The Relative Positions of the Roots of a Polynomial and its Derivative

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Abstract. Complex polynomials are probably one of the simplest complex functions constructible. However there is a fascinating relationship between the relative positions of the roots of a polynomial and its derivative when speaking from a (mostly) geometric point of view. This paper goes over these relationships, beginning with a proof of Marden's theorem, given by Dan Kalman in *An Elementary Proof of Marden's Theorem*. We then generalize this to a proof of the Gauss-Lucas theorem, given by Morris Marden in *Geometry of Polynomials*.

1 Introduction

Let us begin with a precise definition of polynomials[1].

Definition. A **complex polynomial of degree** $n \geq 0$ is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad z \in \mathbb{C}$$

where $a_0, \ldots, a_n \in \mathbb{C}$ are complex numbers and $a_n \neq 0$. [**5**]

Let us also use the following variant of the fundamental theorem of algebra.

Theorem 1 (Fundamental Theorem of Algebra). *A complex polynomial of degree $n \geq 0$ has precisely n complex roots when counting multiplicity.*

For instance, a 3-degree complex polynomial has 3 complex roots. With just these tools, let us begin by looking at the relationship between the roots of a complex quadratic and the roots of its derivative.

Note that for each of the sections from section 3 onwards, the primary source is referenced to in the title. For the remainder of the section, all theorems were referenced from the very same primary source unless explicitly referenced otherwise.

[1] Note that every polynomial in this text is a one-variable polynomial.

2 The Complex Quadratic

The complex quadratic can be written as $p(z) = az^2 + bz + c$ where $a \neq 0$. The derivative of this quadratic is therefore $p'(z) = 2az + b$. Then the roots of $p(z)$, z_0 and z_1 are given by

$$z_0 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad z_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and the root of $p'(z)$ is simply $z_0' = -b/2a$. We can see the relationship between the two if we placed z_0 and z_1 on the complex plane and find the midpoint to be

$$\frac{z_0 + z_1}{2} = \frac{\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}}{2} = \frac{-b}{2a} = z_0'$$

3 Marden's Theorem [2]

Having looked at the simple case of the geometric relationship between the roots of a quadratic and the roots of its derivative, we will now look at the same for a 3-degree polynomial. This relationship is excellently described by Marden's theorem.

Theorem 2 (Marden's Theorem). *Let $p(z)$ be a third-degree polynomial with complex coefficients, and whose roots z_1, z_2, and z_3 are noncollinear points in the complex plane. Let T be the triangle with vertices at z_1, z_2, and z_3. There is a unique ellipse inscribed in T and tangent to the sides at their midpoints. The foci of this ellipse are then the roots of $p'(z)$.*

The proof for Marden's theorem which is described in this text was originally given by Dan Kalman in 2008. Before we can look at Marden's theorem however, we need some geometrical background information.

3.1 Background Information

Before beginning with the proof of Marden's theorem, I will list a few preliminary items that are required knowledge and are proved by Kalman in his paper.

Fact 1. *A triangle can be translated, rotated and scaled in any manner using the linear function $M : \mathbb{C} \to \mathbb{C}$ where $M(z) = \alpha z + \beta$ and $a \neq 0, \beta$ are fixed complex numbers. To rotate or scale the triangle if $a = re^{i\theta}$ we can change r and θ respectively. To translate the triangle we can change β.*

Fact 2. *Marden's theorem holds for a triple $\{z_1, z_2, z_3\}$ if and only if it also holds for the transformed triple $\{M(z_1), M(z_2), M(z_3)\}$ where M is as described in the previous fact.*

Brief outline of the proof for Fact 2. Let $p(z) = a(z - z_1)(z - z_2)(z - z_3)$ be an arbitrary polynomial and let $p_M(z) = b(z - M(z_1))(z - M(z_2))(z - M(z_3))$ where p_M is the transformed configuration for p after applying $M(z) = \alpha z + \beta$. Then

$$p_M(M(z)) = b(M(z) - M(z_1))(M(z) - M(z_2))(M(z) - M(z_3)) = b\alpha^3 p(z)$$

We differentiate both sides and divide by $\alpha \neq 0$ to get

$$p'_M(M(z)) = \alpha^2 p'(z)$$

This shows that if z is a root of p' then $M(z)$ is a root of p'_M. This is one direction of what we wished to show. The other direction comes from this directly due to the invertibility of linear functions. ■

The above two facts combined allow us to translate, rotate and scale the triangle without loss of generality when proving Marden's theorem.

