# Spectral Theorem of Bounded Self-adjoint Operators

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# 1 Introduction

From linear algebra, a student learns diagonalization in finite dimensional space where the linear map can be represented using finite matrix, and determining whether a matrix is diagonalizable is fairly simple. In a infinite dimensional space, the concept of diagonalization need be extended and the identification can be done using spectral theorem. This paper explores an elementary version of proof of spectral theorem of Bounded Self-adjoint Operators in Hilbert space using only the intrinsic properties in the Hilbert space and real number system [4]

# 2 Preliminaries

## 2.1 Complex Euclidean Spaces

**Definition 2.1** (Metric)**.** *A metric on a set $X$ is a function $\rho : X \times X \to [0, \infty)$ such that*

- $\rho(x, y) = 0 \iff x = y$;
- $\rho(x, y) = \rho(y, x)$ *for all* $x, y \in X$;
- $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ *for all* $x, y, z \in X$;

**Definition 2.2** (Complete)**.** *A subset $E$ of $X$ is complete if every Cauchy sequence in $E$ converges and its limit is in $E$.*

**Definition 2.3** (Inner Product)**.** *An inner product on a complex vector space $\mathcal{H}$ is a map $(x, y) \mapsto \langle x, y\rangle$ from $\mathcal{X} \times \mathcal{X} \to \mathbb{C}$ where $\mathcal{X}$ is complex vector space such that:*

- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ *for all* $x, y, z \in \mathcal{H}$ *and* $a, b \in \mathbb{C}$
- $\langle y, x\rangle = \overline{\langle x, y\rangle}$ *for all* $x, y \in \mathcal{H}$
- $\langle x, x\rangle \in (0, \infty)$ *for all nonzero* $x \in \mathcal{X}$

**Definition 2.4** (Norm)**.** *Let $K$ denote either $\mathbb{C}$ or $\mathbb{R}$, and let $\mathcal{X}$ be a vector space over $K$. A norm on $\mathcal{X}$ is a function $x \mapsto \|x\|$ from $\mathcal{X}$ to $[0, \infty)$ such that*

- $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in \mathcal{X}$
- $\|\lambda x\| = |\lambda| \|x\|$ *for all* $x \in \mathcal{X}$ *and* $\lambda \in K$
- $\|x\| = 0 \iff x = 0$

**Definition 2.5** (Complex Euclidean Spaces)**.** *A complex Euclidean space $\mathfrak{L}$ is a complex vector space with a metric derived from a complex-valued inner product and it is complete with respect to metric convergence.*

**Definition 2.6** (Unitary Spaces)**.** *A unitary space $\mathfrak{L}$ is a complex Euclidean space that has a finite basis with respect to linear combination.*

**Definition 2.7** (Hilbert Spaces)**.** *A Hilbert space $\mathfrak{L}$ is a complex Euclidean space without the basis described above but is separable.*

**Definition 2.8** (Strong and Weak Convergence)**.** *A sequence $\{f_n\}$ converges strongly to the limit $f$ if $|f - f_n| \to 0$ with $1/n$.*

*A sequence $\{f_n\}$ converges weakly to the limit $f$ if $\langle f - f_n, g \rangle = \langle f, g \rangle - \langle f_n, g \rangle \to 0$ with $1/n$ for every $g \in \mathfrak{L}$*

**Theorem 2.1** (Properties of weak convergence)**.**

- *If $\{f_n\}$ converges weakly to $f$, then the sequence $\{|f_n|\}$ is bounded and $|f| \leq \liminf_{n \to \infty} |f_n|$*
- *If $\{|f_n|\}$ is bounded, then $\{f_n\}$ contains a subsequence converging weakly to a limit $f$*
- *If the sequence $f_n$ lies in a closed linear manifold $\mathfrak{M}$ and converges weakly to $f$, then $f$ lies in $\mathfrak{M}$*
- *In a unitary space, weak and strong convergence are equivalent*
- *If $\{f_n\}$ converges weakly to $f$ and $\{g_n\}$ strongly to $g$, then $\langle f_n, g_n \rangle \to \langle f, g \rangle$*

**Theorem 2.2.** *If $A_n, B_n, A, B$ are linear operators defined over $\mathfrak{L}$ and meets the conditions*

$$\begin{aligned} |A_n f| \leq \alpha |f|; \quad A_n f \to Af \text{ strongly };\\ |B_n f| \leq \beta |f|; \quad B_n f \to Bf \text{ strongly }; \end{aligned}$$

