# The Constructability of the Regular Heptadecagon

Daniel Humphreys

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## 1 Introduction

The construcability of various geometric figures using only a compass and unmarked straightedge is an area of investigation that dates back to the time of Euclid in ancient Greece. It was known at that time that regular polygons with sides lengths $3, 4, 5, 6, 8, 10, 12, 15...$ could be constructed, but no further progress was made on this problem for a couple thousand years. In fact no progress was made for such a long time that it was thought that the Greeks had found all the regular polygons that could be constructed. However, on March 30th, 1796, a 19 year old Carl Gauss rose from bed and was struck by an idea regarding how to prove that the regular 17-gon was constructable. It is said that Gauss was so pleased by this result that it not only made him decide to peruse a career in mathematics (lucky for us) but that he also requested that a 17 sided polygon be inscribed on his tombstone.

This paper will follow the proof that the regular 17-gon is constructable that Gauss wrote in his diary, given in [2]. Later on, we will briefly discuss the broader theory on constructable $n$-gons including a theorem given by Gauss that gives a necessary and sufficient condition for a regular $n$-gon to be construable.

## 2 Definitions and Proofs

**Definition 2.1.** (Constructable Number) A number is said to be constructable if it has a representation that consists of only finitely many additions, subtractions, multiplications, divisions, and square roots of positive integers.

Note that this definition implies that the sum, difference, product, quotient and square root of a finite number of constructable numbers is also a constructable number.

**Definition 2.2.** (Constructable $n$-gon) A regular $n$-gon is said to be constructable if the number $\cos(2\pi/n)$ is construable.

The reasoning behind this definition is that the problem of constructing a regular $n$-gon is equivalent to the problem of dividing the unit circle (or any circle really) into $n$ equal arcs. Once this is done, it becomes easy to construct the regular $n$-gon by simply connecting the points that divide the circle. The number $\cos(2\pi/n)$ represents the $x$-coordinate of the first of these points (note that the cosine of any integer multiple of the angle $2\pi/n$ will be the $x$-coordinate of one the vertices of the $n$-gon), so if it is a constructable number then the first vertex of the $n$-gon is constructable. The rest of the vertices can then be added using the compass and the first vertex.

We now prove two lemmas which will be needed in our main proof, the first of which will turn out to be as useful as its proof is straightforward.

**Lemma 2.1.** *If two non zero real numbers $r_1$ and $r_2$ satisfy $r_1 r_2 = \frac{c}{a}$ and $r_1 + r_2 = -\frac{b}{a}$ for real numbers $a, b, c$, with $a$ non zero, then $r_1$ and $r_2$ are the roots of the quadratic $x^2 + \frac{b}{a}x + \frac{c}{a}$.*

*Proof.* We will only prove the case for $r_1$, as the case for $r_2$ is essentially identical.

From the first condition, we have $r_2 = \frac{c}{ar_1}$, which, when substituted into $r_1 + r_2 = -\frac{b}{a}$, yields

$$r_1 + \frac{c}{ar_1} = -\frac{b}{a}$$

Multiplying by $r_1$ and rearranging we see that

$$r_1^2 + \frac{b}{a}r_1 + \frac{c}{a} = 0$$

and hence $r_1$ is a root of the quadratic $x^2 + \frac{b}{a}x + \frac{c}{a}$ as desired. □

**Lemma 2.2.** *Denote the sum $\sum_{k=1}^{n} \cos(k\theta)$ by $S_n$. Then*

$$S_n = \frac{\sin(\frac{2n+1}{2}\theta) - \sin(\frac{\theta}{2})}{2\sin(\frac{\theta}{2})}$$

*whenever $\theta \neq 2n\pi$ for some integer $n$.*

*Proof.* Using the product to sum formula for $\cos(\alpha)\sin(\beta)$ (that is $\cos(\alpha)\sin(\beta) = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta)))$, we have

$$2\cos(k\theta)\sin(\theta/2) = \sin\left(\frac{2k+1}{2}\theta\right) - \sin\left(\frac{2k-1}{2}\theta\right)$$

Summing over $k$ we have

$$\begin{aligned} 2\sin(\theta/2)\sum_{k=1}^{n}\cos(k\theta) &= \sum_{k=1}^{n}\sin\left(\frac{2k+1}{2}\theta\right) - \sin\left(\frac{2k-1}{2}\theta\right) \\ &= (\sin\left(\frac{3}{2}\theta\right) - \sin\left(\frac{1}{2}\theta\right)) + (\sin\left(\frac{5}{2}\theta\right) - \sin\left(\frac{3}{2}\theta\right)) + ... \\ &+ (\sin\left(\frac{2n-1}{2}\theta\right) - \sin\left(\frac{2n-3}{2}\theta\right)) + (\sin\left(\frac{2n+1}{2}\theta\right) - \sin\left(\frac{2n-1}{2}\theta\right)) \end{aligned}$$

