Generalized Stokes' Theorem

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Contents

Abstract

We introduce and develop the necessary tools needed to generalize Stokes' Theorem. We begin our discussion by introducing manifolds and differential forms. Going beyond our treatment of integrable functions in 3 dimensions we go through what it means to integrate a smooth differential form on a smooth manifold with boundary. To do this we also introduce helpful tools like the wedge product and the exterior derivative. We do not give a proof of every theorem in this construction, but rather give the reader a detailed guide on how those theorems would be used to prove the main result of the paper: Stokes' Theorem. Unless otherwise noted all references are from [3].

1 The Essentials and Manifolds

The theory of Topological manifolds does not explicitly mention calculus. To understand what we mean by derivatives of functions, curves or maps we introduce a "smooth manifold". For this formalism we restrict our study to the subsets of $\mathbb{R}^n$. We begin by recalling the definition of a smooth map.

Definition 1.1. Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A map $F : U \to V$ is said to be *smooth* if each of the component functions of F has continuous partial derivatives of all orders.

To extend the idea of smoothness we introduce a *diffeomorphism*.

Definition 1.2. If a smooth map F is in addition bijective – one-to-one and onto– and has a smooth inverse map, it is called a *diffeomorphism* .

With the goal to additionally give subsets of $\mathbb{R}^n$ a smooth structure we must base our construction on the calculus of maps between Euclidean spaces. To study this calculus we introduce the following definitions.

Definition 1.3. Let M be a subset of $\mathbb{R}^n$. A *coordinate chart* on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to \widetilde{U}$ is a homeomorphism from U to an open subset $\tilde{U} = \varphi(U) \subset \mathbb{R}^n$.

Definition 1.4. Let M be a subset of $\mathbb{R}^n$. If $(U, \varphi), (V, \psi)$ are two charts such that $U \cap V \neq \varnothing$, then the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a composition of homeomorphisms, and is therefore itself a homeomorphism. Two charts (U, φ) and (V, ψ) are said to be *smoothly compatible* if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.

Loosely speaking, we need a way to keep track of what the charts are communicating about that subset through the so-called atlas.

Definition 1.5. We define an atlas for M to be a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a smooth atlas if any two charts in $\mathcal{A}$ are smoothly compatible with each other.

In practice, many different choices of an atlas would yield the same "smooth structure"(which we have not officially defined yet). That is, it gives us the same set of smooth functions on the manifold. To resolve this, we have a maximal atlas.

Definition 1.6. A smooth atlas $\mathcal{A}$ on M is called maximal if it is not contained in any strictly larger smooth atlas. This just means every chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$.

With this, we let a *smooth structure* on a manifold in $\mathbb{R}^n$ be a maximal smooth atlas.

Definition 1.7. A *smooth manifold* is a manifold together with a smooth structure

In the literature, the smooth structure is commonly omitted. In many cases we will not be able to find a smooth structure in which case the manifold is not smooth or we could possibly find a smooth structure of a given manifold. The difference is an interesting area of study but we will, for the purposes of this paper, just refer to manifolds as smooth or not smooth. In addition to a smooth manifold, we have the notion of a smooth manifold with boundary. Just as the name suggests, a smooth manifold with boundary is exactly like the smooth manifolds we have already defined but with additional structure known as the boundary. In order to rigorously define such an object, we introduce an analogue to $\mathbb{R}^k$.

Definition 1.8. The closed upper half space $\{(x_1, \ldots, x_n) : x^n \geq 0\}$ is denoted $\mathbb{H}^n$.

Definition 1.9. A *smooth manifold with boundary* is smooth manifold M in which every point has a neighbourhood homeomorphic to an open subset of the closed n-dimensional upper half space $\mathbb{H}^n$.

This definition will be essential in generalizing Stokes' Theorem. We also recall from our previous 3-dimensional treatment of Stokes' Theorem that it was important to have an orientable surface, or to orientate the surface. How do we go about giving a manifold in n dimensions an *orientation*? We recall that a *pointwise orientation* is given by choice of orientation of each tangent space in the 3 dimensional case and this definition is generalized to any number of dimensions. The same theorem applies as well.