*then the sequence $\{A_n B_n\}$ has the properties*

$$|A_n B_n f| \leq \alpha\beta |f|; \quad A_n B_n f \to ABf \text{ strongly };$$

*Proof.*

$$|A_n B_n f| = |A_n(B_n f)| \leq \alpha |B_n f| \leq \alpha\beta |f|$$

Since we know $|\cdot|$ is a metric defined on $\mathfrak{L}$, we have

$$\begin{aligned} |ABf - A_n B_n f| &\leq |A(Bf) - A_n(Bf)| + |A_n(Bf - B_n f)| \\ &\leq |A(Bf) - A_n(Bf)| + \alpha |Bf - B_n f| \end{aligned}$$

Since we know both $|Af - A_n f|, |Bf - B_n f|$ are convergent, we have $|Af - A_n f| < \epsilon/2, |Bf - B_n f| < \epsilon/(2\alpha)$ for $\epsilon > 0$. Then, we have $|ABf - A_n B_n f| < \epsilon$, so it is $A_n B_n f \to ABf$ strongly □

## 2.2 Operators

**Definition 2.9** (Linear Operators). *A linear operator $T$ is an mapping in vector space $\mathfrak{D}(T) \in X, \mathfrak{R} \in Y$ where $X, Y$ are vector spaces*

$$T : \mathfrak{D}(T) \to \mathfrak{R}(T)$$

*For all $x, y \in \mathfrak{D}(T)$ and scalar $\alpha$,*

$$T(x + y) = Tx + Ty$$
$$T(\alpha x) = \alpha Tx$$

**Definition 2.10** (Symmetric). *An operator $A$ is said to be symmetric if its domain generates a closed linear manifold coincident with $\mathfrak{L}$ and if $\langle Af, g\rangle = \langle f, Ag\rangle$ for all $f, g$ in the domain.*

**Definition 2.11** (Self-adjoint Operators).

- *For operators $A$ on $\mathfrak{L}$, the adjoint of $A$ on $\mathfrak{L}$, namely $A^*$, satisfy that*

  $$\langle Tx, y\rangle = \langle x, T^* y\rangle$$

  *for all $x \in \mathfrak{D}_1 \subset \mathfrak{L}$ and $y \in \mathfrak{D}_2 \subset \mathfrak{L}$*

- *A self-adjoint operator $A$ is symmetric and $A = A^*$*

**Theorem 2.3.** *Let $A$ be linear and symmetric. If $A$ has bounds $\alpha$ and $\beta$, then $A + \lambda I$ has bound $\alpha + \lambda$ and $\beta + \lambda$ for arbitrary real $\lambda$; if $\rho$ is real and positive, $\rho A$ has bounds $\rho\alpha$ and $\rho\beta$ and $-A$ has bounds $-\beta$ and $-\alpha$.*

**Theorem 2.4.** *The least upper bounds of quotient $\langle Af, g\rangle/|f||g|$ and $|Af|/|f|$, where $f \neq 0$ and $g \neq 0$, have $\max(|\alpha|, |\beta|)$ as their common value when $A$ has bounds $\alpha$ and $\beta$.*

**Theorem 2.5.** *In order that a bound of $A$ be attained, it is necessary and sufficient that it be a characteristic value of $A$; and the elements for which the bound is attained are precisely the corresponding characteristic elements of $A$.*

**Definition 2.12** (Projection). *$P$ is called the Projection of $H$ onto $Y$, where $H$ is Hilbert space and $Y$ is subspace of $H$. If $P$ is a bounded linear operator. $P$ maps $H$ onto $Y$, $Y$ onto itself, $Z = Y^{\perp}$ onto 0 and is idempotent ($P^2 = P$)*

**Definition 2.13** (Linear manifold).

# 3 Spectral Theory for Bounded Definite Self-Adjoint Operators

Consider the self-adjoint operator $H$ defined over entire space $\mathfrak{L}$ with the bounds $\alpha$ and $\beta$ satisfy that

$$0 \leq \alpha \leq \beta < +\infty$$

By Theorem 2.3 defined above, we are able to manipulate the operator and get the bounds that fit into a more general case. For now, without lose of generality, we will work with the bounds defined above.