This sum clearly telescopes, and so we are left with

$$2\sin(\theta/2)\sum_{k=1}^{n}\cos(k\theta) = \sin\left(\frac{2n+1}{2}\theta\right) - \sin\left(\frac{\theta}{2}\right)$$

Since $\theta \neq 2n\pi$ for some integer $n$, $\sin(\theta/2)$ is non zero, and so we can divide to obtain

$$S_n = \sum_{k=1}^{n}\cos(k\theta) = \frac{\sin(\frac{2n+1}{2}\theta) - \sin(\frac{\theta}{2})}{2\sin(\frac{\theta}{2})}$$

as desired. □

We can now use these results to prove the main theorem, that the regular heptadecagon is constructable. We will follow the proof that Gauss wrote in his diary given in [2].

**Theorem 2.3.** *The regular heptadecagon is constructable.*

*Proof.* In order to prove construcability, we must show that $\cos(\frac{2\pi}{17})$ is a constructable number. To do so, we set $\theta = \frac{2\pi}{17}$ and make the following definitions:

$$\begin{aligned} a &= \cos(\theta) + \cos(4\theta) \\ b &= \cos(2\theta) + \cos(8\theta) \\ c &= \cos(3\theta) + \cos(5\theta) \\ d &= \cos(6\theta) + \cos(7\theta) \\ e &= a + b \\ f &= c + d \end{aligned}$$

By lemma 2.2, we have

$$\begin{aligned} a + b + c + d = e + f = \sum_{k=1}^{8}\cos(k\theta) &= \frac{\sin(\frac{17}{2}\frac{2\pi}{17}) - \sin(\frac{\pi}{17})}{2\sin(\frac{\pi}{17})} \\ &= \frac{\sin(\pi) - \sin(\pi/17)}{2\sin(\pi/17)} \\ &= -\frac{1}{2} \end{aligned}$$

We now must form several products.

$$\begin{aligned} 2ab &= 2(\cos(\theta) + \cos(4\theta))(\cos(2\theta) + \cos(8\theta)) \\ &= 2(\cos(\theta)\cos(2\theta) + \cos(\theta)\cos(8\theta) + \cos(4\theta)\cos(2\theta) + \cos(4\theta)\cos(8\theta)) \end{aligned}$$

We again use the product to sum formula to obtain

$$\begin{aligned} 2\cos(\theta)\cos(2\theta) &= \cos(\theta) + \cos(3\theta) \\ 2\cos(\theta)\cos(8\theta) &= \cos(7\theta) + \cos(9\theta) \\ 2\cos(4\theta)\cos(2\theta) &= \cos(6\theta) + \cos(2\theta) \\ 2\cos(4\theta)\cos(8\theta) &= \cos(12\theta) + \cos(4\theta) \end{aligned}$$

We now note that, for any integer $n$

$$\begin{aligned}\cos\left((17-n)\frac{2\pi}{17}\right) &= \cos\left(2\pi - \frac{\pi n}{17}\right) \\ &= \cos(2\pi)\cos(2\pi n/17) + \sin(2\pi)\sin(2\pi n/17) \\ &= \cos\left(\frac{2\pi n}{17}\right)\end{aligned}$$

and so we have $\cos(12\theta) = \cos(5\theta)$ and $\cos(9\theta) = \cos(8\theta)$. Combining all this, we have

$$\begin{aligned}2ab &= \cos(\theta) + \cos(2\theta) + \cos(3\theta) + \cos(4\theta) + \cos(5\theta) + \cos(6\theta) + \cos(7\theta) + \cos(8\theta) \\ &= e + f \\ &= -\frac{1}{2}\end{aligned}$$

Similarly, we have

$$\begin{aligned}2ac &= 2(\cos(\theta) + \cos(4\theta)(\cos(3\theta) + \cos(5\theta)) \\ &= 2(\cos(\theta)\cos(3\theta) + \cos(\theta)\cos(5\theta) + \cos(3\theta)\cos(4\theta) + \cos(4\theta)\cos(5\theta)) \\ &= (\cos(4\theta) + \cos(2\theta)) + (\cos(6\theta) + \cos(4\theta)) + (\cos(7\theta) + \cos(\theta)) + (\cos(9\theta) + \cos(\theta)) \\ &= 2(\cos(4\theta) + \cos(\theta)) + (\cos(2\theta) + \cos(8\theta)) + (\cos(6\theta) + \cos(7\theta)) \\ &= 2a + b + d\end{aligned}$$