Fact 3. *If the quadratic $z^2 + bz + c$ has roots z_1 and z_2 then $b = -(z_1 + z_2)$ and $c = z_1 z_2$.*

Fact 4. *At any point of an ellipse, the tangent line makes equal acute angles with the lines to the two foci.*

Fact 5. *At any point of an ellipse, the sum of the distances from the foci to any point of the ellipse is constant.*

Fact 6. *Let T be a triangle. Then there exists a unique ellipse inscribed in T which is tangent to the sides of T at their midpoints. This ellipse is referred to as the Steiner Inellipse.*

3.2 Proof

Most of the proof consists of proving 3 lemmas which we do below. While all three are proved by Kalman, they originate from three different places. The first lemma comes from classic Greek geometry, the second lemma comes from Marden's original paper on the subject and the third lemma comes from a proof for Marden's theorem by Bôcher.

Lemma 1. *Consider an ellipse with foci F_1 and F_2, and a point A outside the ellipse. There are two lines through A that are tangent to the ellipse. Let G_1 and G_2 be the points of tangency of these lines with the ellipse. Then $\angle F_1AG_1 = \angle F_2AG_2$.*

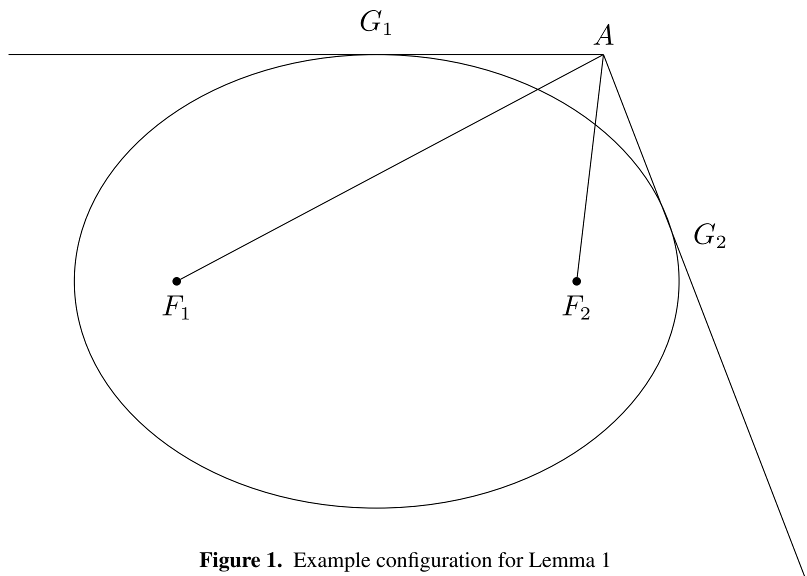

Figure 1. Example configuration for Lemma 1

Proof. Notice that there is no loss in generality to label as illustrated in the figure. Now begin by reflecting F_1 through AG_1 to define H_1 and let K_1 be the intersection of AG_1 with F_1H_1. Then $\triangle AK_1F_1$ and $\triangle AK_1H_1$ are congruent right triangles, so $AF_1 = AH_1$ and $\angle F_1AK_1 = \angle H_1AK_1$. We can perform a similar construction and receive the same equalities on G_2 with new point H_2. Then it suffices to show $\angle F_1AH_1 = \angle F_2AH_2$.

Draw the lines FG_1 and FG_2. Due to Fact 4 we can see that $\angle F_1G_1K_1 = \angle F_2G_1A$. Also notice that $\angle F_1G_1K_1 = \angle H_1G_1K_1$ since AK_1 is the perpendicular bisector of F_1H_1. Thus we get that F_2, G_1, and H_1 are collinear. We can once again perform a similar construction and receive the same equalities on G_2.

We have already seen that $AH_1 = AF_1$ and $AF_2 = AH_2$. Also we have that $H_1F_2 = F_1G_1 + G_1F_2 = F_1G_2 + G_2F_2 = F_1H_2$ where the second equality comes from Fact 5. Thus we get that $\triangle AH_1F_2$ is congruent to $\triangle AH_2F_1$ whcih implies that $\angle H_1AF_2 = \angle F_1AH_2$. Since both angles contain $\angle F_1AF_2$, we can see that $\angle H_1AF_1 = \angle F_2AH_2$ which is sufficient. ∎

Lemma 2. *Let the polynomial $p(z)$, it's roots z_1, z_2, z_3 and the triangle T be as in Marden's Theorem. Then the ellipse with foci at the roots of p' and passing through the midpoint of one side of T is actually tangent to that side of T.*