**Theorem 3.1.** *Let $H$ be a self-adjoint operator defined as above. the class $\mathfrak{M}(\lambda)$, $-\infty < \lambda < \infty$, of all elements in $\mathfrak{L}$ for which the condition*

$$|H^k f| \leq \lambda^k |f| \quad \text{for } k \geq 1, \tag{1}$$

*or, when $\lambda > 0$, the equivalent condition*

$$\lim_{k\to\infty} \sup |H^k f|/\lambda^k < +\infty \tag{2}$$

*is satisfied, is a closed linear manifold; Its projection operator $F(\lambda)$ is permutable with $H$ and also with every bounded linear operator which is permutable with $H$; and in $\mathfrak{M}(\lambda)$ the upper bound of $H$ does not exceed $\lambda$. In the range $\mathfrak{N}(\lambda)$ of $I-F(\lambda)$ the lower bound of $H$ is not less than $\lambda$; and, if it is equal to $\lambda$, is not attained. The inequality $\mu \leq \lambda$ implies the equivalent relation $\mathfrak{M}(\mu) \subset \mathfrak{M}(\lambda), F(\mu)F(\lambda) = F(\mu)$; the inequality $\lambda < 0$ implies the equivalent relations $\mathfrak{M}(\lambda) = \mathfrak{O}, F(\lambda) = O$; amd the inequality $\lambda > \beta$, where $\beta$ is the upper bound of $H$, implies the equivalent relations $\mathfrak{M}(\lambda) = \mathfrak{L}, F(\lambda) = I$. When $\epsilon > 0$ tends to zero, $F(\lambda+\epsilon)$ tends strongly to $F(\lambda)f$ for every $f$ in $\mathfrak{L}$.*

*Proof.* First Consider the case $\lambda < 0$. Then, the inequality $|H^k f| \leq \lambda^k |f|$ in condition (1) implies that $f = 0$. Then, we have $\mathfrak{M}(\lambda) = \mathfrak{O}$ and $F(\lambda) = O$. And statements above follows trivially.

Now, we will consider the last part of the theorem assuming that $\lambda \geq 0, \mu \geq 0$ The inequality $\mu \leq \lambda$ indicates that $f \in \mathfrak{M}(\mu)$ must satisfies

$$|H^k f| \leq \mu^k |f| \leq \lambda^k |f| \quad \text{for } k \geq 1,$$

Thus, we have $f$ must also in $\mathfrak{M}(\lambda)$, so $\mathfrak{M}(\mu) \subset \mathfrak{M}(\lambda)$. Thus, the projection operator $F(\mu)F(\lambda) = F(\mu)$.

If $\lambda \geq \beta$, $\beta$ is the upper bound of $H$, we have that

$$|H^k f| \leq \beta |H^{k-1} f| \leq \lambda |H^{k-1} f| \leq \beta^k |f| \leq \lambda^k |f| \text{ for } k \geq 1, \quad \text{By theorem 2.2}$$

Then, since every elements in $\mathfrak{L}$ satisfies, $\mathfrak{M}(\lambda) = (L)$ and $F(\lambda) = I$.

Going back to the first two statements in the theorem, consider the case $\lambda = 0$. the condition (1) is reduced to $|H^k f| \leq 0$. We know $\mathfrak{M}(0)$ is either the characteristic manifold of $H$ of value 0 or the same as the $\lambda < 0$ cases. Consider the case of $\mathfrak{M}(0)$ and $F(0)$. If $T$ is a bounded linear operator defined on $\mathfrak{L}$ and permutable with $H$, $T^*$, the adjoint of $T$, is also permutable with $H$. It follows that $T, T^*$ both transform $\mathfrak{M}(0)$ into part of itself by $Hf = 0 \implies HTf = THf = 0 = HT^*f = T^*Hf$. We can get $TF(0) = F(0)T$ from the geometric properties of $\mathfrak{M}(0)$ and T. Then, we may take $T = H$. Also, the lower bound of $H$ in $\mathfrak{N}(0)$ will not be less than zero since it cannot be less that the lower bound of $H$ in $\mathfrak{L}$. By theorem 2.5, the lower bound of $H$ in $\mathfrak{N}(0)$ cannot be zero since zero is not a characteristic value of $I$. Then, we are done with $\lambda \leq 0$ cases. For the first two statements, we can substitute the $H$ with $H/\lambda$ and put $\lambda = 1$. The meaning is not changed since the permutablility is the same and the bound will alter as suggested by theorem 2.3. Thus, we will consider $\lambda = 1$ for proving $\lambda > 0$ cases in the following parts. □