Continuing on in this fashion, we see that

$$\begin{aligned}2ad &= b + c + 2d \\ 2bc &= a + 2c + d \\ 2bd &= a + 2b + c \\ 2cd &= -\frac{1}{2}\end{aligned}$$

We then see that

$$\begin{aligned}2ac + 2ad + 2bc + 2bd &= (2a + b + d) + (b + c + 2d) + (a + 2c + d) + (a + 2b + c) \\ &= 4(a + b + c + d) \\ &= 4(e + f) \\ &= -2\end{aligned}$$

Factoring the LHS we have

$$\begin{aligned}2(ac + ad + bc + bd) &= 2(a(c + d) + b(c + d)) \\ &= 2(c + d)(a + b) \\ &= 2ef\end{aligned}$$

and so

$$\begin{aligned} 2ef &= -2 \\ \Rightarrow ef &= -1 \end{aligned}$$

Thus we have found that $ef = -1$ and $e + f = -\frac{1}{2}$, and so by lemma 2.1 we know that $e$ and $f$ are the roots pf the quadratic $x^2 + \frac{1}{2}x - 1$. We can find these roots explicitly using the quadratic formula, yielding

$$\begin{aligned} r_1 &= -\frac{1}{4} + \sqrt{\frac{17}{16}} \\ r_2 &= -\frac{1}{4} - \sqrt{\frac{17}{16}} \end{aligned}$$

Using some numerical estimates, we have

$$\begin{aligned} e &= \cos(2\pi/17) + \cos(4\pi/17) + \cos(8\pi/17) + \cos(16\pi/17) \\ &\approx 0.93247 + 0.73901 + 0.09226 - 0.98297 \\ &\approx 0.78077 \end{aligned}$$

$$\begin{aligned} f &= \cos(6\pi/17) + \cos(10\pi/17) + \cos(12\pi/17) + \cos(14\pi/17) \\ &\approx 0.44573 - 0.27366 - 0.60263 - 0.85021 \\ &\approx -1.28077 \end{aligned}$$

and so we must have that $r_1 = e$ and $r_2 = f$. Next, we saw before that $ab = -\frac{1}{4}$, and by definition $a+b = e$, so by lemma 2.1 $a$ and $b$ are the roots of the quadratic $x^2 - ex - \frac{1}{4}$. These roots are given explicitly by

$$\begin{aligned} r_1 &= \frac{e}{2} + \sqrt{\frac{1}{4} + \frac{e^2}{4}} \\ r_2 &= \frac{e}{2} - \sqrt{\frac{1}{4} + \frac{e^2}{4}} \end{aligned}$$

We again use some numerical estimates to determine which root is $a$ and which is $b$. We have

$$\begin{aligned} a &= \cos(2\pi/17) + \cos(8\pi/17) \\ &\approx 0.93247 + 0.09226 \\ &\approx 1.0247 \end{aligned}$$

$$\begin{aligned} b &= \cos(4\pi/17) + \cos(16\pi/17) \\ &\approx 0.73900 - 0.98297 \\ &\approx -0.24397 \end{aligned}$$

and so we must have

$$
\begin{aligned}
r_1 = a &= -\frac{1}{8} + \frac{1}{8}\sqrt{17} + \frac{1}{8}\sqrt{34 - 2\sqrt{17}} \\
r_2 = b &= -\frac{1}{8} + \frac{1}{8}\sqrt{17} - \frac{1}{8}\sqrt{34 - 2\sqrt{17}}
\end{aligned}
$$

We can determine $c$ and $d$ in a similar fashion. Recall from earlier that $2cd = -\frac{1}{2}$ and so $cd = -\frac{1}{4}$, and by definition $c + d = f$. Thus, by lemma 2.1, $c$ and $d$ are the roots of the quadratic $x^2 - fx - \frac{1}{4}$, which are given explicitly by

$$
\begin{aligned}
r_1 &= \frac{f}{2} + \sqrt{\frac{1}{4} + \frac{f^2}{4}} \\
r_2 &= \frac{f}{2} - \sqrt{\frac{1}{4} + \frac{f^2}{4}}
\end{aligned}
$$