Proof. Such an ellipse always exists because we can create the ellipse with the foci located at the roots of p' and scale it so it passes through the midpoint of one side of the triangle T. By Fact 1 and Fact 2 we see that we can scale, translate and rotate the triangle in any fashion so it suffices to prove it for the following triangle T': One side of T' lies on the x-axis centered at the origin with length 2 while the third vertex sits in the upper half-plane. Let M be the transformation that transforms T to T'. Thus the vertices of T' (the roots of p_M) are at $1, -1$ and $w = a + bi$ where $b > 0$. Let E be the ellipse that has foci at the roots of p'_M that passes through the origin, which is the midpoint of the x-axis side of T'. We know the roots of p_M and we can assume it is monic without loss of generality so

$$p_M(z) = (z-1)(z+1)(z-w) = z^3 - wz^2 - z + w$$

Differentiating, we see that

$$p'_M(z) = 3z^2 - 2zw - 1 = 3\left(z^2 - \frac{2wz}{3} - \frac{1}{3}\right)$$

Let the roots for p'_M be $z_4 = r_4 e^{i\theta_4}$ and $z_5 = r_5 e^{i\theta_5}$. Then from Fact 3, $z_4 + z_5 = 2w/3$ and $z_4 z_5 = -1/3$. The former equation implies that either z_4 or z_5 is in the upper half-plane and the latter then shows that $\theta_4 + \theta_5 = \pi$ which then shows that both z_4 and z_5 are in the upper half-plane. Since $\theta_4 + \theta_5 = \pi$, then either $\theta_4 = \theta_5 = \pi/2$ or one of the angles is acute and the other is equal when considered on the negative axis. Either way the lines from the foci of E to 0 make equal angles with the x-axis. Thus the x-axis is a tangent line to E. ■

Lemma 3. *Let the polynomial $p(z)$, it's roots z_1, z_2, z_3 and the triangle T be as in Marden's Theorem. Then the ellipse with foci at the roots of p' which is tangent to one side of T at its midpoint is also tangent to the other two sides of T.*

Proof. We can once again position the triangle however we wish. The ellipse E is tangent to one side of the triangle, let this side be placed on the x-axis with one vertex on the origin $O = 0$ and the other at 1. The final vertex is once again placed at $w = a + bi$ with $b > 0$. Once again call this transformed triangle T', transformed using linear transformation M. Then since the vertexes are the roots of p_M we get that

$$p_M(z) = z(z-1)(z-w) = z^3 - (1+w)z^2 + wz$$

Differentiating, we see that

$$p'_M(z) = 3z^2 - 2(1+w)z + w$$

Let the roots for p' be $z_4 = r_4 e^{i\theta_4}$ and $z_5 = r_5 e^{i\theta_5}$. Then from Fact 3, $z_4 + z_5 = 2(1+w)/3$ and $z_4 z_5 = w/3$. The former equation shows that either z_4 or z_5 are in the upper half-plane. Since E is tangent to the x-axis, then if one of the roots is in the upper half-plane, both must be. This implies that without loss of generality $0 < \theta_4 \leq \theta_5 < \pi$.

The equation $z_4 z_5 = w/3$ shows that $\theta_4 + \theta_5$ is equal to the angle between the positive x-axis and the side Ow. This implies that the angle between Oz_5 and Ow is equal to θ_4. Now let us apply Lemma 1 with O being the external point outside E. The x-axis is one of the lines tangent to E passing through O. Let the second such line be L. By Lemma 1, the angle β between Oz_5 (one of the foci of E) and L equals the angle between the x-axis and Oz_4 (the other focus) which is just θ_4. Therefore Ow is on the line L because $\beta + \theta_5 = \theta_4 + \theta_5$ so Ow is tangent to E by the definition of L.

To show the third side is tangent to E we can perform an extremely similar proof but with the triangle translated horizontally so that the vertices are moved from 0 to -1 and 1 to 0. ■

We finally have the tools to prove Marden's Theorem. I will restate it here for convenience.