Theorem 1.1. A connected, in the topological sense, orientable smooth manifold with boundary admits exactly two orientations.

A theorem that we present without proof will become useful for later in the paper.

Theorem 1.2. *If M is any smooth manifold with boundary, there is a smooth outward-pointing vector field along ∂M*

To conclude, we introduce the partition of unity. First, the idea of a *support* and its properties.

Definition 1.10. The *support* of a function f on a smooth manifold M, denoted $\operatorname{supp} f$ is the closure of the set where f is nonvanishing:

$$\operatorname{supp} f = \overline{\{p \in M : f(p) \neq 0\}}$$

Definition 1.11. Now let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an arbitrary open cover of a smooth manifold M. A partition of unity subordinate to $\mathcal{U}$ is a collection of smooth functions $\{\varphi_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$, with the following properties:
(i) $0 \leq \varphi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in M$
(ii) $\operatorname{supp} \varphi_\alpha \subset U_\alpha$
(iii) the set of supports $\{\operatorname{supp} \varphi_\alpha\}_{\alpha \in A}$ is locally finite; and (iv) $\sum_{\alpha \in A} \varphi_\alpha(x) = 1$ for all $x \in M$

2 Introduction to Differential Forms

To begin our discussion on differential forms we discuss multilinear functions – the language of tensors. We have already been introduced to, and are quite familiar with, some mulilinear functions: the cross product, dot product, and determinant. All of these functions are linear in each separate argument. Formally, we introduce the following definition.

Definition 2.1. Let k be a natural number and let V be a finite dimensional vector space. A *k-tensor* on V is a real-valued function T on the cartesian product V^k where T is real-valued multilinear function of k elements of V and

$$T : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}$$

The natural number k is usually referred to as the *rank* of the tensor. A neat way to make other tensors and study their properties is to perform operations on them like multiplying two real numbers. For tensors we have the following definition.

Definition 2.2. Let V be a finite-dimensional real vector space and let $S \in T^k(V), T \in T^l(V)$. Where $T^k(V)$ and $T^l(V)$ are the set of rank k and l tensors in V respectively. Define a map

$$S \otimes T : \underbrace{V \times \cdots \times V}_{k+l \text{ copies}} \to \mathbb{R}$$

by

$$S \otimes T\left(X_1, \ldots, X_{k+l}\right) = S\left(X_1, \ldots, X_k\right) T\left(X_{k+1}, \ldots, X_{k+l}\right)$$

With both S and T both depending linearly on its arguments, the X_i, it follows that $S \otimes T$ is a rank-$(k + l)$ tensor and is called the *tensor product*.

Theorem 2.1. The tensor product is associative; $T \otimes (R \otimes Q) = (T \otimes R) \otimes Q$.

Proof. Let T, R, and Q be rank p, q, and s rank tensors, respectively. By the associativity of the real numbers we have that

$$\begin{aligned}
&((T \otimes S) \otimes R)\left(X_1, \ldots, X_p, X_{p+1}, \ldots, X_{p+q}, X_{p+q+1}, \ldots, X_{p+q+s}\right) \\
&= \left[T\left(X_1, \ldots, X_p\right) \cdot S\left(X_{p+1}, \ldots, X_{p+q}\right)\right] \cdot R\left(X_{p+q+1}, \ldots, v_{p+q+s}\right) \\
&= T\left(X_1, \ldots, X_p\right) \cdot \left[S\left(v_{p+1}, \ldots, X_{p+q}\right) \cdot R\left(X_{p+q+1}, \ldots, X_{p+q+s}\right)\right] \\
&= (T \otimes (S \otimes R))\left(X_1, \ldots, X_p, X_{p+1}, \ldots, X_{p+q}, X_{p+q+1}, \ldots, X_{p+q+s}\right)
\end{aligned}$$

□

This type of product will eventually formulate the wedge product of differential forms. To develop this mathematical operation we need to know about a special kind of tensor– the alternating tensor.