Estimating $c$ and $d$ numerically we have

$$
\begin{aligned}
c &= \cos(6\pi/17) + \cos(10\pi/17) \\
&\approx 0.44573 - 0.27366 \\
&\approx 0.17206
\end{aligned}
$$

$$
\begin{aligned}
d &= \cos(12\pi/17) + \cos(14\pi/17) \\
&\approx -0.60263 - 0.85021 \\
&\approx -1.45284
\end{aligned}
$$

and so we must have

$$
\begin{aligned}
r_1 = c &= -\frac{1}{8} - \frac{1}{8}\sqrt{17} + \frac{1}{8}\sqrt{34 + 2\sqrt{17}} \\
r_2 = d &= -\frac{1}{8} - \frac{1}{8}\sqrt{17} - \frac{1}{8}\sqrt{34 + 2\sqrt{17}}
\end{aligned}
$$

Finally, we have, by the product to sum formula,

$$
\cos(\theta)\cos(4\theta) = \frac{1}{2}(\cos(3\theta) + \cos(5\theta)) = \frac{1}{2}c
$$

and since $\cos(\theta) + \cos(4\theta) = a$ by definition, we have, by lemma 2.1, that $\cos(\theta)$ and $\cos(4\theta)$ are the roots of the quadratic $x^2 - ax + \frac{1}{2}c$. These roots are

$$
\begin{aligned}
r_1 &= \frac{a}{2} + \sqrt{\frac{a^2}{4} - \frac{1}{2}c} \\
r_2 &= \frac{a}{2} - \sqrt{\frac{a^2}{4} - \frac{1}{2}c}
\end{aligned}
$$

Since both $\theta$ and $4\theta$ are in the interval $[0, \pi/2]$, we see that $\cos(\theta) > \cos(4\theta)$ since cos is decreasing on this interval. Thus we must have

$$\cos(\theta) = r_1 = \frac{a}{2} + \sqrt{\frac{a^2}{4} - \frac{1}{2}c}$$

This can be simplified somewhat by noting that

$$\begin{aligned} 2a^2 &= 2(\cos(\theta) + \cos(4\theta))^2 \\ &= 2\cos^2(\theta) + 4\cos(\theta)\cos(4\theta) + 2\cos^2(4\theta) \\ &= 1 + \cos(2\theta) + 2\cos(3\theta) + 2\cos(5\theta) + 1 + \cos(8\theta) \\ &= 2 + 2c + b \end{aligned}$$

and so we obtain

$$\cos(2\pi/17) = \frac{a}{2} + \sqrt{\frac{1}{4} + \frac{1}{8}b - \frac{1}{4}c}$$

which, after begin expanded, is

$$\cos(2\pi/17) = -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} + \frac{1}{16}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}$$

Thus $\cos(2\pi/17)$ is constructable, as desired. □

# 3 Broader Theory

Here we will briefly discuss some of the general theory on which $n$-gons are constructable and which are not. It relies on somewhat advanced abstract algebra, so we will try to limit the discussion to what we are already familiar with. A more in depth look at the concepts and proofs can be found in [1].

**Definition 3.1.** (Fermat Number) A Fermat number is a number of the form

$$2^{2^k} + 1$$

for some $k \geq 1$. If such a number happens to be prime, it is called a Fermat prime.

**Theorem 3.1.** *A regular $p^k$-gon for some prime $p$ is constructable if and only if $p^k$ has the form $2^a$ for some integer $a$, or is a Fermat prime.*

The proof is omitted, but can be found in [1]. We now prove what is called the Composition lemma.

**Theorem 3.2.** *Given two positive integers $m$ and $n$ such that $gcd(m, n) = 1$, the regular $m$ and $n$-gons are constructable if and only if a regular $mn$-gon is constructable.*

*Proof.* We first let the regular $mn$-gon be constructable. Thus we know by definition that the number $\cos(2\pi/mn)$ is a constructable number, and so we can split the unit circle into $mn$ equal arcs. Hence we can connect every $m$th point to construct a regular $n$-gon and we can connect every $n$th point to construct a regular $m$-gon. That is, if there are $mn$ total points on the unit circle, there will be a total of $n$ equally spaced $m$th points and hence connecting them will yield a regular $n$-gon, and vice versa. Thus we see that if the regular $mn$-gon is constructable then both the regular $n$-gon and $m$-gon are constructable.