Theorem (Marden's Theorem). *Let $p(z)$ be a third-degree polynomial with complex coefficients, and whose roots z_1, z_2, and z_3 are noncollinear points in the complex plane. Let T be the triangle with vertices at z_1, z_2, and z_3. There is a unique ellipse inscribed in T and tangent to the sides at their midpoints. The foci of this ellipse are then the roots of $p'(z)$.*

Proof. Let us assume p, z_1, z_2, z_3, T as in the statement of the theorem. Using roots of p' as foci, draw an ellipse E that passes through the midpoint of one side of T. By Lemma 2, E is tangent to that side of T. By Lemma 3, E is also tangent to the other two sides of T. Now we claim that the points of tangency with these other two sides must be midpoints. Suppose this is not true. Repeat the construction above with a new side, producing E'. Since E and E' have the same foci, and are both tangent to the same line, they must actually coincide. This shows that both E and E' would contact the new side in the midpoint. By symmetry, this holds for the third side of the triangle. Thus the original ellipse E is tangent to all three sides at their midpoints. This ellipse is unique from Fact 6. Thus creating an ellipse such that is tangent to all three sides of midpoint must have the roots of p' as the focii because of uniqueness, completing the proof. ∎

3.3 Remarks

The remarkable aspect of this theorem is the precision with which it gives the locations for the roots of a polynomial without having to calculate the derivative explicitly. Kalman's proof is exceptional in its own right, because of its use of extremely elementary concepts in both complex analysis and geometry and not much else.

4 The Gauss-Lucas Theorem [4]

Both two degree and three degree polynomials have a pretty accurate way of locating the roots of their derivatives. From this a natural next step is to wonder if the same can be said about any polynomial, regardless of degree. This is the Gauss-Lucas theorem, stated below.

Theorem 3 (Gauss-Lucas Theorem). *All the critical points of a non-constant polynomial f (i.e. the roots of f') lie in the convex hull H of the set of zeros of f.*

4.1 Background Information

Before looking at the facts needed for the proof, let us first try to understand the theorem. The following two facts are standard definitions:

Definition. A set is convex if given any two points in the set, it contains the whole line segment that joins them.

Definition. The convex hull H of a set of points is the smallest convex set that contains all the points in the set.

A way I like to think about the convex hull of a set of points in the complex plane is like a rubber brand that stretches around all of the points of the set. So from these definitions we can understand that the Gauss-Lucas theorem basically refines the location of the roots of the derivative of the polynomial to a much smaller bounded subset of the complex plane. The following is a lemma that proves quite useful towards the end of the proof.

Lemma 4. *If each complex number w_j, $j = 1, 2, \ldots, p$, has the properties that $w_j \neq 0$ and*

$$\gamma \leq \arg w_j < \gamma + \pi \qquad j = 1, 2, \ldots, p$$

where $\gamma \in \mathbb{R}$, then $w = \sum_{j=1}^{p} w_j \neq 0$.

Proof. If $\gamma = 0$ then there are two cases. If $\arg w_j = 0$ for all j, then $\text{Re}(w_j) > 0$ for all j since $w_j \neq 0$ which directly implies that $\text{Re}(w_j) > 0$. If $\arg w_j \neq 0$ for some j then since $\gamma = 0$, $0 < \arg w_j < \pi$. Since $w_j \neq 0$, this implies that $\text{Im}(w_j) > 0$ which implies that $\text{Im}(w) > 0$. Therefore $w \neq 0$.

Now if $\gamma \neq 0$ then consider $w'_j = e^{-i\gamma} w_j$. These satisfy $0 \leq w'_j < \pi$ for all j and therefore from the previous paragraph, $\sum_{j=1}^{p} w'_j \neq 0$ which means that $\sum_{j=1}^{p} w'_j = e^{-i\gamma} \sum_{j=1}^{p} w_j = e^{-i\gamma} w \neq 0$ so $w \neq 0$. ∎

4.2 Proof

Now we prove the theorem. For convenience, I have restated it here.

Theorem 4 (Gauss-Lucas Theorem). *All the critical points of a non-constant polynomial f (i.e. the roots of f') lie in the convex hull H of the set of zeros of f.*

Proof. Let $f(z)$ be an arbitrary non-constant polynomial with zeros $z_1, \ldots, z_p$ in the following form

$$f(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \ldots (z - z_p)^{m_p}, \quad n = \sum_{j=1}^{p} m_j, \forall j \; m_j > 0$$

Let $F(z)$ be defined as follows for all z such that $f(z) \neq 0$,

$$F(z) = \frac{f'(z)}{f(z)} = \frac{d \ln f(z)}{dz} = \frac{d}{dz}\left(\sum_{j=1}^{p} m_j \ln(z - z_j)\right) = \sum_{j=1}^{p} \frac{m_j}{z - z_j}$$

Let the conjugate imaginary of F be $\overline{F}(z)$. Then we see that

$$\overline{F}(z) = \sum_{j=1}^{p} \frac{m_j}{\overline{z} - \overline{z_j}} = \sum_{j=1}^{p} m_j w_j \qquad w_j = 1/(\overline{z} - \overline{z_j})$$