Definition 2.3. A covariant k-tensor T on a finite-dimensional vector space V is said to be *alternating* if it has the following property:

$$T\left(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_k\right) = -T\left(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_k\right)$$

To describe the algebra and applications of alternating tensors we recall the definition of the sign of a permutation– a function that, in this context, will rearrange the arguments of a tensor. The sign is $+1$ if the the permutation can be written as a even number of compositions and -1 otherwise. Alternating tensors are interesting objects of study, but is there a way to make alternating tensors out of ordinary tensors? This process is called the *alternating projection*.

Definition 2.4. Let T be a k-tensor belonging to a finite-dimensional vector space V we define

$$\operatorname{Alt} T = \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) T^\sigma$$

where S_k is the set of permutations of the arguments of T. And T^σ denotes the permuted arguments of T so

$$(\text{ Alt } T)\left(X_1, \ldots, X_k\right) = \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) T\left(X_{\sigma(1)}, \ldots, X_{\sigma(k)}\right)$$

The following theorem is from [2]:

Theorem 2.2. Let T be a k-tensor. Then $\operatorname{Alt} T$ is an alternating tensor

Proof.

$$[\mathrm{Alt(T)}]^\sigma = \left[\frac{1}{k!}\sum_{\pi\in S_k}(-1)^\pi T^\pi\right]^\sigma = \frac{1}{k!}\sum_{\pi\in S_k}(-1)^{\pi\circ\sigma\circ\sigma}\left(T^\pi\right)^\sigma = \frac{1}{k!}(-1)^\sigma\sum_{\pi\in S_k}(-1)^{\pi\circ\sigma}T^{\pi\circ\sigma}$$

Now let $\tau = \pi\circ\sigma$. This yields

$$[\mathrm{Alt(T)}]^\sigma = (-1)^\sigma\left[\frac{1}{k!}\sum_{\tau\in S_k}(-1)^\tau T^\tau\right] = (-1)^\sigma[\mathrm{Alt(T)}]$$

□

Now having the ability to make alternating tensors out of ordinary ones and being able to multiply the results together we can introduce the wedge product.

3 The Wedge Product

With our preliminary work on tensors now complete we are ready to give a useful mathematical operation: the wedge product.

Definition 3.1. Let S and T be alternating tensors of rank k and l, respectively. The wedge product $S\wedge T$ is defined as

$$S\wedge T = \frac{(k+l)!}{k!l!}\,\mathrm{Alt}(S\otimes T)$$

With this product we summarize some of the most important properties of the wedge product with the following lemma and subsequent theorem. First a definition:

Definition 3.2. Let V be an n -dimensional vector space, and suppose $(\varepsilon^1,\ldots,\varepsilon^n)$ is any basis for V^*. We will define a collection of alternating tensors on V that generalize the determinant function on $\mathbb{R}^n$. For each multi-index $I=(i_1,\ldots,i_k)$ of length k such that $1\le i_1,\ldots,i_k\le n$, define a covariant k -tensor ε^I by

$$\begin{aligned}\varepsilon^I\left(X_1,\ldots,X_k\right) &= \det\begin{pmatrix}\varepsilon^{i_1}\left(X_1\right) & \ldots & \varepsilon^{i_1}\left(X_k\right)\\ \vdots & & \vdots\\ \varepsilon^{i_k}\left(X_1\right) & \ldots & \varepsilon^{i_k}\left(X_k\right)\end{pmatrix}\\ &= \det\begin{pmatrix}X_1^{i_1} & \ldots & X_k^{i_1}\\ \vdots & & \vdots\\ X_1^{i_k} & \ldots & X_k^{i_k}\end{pmatrix}\end{aligned}$$

Lemma 3.1. For any multi-indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}$$

where IJ is the multi-index $(i_1, \dots, i_k, j_1, \dots, j_l)$ obtained by concatenating I and J

Theorem 3.2. Let ω, η, ξ be alternating tensors. Then the wedge product is *associative* :

