_0 \frac{c_k z_0^k + (-1)^m b_m z_0^{1-2m} k^{m-1} + \dots}{c_{k+1} z_0^{k+1} + (-1)^m b_m z_0^{1-2m} k^{m-1} + \dots}.\end{aligned}$$

Since $\sum c_k z_0^k$ converges, $c_k z_0^k \to 0$ as $k \to \infty$. Now let $k \to \infty$ and see that

$$\frac{a_k}{a_{k+1}} \to z_0.$$

□

References

[1] Theodore Gamelin, *Complex Analysis*, Springer-Verlag, 2001.