Also let H be the convex hull of the set of zeros of f. Now let us begin the actual proof. For the sake of contradiction assume $z_0 \in \mathbb{C}$ is such that $f'(z_0) = 0$ but $z_0 \notin H$. Since $z_0 \notin H$, we get that $f(z_0) \neq 0$ so $F(z_0)$ is defined. We can then see that

$$\gamma \leq \arg(z_j - z_0) < \gamma + \pi \quad j = 1, 2, \ldots, p$$

where γ is a real constant. This is shown by considering the fact that if there was $\arg(z_j - z_0)$ that falls outside this range, then z_0 would be inside H because H is convex. Remember that $w_j = 1/(\overline{z} - \overline{z_j})$ so

$$\arg(-w_j) = \arg(1/(\overline{z_0} - \overline{z_j})) = -\arg(\overline{z_0} - \overline{z_j}) = \arg(z_0 - z_j)$$

when evaluated at z_0. Therefore at z_0 we get that $\gamma \leq \arg -w_j < \gamma + \pi$ for all j and so $\gamma \leq \arg -m_j w_j < \gamma + \pi$ for all j . Therefore from Lemma 4 we get that

$$\frac{f'(z_0)}{f(z_0)} = -1 \cdot -\overline{F}(z_0) = -\sum_{j=1}^{p} -m_j w_j \neq 0$$

Therefore $f'(z_0) \neq 0$ which contradicts the definition of z_0. ■

4.3 Remarks

Although not nearly as precise as Marden's theorem, the Gauss-Lucas theorem provides an excellent idea for where the critical points of a complex polynomial are located. An interesting aspect of this theorem is the history behind the theorem, and how it was named. The Gauss-Lucas theorem is named after Johann Carl Friedrich Gauss and Felix Lucas [**4**]. The first proof for the theorem was given by Lucas. Gauss played quite a different role in this theorem. He provided a mechanical interpretation for the zeros of the derivative of a polynomial.

Theorem 5. *The zeros of the function $F(z) = \sum_1^p m_j/(z - z_j)$ with all m_j real are the points of equilibrium in the field of force due to the system of p masses (point charges) m_j at the fixed points z_j repelling a movable unit mass at z according to the inverse distance law.*

If we restrict the m_j to be positive integers, then we can see how this theorem implies the Gauss-Lucas theorem in a physical way. Note that this is not necessarily a proof for the Gauss-Lucas theorem, just an interesting way of thinking about it.

5 Interesting Extensions

These are a couple ideas that, while I do not prove, are relevant to the content considered above and I thought seemed interesting.

5.1 Generalization of Marden's [1]

Theorem 6. *Let $f(z) = \Pi_{i=1}^{n}(z - z_i)^{\mu_i}$ where $\mu_i > 0$ for all $i \in \{1, \ldots, n\}$. Then the roots of f' are either the multiple roots of $f(z)$ to one lower order, or are the foci of a curve of class $n-1$ which touches each segment $z_i z_j$ (where $i \neq j$ and $i, j \in \{1, \ldots, n\}$ at a point dividing it in the ratio of μ_i to μ_j.*

This theorem is clearly a direct generalization of Marden's theorem, and also provides a quite specific location for all polynomials. This theorem was given by Ben-Zion Linfield in 1919, who was inspired by a few of Bôcher's ideas which were written down as an afterthought to Bôcher's proof to Marden's theorem. An interesting tidbit is that this is the same Bôcher who was responsible for Lemma 3.

5.2 Sendov's Conjecture [3]

Conjecture (Sendov's Conjecture). Let $P(z) = (z - a)\Pi_{k=1}^{n-1}(z - z_k)$ be a monic complex polynomial such that all of its zeros are in the closed unit disk $\{z : |z| \leq 1\}$. Then there exists a zero ξ of P' such that $|\xi - a| \leq 1$.

What's so interesting about this conjecture is that it provides a relatively precise location for just one critical point, not all of them. Notice that it is obvious that all critical points of P' are such that $|\xi - a| \leq 2$ because of the Gauss-Lucas theorem. This conjecture has been proven to be true under various restrictions which is what makes me suspect it may be true in general. Some such restrictions were: polynomials with at most 8 distinct zeros, when $|a| = 1$, if P vanishes at 0. My favorite is probably that proves it for $0 < a < 1$ for all n large enough [**3**].

6 Conclusion

Thus we see the location of the critical points of a polynomial can be approximated to a reasonable extent without ever having to compute the derivative, which is quite an intuitive result all things considered. I will also recommend to the still curious reader the book *Geometry of Polynomials* by Morris Marden [**4**] which is quite a consolidated book on similar results to the ones above.

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