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi.$$

The product is also *anticommunative* so if ω is a k-alternating-tensor and η is an l-alternating-tensor then

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$$

Proof. To prove associativity, note that Lemma 3.1 gives

$$\left(\varepsilon^I \wedge \varepsilon^J\right) \wedge \varepsilon^K = \varepsilon^{IJ} \wedge \varepsilon^K = \varepsilon^{IJK} = \varepsilon^I \wedge \varepsilon^{JK} = \varepsilon^I \wedge \left(\varepsilon^J \wedge \varepsilon^K\right)$$

The general case follows from bilinearity. Similarly, using Lemma 3.1 again, we get

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ} = (\operatorname{sgn} \tau)\varepsilon^{JI} = (\operatorname{sgn} \tau)\varepsilon^J \wedge \varepsilon^I$$

where τ is the permutation that sends IJ to JI. It is easy to check that $\operatorname{sgn} \tau = (-1)^{kl}$, because τ can be decomposed as a composition of kl transpositions (each index of I must be moved past each of the indices of J) Anticommutativity then follows from bilinearity. $\square$

4 Forms on Manifolds and Exterior Derivative

Returning our attention to smooth manifolds we consider the subset of alternating k-tensors on a smooth n-dimensional manifold M. We take the following definition from [2].

Definition 4.1. Let M be a smooth manifold. A k-form on M is a function ω that assigns to each point $x \in M$ an alternating k-tensor $\omega(x)$ on the tangent space of X at x.

By choice of coordinate chart(see Definition 1.4) we can write a k-form ω locally as

$$\omega = \sum_I{}' \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

What kind of operations can we make on differential forms? We have already seen how we can take the wedge product but there is also the exterior derivative. This operation generalizes the differential of a function.

Definition 4.2. Let ω be some smooth k-form on an open subset of $\mathbb{R}^k$. The *exterior derivative* of ω is then defined as the following $(k+1)$ form:

$$d\left(\sum_I{}' \omega_I dx^I\right) = \sum_I{}' d\omega_I \wedge dx^I$$

or equivalently

$$d\left(\sum_I{}' \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) = \sum_I{}' \sum_i \frac{\partial \omega_I}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

We summarize the most important properties of the exterior derivative in the following theorem but do not give proof of each property.

Theorem 4.1. *The exterior differentiation operator defined for forms on arbitrary manifolds with boundary has the following properties:*
1. $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$
2. $d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$, *where* ω_1 *is a* k*-form.*
3. $d(d\omega) = 0$

With these properties and definition we know, for example, the exterior derivative of a real-valued function f is just its differential df. We emphasize this result because of its importance later in the paper.

5 Integration of Differential Forms

Our study of the integration of differential forms will just consider forms on the subsets of $\mathbb{R}^n$. We begin by defining what we mean by a smooth n-form on a compact set. We know from our study of complex functions this quarter that a smooth form ω on a set C has a smooth extension to some open set containing C.

Definition 5.1. Let C be a compact set in $\mathbb{R}^n$. Let ω be a smooth n-form on C.

$$\int_C f\, dx^1 \wedge \cdots \wedge dx^n = \int_C f\, dx^1 \cdots dx^n$$

Recalling Folland's change of variable theorem for integrable functions of n variables we have from [1]:

Theorem 5.1. *Let A be an invertible $n \times n$ matrix, and let $\mathbf{G}(\mathbf{u}) = Au$ be the corresponding linear transformation of $\mathbb{R}^n$. Suppose S is a measurable region in $\mathbb{R}^n$ and f is an integrable function on S. Then $\mathbf{G}^{-1}(S) = \{A^{-1}\mathbf{x} : \mathbf{x} \in S\}$ is measurable and $f \circ G$ is integrable on $\mathbf{G}^{-1}(S)$, and*

$$\int \cdots \int_S f(\mathbf{x}) d^n\mathbf{x} = |\det A| \int \cdots \int_{\mathbf{G}^{-1}(S)} f(A\mathbf{u}) d^n\mathbf{u}$$

To prove the following theorem would require additional definitions and ideas that don't help the reader understand the main result. We use Theorem 5.2 to prove the more "useful" Corollary that follows.