We now let the regular $m$ and $n$-gons be constructable. By definition we know that $\cos(2\pi/n)$ and $\cos(2\pi/m)$ are both constructable numbers. Since we assumed that $\gcd(m, n) = 1$, then we know by the Euclidean Algorithm that there exists integers $a$ and $b$ such that $an + bm = 1$ and so

$$\begin{aligned}\cos\left(\frac{2\pi}{nm}\right) &= \cos\left(\frac{2\pi(an+bm)}{mn}\right)\\ &= \cos\left(2\pi\left(\frac{a}{m}+\frac{b}{n}\right)\right)\\ &= \cos\left(\frac{2\pi a}{m}\right)\cos\left(\frac{2\pi b}{n}\right) - \sin\left(\frac{2\pi a}{m}\right)\sin\left(\frac{2\pi b}{n}\right)\end{aligned}$$

The sine of any angle is easily constructed by simply drawing the line that intersects the circle and the $x$-axis at the point given by the cosine of the same angle. Thus if the two cosine terms in the expression above are constructable, then so are the sine terms. To show that the cosine terms are constructable, we note that since $a$ and $b$ are integers, the numbers $\cos\left(\frac{2\pi a}{m}\right)$ and $\cos\left(\frac{2\pi b}{n}\right)$ represent the $x$-coordinates of two of the vertices of the regular $m$ and $n$-gons. Since we know that these polygons are constructable, all of there vertices are, and so the cosine terms above are constructable. Thus the number $\cos\left(\frac{2\pi}{nm}\right)$ is constructable and so by definition the regular $mn$-gon is constructable, as desired. □

As an example of the composition lemma, consider the 15-gon. This polygon will split the unit circle into 15 equal arcs with 15 evenly spaced points, the vertices of the polygon. If we keep every third point but remove the rest, we are left with 5 equally spaced points, and so connecting them will yield a pentagon. If instead we remove all but every fifth point, we are left with 3 evenly spaced points, and so connecting them will yield an equilateral triangle.

Further, note that this result implies that if the $p_1p_2...p_k$-gon where the $p_i$'s are prime is constructable then each of the $p_i$-gons is also constructable, as we can set $p_1 = m$ and $n = p_2p_3...p_k$ to see that these are constructable. We then repeat this by setting $m = p_2$ and $n = p_3p_4...p_k$ to see that these are constructable. Carrying on we see that each of the $p_i$-gons is constructable. In the other direction, we see that if the $p_i$'s represent constructable regular polygons, then their product is a constructable regular polygon. To see this, take $m = p_1$, $n = p_2$ and so $p_1p_2$ is constructable. Next take $m = p_1p_2$ and $n = p_3$ and carry on in this fashion.

With these two results in hand, we can give a necessary and sufficient condition for a regular $n$-gon to be constructable.

**Theorem 3.3.** *A regular $n$-gon is constructable if and only if $n$ is of the form*

$$n = 2^a p_1 p_2 ... p_k$$

*where $a$ is a positive integer and the $p_i$'s are distinct Fermat primes.*

*Proof.* First suppose that $n$ has the form $2^a p_1 p_2 ... p_k$. It follows from theorem 3.1 that each number in the factorization of $n$ represents a regular constructable polygon, and so, by the composition lemma (since the Fermat primes are distinct and 2 is not a Fermat prime, the gcd of any of the numbers in $n$'s factorization will be 1), we see that the regular $n$-gon is also constructable.

Next suppose that the regular $n$-gon is constructable. We know from the composition lemma that each of the prime powers in the factorization of $n$ represents a constructable regular polygon. Further, by theorem 3.1 we see that each of these factors must have the form $2^a$ or be a Fermat prime. Thus we must have $n = 2^a p_1 p_2 ... p_k$ as desired. □

From this result we can automatically see that $17 = 2^{2^2} + 1$ must be constructable. Some other Fermat primes give us that the 257-gon and the 65537-gon are also constructable.

# 4 Conclusion

We have seen some of the theoretical aspects of which regular $n$-gons are constructable, but as with many things in math, none of it gives us any indication about how to actually construct a regular $n$-gon. Not even Gauss gave an actual construction of the heptadecagon. For that the mathematics community had to wait until 1893 when Herbert William Richmond gave one. It is also worth noting that an explicit construction for the 65537-gon does actually exist. It was given in a 200 page book by Johann Gustav Hermes in 1894 after 10 years of work. A more detailed look into the construction on the heptadecagon can be found in [1].

# References


[1] Devin Kuh. *Constructable Regular n-gons.* URL: `https://www.whitman.edu/Documents/Academics/Mathematics/Kuh.pdf`. (accessed: 05.25.2020).

[2] Yutaka Nishiyama. "Gauss' Method of Constructing a Regular Heptadecagon". In: *International Journal of Pure and Applied Mathematics* 82.5 (2013), pp. 695–707. DOI: `http://dx.doi.org/10.12732/ijpam.v82i5.3`.