**Theorem 3.2** (part a)**.** *Let $H$ be an operator describe in theorem 3.1. The class $\mathfrak{M}$ of all elements in $\mathfrak{L}$ for which the equivalent conditions*

$$|H^k f| \leq |f| \quad \text{for } k \geq 1 \tag{3}$$

$$\lim_{k\to\infty} \sup |H^k f| < +\infty \tag{4}$$

*are satisfied is a closed linear manifold; its projection operator $F$ is permutable with $H$ and also with every bound linear operator which is permutable with $H$; and in $\mathfrak{M}$ the upper bound of $H$ does not exceed 1. In the range $\mathfrak{N}$ of $I - F$ the relation $\lim_{k\to\infty} \sup |H^k f| = \infty$ holds when $f \neq 0$.*

*Proof.* Let's denote $\sigma(f) = \limsup_{k\to\infty} |H^k f|$, where $f \in \mathfrak{L}$. $\sigma(f)$ satisfies that $0 \leq \sigma(f) \leq \infty$. Since we know $\limsup_{k\to\infty} |\cdot|$ is infinite norm, we see that $\sigma(0) = 0$, $\sigma(\alpha f) = \limsup_{k\to\infty} |H^k \alpha f| = |\alpha| \limsup_{k\to\infty} |H^k f| = |\alpha|\sigma(f)$ when $\alpha \neq 0$, and $\sigma(f+g) \leq \sigma(f) + \sigma(g)$. Then, the inequality (4) implies $\mathfrak{M}$ is a linear manifold. Let $T$ be a bounded linear operator defined on $\mathfrak{L}$ and permutable with $H$, then we have $|Tf| \leq \gamma$ by its boundedness. $|H^k T f| = |TH^k f| \leq \gamma |H^k f|$ follows immediately. By definition, $\sigma(Tf) \leq \gamma\sigma(f)$. It follows that $T$ is a projection operator on $\mathfrak{L}$.

Since we have

$$\begin{aligned} 0 \leq |H^k f|^2 = \langle H^k f, H^k f\rangle = \langle H^{k-1} f, H^{k+1} f\rangle = |H^{k-1} f||H^{k+1} f| \\ \implies |H^k f|/|H^{k-1} f| \leq |H^{k+1} f|/|H^k f| \end{aligned}$$

for $k \geq 1$ and $H^k f \neq 0$. Let $\tau_k(f) = |H^k f|/|H^{k-1} f|$ be a increasing sequence. Since we know $|H^k f|$ is bounded, there is a positive limit $\tau(f)$. We also define that when $H^k f = 0$, $\tau(f) = 0$.

We have

$$|H^k f|/|f| = \tau_1(f)\tau_2(f)\dots\tau_k(f) \leq \tau^k(f)$$

If $\tau(f) \leq 1$, the above equality is less than one and we get condition (3). If $\tau(f) > 1$, there is a real number $\rho$ such that $\tau(f) > \rho > 1$ and an integer $p$ such that $\tau_k(f) > \rho$ for $k > p$. Then,

$$|H^{n+p} f|/|H^p f| = \tau_{p+1}(f)\dots\tau_{n+p}(f) > \rho^n |H^p f|$$

we know then $\sigma(f) \to +\infty$. so we have $\sigma(f) < \infty$ if and only if condition (3) is met.

Since $H^k$ is continuous operator, we have the class specified by $|H^k f| \leq |f|, k \geq 1$ is a closed subset of $\mathfrak{L}$. The manifold $\mathfrak{M}$ is closed. In the case of $k = 1$, the condition (3) suggest that $|Hf| \leq |f|$ where $H$ is defined on $\mathfrak{M}$, we have the upper bound cannot exceed 1 by theorem 2.4.

The manifold $\mathfrak{N}$ is the orthogonal complement of $\mathfrak{M}$ where it has no element except null element in common. Thus, we have $\sigma(f) = \infty$ for $f$ on $\mathfrak{N}$ □

**Theorem 3.3** (part b)**.** *With the same setting as part a, the lower bound of $H$ in $\mathfrak{N}$ is not less than 1; and, if equal to 1, is not attained. It means that if $\limsup_{k\to\infty} |H^k f| = \infty$ for every $f \neq 0$ in $\mathfrak{N}$, then $\langle Hf, f\rangle > \langle f, f\rangle$. Let $\alpha$ be the lower bound in $\mathfrak{N}$. We know $\limsup_{k\to\infty} |H^k f| = \infty$ for every $f \neq 0$ in $\mathfrak{N}$. We need show that $\alpha \geq 1$.*