Theorem 5.2. Let D and E be domains of integration in $\mathbb{R}^n$, and let ω be an n -form on E. If $G : D \to E$ is a smooth map whose restriction to Int D is an orientation-preserving($\det(dG) > 0$) or orientation-reversing($\det(dG) < 0$)) diffeomorphism onto Int E, then

$$\int_E \omega = \begin{cases} \int_D G^*\omega & \text{if } G \text{ is orientation-preserving,} \\ -\int_D G^*\omega & \text{if } G \text{ is orientation-reversing.} \end{cases}$$

Corollary 5.2.1. Suppose U, V are open subsets of $\mathbb{R}^n$, $G : U \to V$ is an orientation-preserving diffeomorphism, and ω is a compactly supported n-form on V. Then

$$\int_V \omega = \int_U G^*\omega$$

Proof. Let $E \subset V$ be a domain of integration containing supp ω(just the support of ω). Since smooth maps take sets of measure zero to sets of measure zero, $D = G^{-1}(E) \subset U$ is a domain of integration containing supp $G^*\omega$. Therefore, the result follows from the preceding theorem. □

To make sense of integrating a form over a whole manifold we have to consider a partition of unity(see Definition 1.11) and arrive at the following:

Definition 5.2. Let $\{(U_i, \phi_i)\}$ be a finite cover of the support of ω and let $\{\psi_i\}$ be a subordinate partition of unity. Then the integral of ω over M is given by

$$\int_M \omega = \sum_i \int_M \psi_i \omega$$

Theorem 5.3. *The definition of $\int_M \omega$ given above does not depend on the choice of oriented charts or partition of unity.*

6 Generalized Stokes' Theorem

All the previous sections now come together to give a beautiful and general result.

Theorem 6.1. *Let M be an oriented n dimensional manifold with boundary, and let ω be a compactly supported $(n-1)$-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega$$

Proof. We begin by considering a very special case: Suppose M is the upper half space $\mathbb{H}^n$ itself. Then the fact that ω has compact support means that there is a number $R > 0$ such that supp ω is contained in the rectangle $A = [-R, R] \times \cdots \times [-R, R] \times [0, R]$. We can write ω in standard coordinates as

$$\omega = \sum_{i=1}^{n} \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

where the hat means that dx^i is omitted. Therefore,

$$\begin{aligned} d\omega &= \sum_{i=1}^{n} d\omega_i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \end{aligned}$$

Thus we compute

$$\begin{aligned} \int_{\mathbb{H}^n} d\omega &= \sum_{i=1}^{n} (-1)^{i-1} \int_A \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) dx^1 \cdots dx^n \end{aligned}$$

We can rearrange the order of integration in each term so as to do the x^i integration first. By the fundamental theorem of calculus, the terms for which $i \neq n$ reduce to

$$
\begin{aligned}
&\sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^R\cdots\int_{-R}^R\frac{\partial\omega_i}{\partial x^i}(x)dx^1\cdots dx^n\\
&\quad=\sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^R\cdots\int_{-R}^R\frac{\partial\omega_i}{\partial x^i}(x)dx^idx^1\cdots\widehat{dx}^i\cdots dx^n\\
&\quad=\sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^R\cdots\int_{-R}^R\omega_i(x)\Big|_{x^i=-R}^{x^i=R}dx^1\cdots\widehat{dx}^i\cdots dx^n\\
&\quad=0
\end{aligned}
$$

because we have chosen R large enough that $\omega = 0$ when $x^i = \pm R$. The only term that might not be zero is the one for which $i = n$. For that term we have

$$
\begin{aligned}
\int_{\mathbb{H}^n} d\omega &= (-1)^{n-1}\int_{-R}^R\cdots\int_{-R}^R\int_0^R\frac{\partial\omega_n}{\partial x^n}(x)dx^ndx^1\cdots dx^{n-1}\\
&= (-1)^{n-1}\int_{-R}^R\cdots\int_{-R}^R\omega_i(x)\Big|_{x^n=0}^{x^n=R}dx^1\cdots dx^{n-1}\\
&= (-1)^n\int_{-R}^R\cdots\int_{-R}^R\omega_i\left(x^1,\ldots,x^{n-1},0\right)dx^1\cdots dx^{n-1}
\end{aligned}
$$