*First case is that $\mathfrak{N}$ is a unitary space. Since we know its complete, there exists a sequence $\{f_n\}$ such that $\langle Hf_n, f_n\rangle \to \alpha, |f_n| = 1$. Since we know $\{|f_n|\}$ is bounded, there exists a weakly convergent subsequence of $\{f_n\}$ with limit $f$. Since in unitary space, strong convergence and weak convergence are equivalent. We have $\langle Hf, f\rangle = \alpha, |f| = 1$. Since we attatined such bound $\alpha$, by theorem 2.5 we have $Hf = \alpha f$ and $|H^k f| = \alpha^k |f|$ for $k \geq 1$ and $\alpha > 1$*

*Then, consider the case where $\mathfrak{N}$ is Hilbert space. Let $\epsilon > 0, \epsilon$. By definition, there is an element $\varphi_1$ in $\mathfrak{N}$ such that $\langle H\varphi_1, \varphi_1\rangle \leq \alpha + \epsilon, |\varphi_1| = 1$. Then, we get a complete orthonormal set $\{\varphi_n\}$ in $\mathfrak{N}$. We denote the projection generated by $\{\varphi_n\}$ by $E_n$. Then, let $H_n = E_n H E_n/(\alpha + 2\epsilon)$. $H_n$ is also self-adjoint since $E_n$ is the projection to $\varphi_n$'s and $H$ is self-adjoint. Since $\langle E_n H E_n f, f\rangle = \langle HE_n f, E_n f\rangle \geq 0$ for all $f$ in $\mathfrak{N}$*

# 4 Spectral Theory for Bounded Self-Adjoint Operators

**Theorem 4.1.** *If $A$ is a self-adjoint operator with finite bounds $\alpha$ and $\beta$, there exists a family of Projection $E(\lambda)$, $-\infty < \lambda < +\infty$, with the properties*

*(1) $E(\lambda)$ is permutable with $A$ and also with every bounded linear operator which is permutable with $A$;*

*(2) $\mu \leq \lambda$ implies $E(\mu)E(\lambda) = E(\mu)$;*

*(3) $E(\lambda) = O$ for $\lambda < \alpha$; $E(\lambda) = I$ for $\lambda \geq \beta$;*

*(4) if $\epsilon > 0$, then $E(\lambda + \epsilon)f \to E(\lambda)f$ strongly as $\epsilon \to 0$ for every $f$ in $\mathfrak{L}$*

*(5) in the range of $E(\lambda)$, the upper bound of $A$ does not exceed $\lambda$;*

*(6) in the range of $I - E(\lambda)$, the lower bound of $A$ is not less than $\lambda$; and, if equal to $\lambda$, is not attained.*

*The projection $E(\lambda)$ is uniquely determined by properties (1),(5),(6)*

**Theorem 4.2.** *If $A$ and $E(\lambda)$ have the significance indicated in Theorem 4.1; if number $\lambda_0 = \alpha, \lambda_1, \dots \lambda_n = \beta$, arranged in order of magnitude, determine a partition of the interval $(\alpha, \beta)$, where $\alpha$ and $\beta$ are the bounds of $A$; if $\lambda'_1, \dots, \lambda'_n$ are arbitrary numbers such that $\lambda_{k-1} \leq \lambda'_k \leq \lambda_k$ for $k = 1, \dots, n$; if*

$$\delta = \max_{k=1}, \dots, n(\lambda_k - \lambda_{k-1});$$

*and if $E(\triangle_k)$ is defined for the interval $\triangle_k = (\lambda_{k-1}, \lambda_k)$ by the relations*

$$E(\triangle_1) = E(\lambda_1), E(\triangle_k) = E(\lambda_k) - E(\lambda_{k-1}) \quad \textit{for } k = 2, \dots, n;$$

*then $|Af - \sum_1^n \lambda'_k E(\triangle_k) f| \leq \delta |f|$ for every $f$ in $\mathfrak{L}$.*

**Theorem 4.3.** *If $A$ and $E(\lambda)$ have the significance indicated in Theorem 4.1, then a bounded linear operator $T$ defined over $\mathfrak{L}$ is permutable with $A$ if and only if it is permutable with $E(\lambda)$ for every $\lambda$.*

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