because $\omega_n = 0$ when $x^n = R$. To compare this to the other side of the above equation we compute as follows:

$$
\int_{\partial\mathbb{H}^n}\omega=\sum_i\int_{A\cap\partial\mathbb{H}^n}\omega_i\left(x^1,\ldots,x^{n-1},0\right)dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n
$$

Because x^n vanishes on $\partial\mathbb{H}^n$, the restriction of dx^n to the boundary is identically zero. Thus, the only term above that is nonzero is the one for which $i = n$, which becomes

$$
\int_{\partial\mathbb{H}^n}\omega=\int_{A\cap\partial\mathbb{H}^n}\omega_n\left(x^1,\ldots,x^{n-1},0\right)dx^1\wedge\cdots\wedge dx^{n-1}
$$

Taking into account the fact that the coordinates $(x^1,\ldots,x^{n-1})$ are positively oriented for $\partial\mathbb{H}^n$ when n is even and negatively oriented when n is odd, this becomes

$$
\int_{\partial\mathbb{H}^n}\omega=(-1)^n\int_{-R}^R\cdots\int_{-R}^R\omega_n\left(x^1,\ldots,x^{n-1},0\right)dx^1\cdots dx^{n-1}.
$$

Next, let M be an arbitrary manifold with boundary, but consider an $(n-1)$-form ω that is compactly supported in the domain of a single chart (U,φ). Assuming without loss of generality that φ is an oriented chart, the definition(with the right interpretation of charts in this case) yields

$$\int_M d\omega = \int_{\mathbb{H}^n} \left(\varphi^{-1}\right)^* d\omega = \int_{\mathbb{H}^n} d\left(\left(\varphi^{-1}\right)^* \omega\right)$$

since $\left(\varphi^{-1}\right)^* d\omega$ is compactly supported on $\mathbb{H}^n$. By the computation above, this is equal to

$$\int_{\partial\mathbb{H}^n} \left(\varphi^{-1}\right)^* \omega$$

where $\partial\mathbb{H}^n$ is given the induced orientation. since φ_* takes outward-pointing vectors on ∂M to outward-pointing vectors on $\mathbb{H}^n$ (by Theorem 1.2), it follows that $\varphi|_{U\cap\partial M}$ is an orientation-preserving diffeomorphism onto $\varphi(U)\cap\partial\mathbb{H}^n$, and thus is equal to $\int_{\partial M}\omega$. This proves the theorem in this case.

Finally, let ω be an arbitrary compactly supported $(n-1)$-form. Choosing a cover of supp ω by finitely many oriented coordinate charts $\{(U_i,\varphi_i)\}$, and choosing a subordinate partition of unity $\{\psi_i\}$, we can apply the preceding argument to $\psi_i\omega$ for each i and obtain

$$\begin{aligned}
\int_{\partial M}\omega &= \sum_i \int_{\partial M} \psi_i\omega \\
&= \sum_i \int_M d\left(\psi_i\omega\right) \\
&= \sum_i \int_M d\psi_i \wedge \omega + \psi_i d\omega \\
&= \int_M d\left(\sum_i \psi_i\right) \wedge \omega + \int_M \left(\sum_i \psi_i\right) d\omega \\
&= 0 + \int_M d\omega
\end{aligned}$$

because $\sum_i \psi_i \equiv 1$. This proves the result. □

7 Conclusion

With the presentation of this general result we immediately obtain the higher dimensional analogue of the Fundamental Theorem of Line Integrals, Green's Theorem and the Divergence Theorem. This result has important implications in the study of cohomology of forms, complex analysis, and other areas of differential geometry. The generalized Stokes' Theorem also has rich applications in active areas of Physics research.

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