On the Prime Number Theorem

Alice Yang

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1 Introduction

In this paper, we will discuss the prime number theorem. The proof we will give is based on a concise version of Newman's proof by Zagier [7]. It relies on some properties of the Riemann zeta function, the most essential of which is the fact that $\zeta(s)$ has no zeros on $\mathrm{Re}(s) \geqslant 1$. In addition, it relies on an analytic theorem commonly known as "Tauberian theorem", which is of interest in its own right.

Since this is a topic in complex analysis, the reader should expect to be familiar with the convergence and uniform convergence of infinite series and infinite products of real and complex numbers, complex integration, and analyticity.

We should note that there are also "elementary" proofs of the prime number theorem obtained independently by Erd*ös* [2] and Selberg [5] in the 1940s that do not use the Riemann zeta function, but they are only elementary in the sense that they do not involve complex analysis; these "elementary" proofs are actually much more complicated than those that use complex analysis.

2 Analysis background

In this section, we will give functions, transforms, and theorems that we will be using in our proofs for the prime number theorem. We will not prove any of the results. Most results can be found in Gamelin's *Complex Analysis* [3].

2.1 Riemann Zeta Function

The Riemann Zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

where the series converges absolutely and uniformly for $\mathrm{Re}(s) \geqslant 1+\epsilon, \epsilon > 0$ by Weierstrass M-test, setting $M_n = n^{-(1+\epsilon)}$. Thus it defines an analytic function on this domain since n^{-s} is analytic for $\mathrm{Re}(s) \geqslant 1, n \geqslant 1$.

2.2 Chebyshev function

$$\vartheta(x) = \sum_{p \leqslant x} \log p.$$

This is also called the log-weighted prime counting function. The sum above range over primes p: the symbol p will always denote a prime number, and any sum or product over p is understood to be over primes even if it is not specifically stated.

The function $\vartheta(x)$ is an increasing, piecewise continuous function of x.

2.3 Auxiliary function

$$\Phi(s) = \sum_{p} \frac{\log p}{p^s}.$$

We can easily verify that the series converges absolutely and locally uniformly for $\mathrm{Re}(s) \geqslant 1$, thus it defines an analytic function in this domain given that $\dfrac{\log p}{p^s}$ is analytic for $\mathrm{Re}(s) \geqslant 1$, p prime.

2.4 Laplace Transform

Let $h(t)$ be a real-valued piecewise continuous function that is defined for all $t \geqslant 0$. The Laplace transform $\mathscr{L}h(s)$ is the complex function defined by

$$\mathscr{L}h(s) = \int_0^{\infty} h(t)e^{-st}dt.$$

The following properties of Laplace transform are easy to verify, and is left to the reader:

- $\mathscr{L}(g+h) = \mathscr{L}g + \mathscr{L}h$.

- For any $a \in \mathbb{R}, \mathscr{L}(ah) = a(\mathscr{L}h)$.
- If $h(t) = a \in \mathbb{R}$ is constant, then $\mathscr{L}h(s) = \dfrac{a}{s}$.
- $\mathscr{L}(e^{at}h(t))(s) = \mathscr{L}h(s-a)$.

2.5 Complex analysis background

Definition 2.5.1(Convergence of infinite product). The infinite product $\prod_{j=1}^{\infty} p_j$ converges if $\lim_{j\to\infty} P_j = P \neq 0$, where P_j is the partial product of p_j.

If the infinite product converges and one of the $p_j{}'s$ is 0, then we define the value of the product to be 0, otherwise, we define it to be

$$\prod_{j=1}^{\infty} p_j = \exp\,(\sum_{j=1}^{\infty} \log p_j),$$

where $\log(p_j)$ is the principal value of the logarithm. Thus, any questions about infinite products can be translated into a question about infinite series by taking the logarithms, $\prod_{j=1}^{\infty} p_j$ converges if $\sum_{j=1}^{\infty} \log p_j$ converges.

Notice that this implies that $p_j \neq 0$ and $p_j = \frac{P_{j+1}}{P_j} \to 1$.

Theorem 2.5.1. $\prod_{j=1}^{\infty} p_j$ converges if and only if $\sum_{j=1}^{\infty} \log(1+a_j)$ converges, where $\log(1+a_j)$ is the principal value of the logarithm.

It is not necessarily true that $\log(\prod_{j=1}^{\infty} p_j) = \sum_{j=1}^{\infty} \log(1+a_j) = S$.

For a more complete discussion of the convergence of the infinite product, along with a proof, see section 2.2 of Ahlfors [1].

Definition 2.5.2 (Logarithmic differentiation). If $G(z) = g_1(z)...g_m(z)$ is a finite product of analytic functions, then by taking the logarithms and differentiating, we obtain $\dfrac{G'(z)}{G(z)} = \dfrac{g_1'(z)}{g_1(z)} + ... + \dfrac{g_m'(z)}{g_m(z)}$. This procedure is called **logarithmic differentiation**.

The logarithmic differentiation also holds for uniformly convergent infinite products of analytic functions.

Theorem 2.5.2. Let $g_k(z)$ $(k \geqslant 1)$ be analytic functions on a domain D such that $\prod_{k=1}^{m} g_k(z)$ converges normally on D to $G(z) = \prod_{k=1}^{\infty} g_k(z)$. Then

$$\frac{G'(z)}{G(z)} = \sum_{k=1}^{\infty} \frac{g_k'(z)}{g_k(z)}$$

where the sum converges normally on D.

Proof: We can apply the logarithmic differentiation to finite subproducts and passing the limit. Note that the function $\frac{G'(z)}{G(z)}$ has poles at zeros of G (z), and the order of pole of $\frac{G'(z)}{G(z)}$ equals to the order of the zero of G(z). Moreover, the hypothesis implies that $g_k(z) \to 1$ uniformly on any compact subset of D, so the summands $\frac{g_k'(z)}{g_k(z)}$ are analytic on any compact subset for k large.

We know that the uniform convergence of the series is not affected by the first terms of the series, the poles has no affect on the uniform convergence of the series. Moreover, we can derive from the theorem that if f is meromorphic on a set S, then so is its logarithmic derivative $\frac{f'}{f}$.

For a more rigorous proof, see section XIII.3 in Gamelin [3].

3 Prime Number Theorem

In this section, we will prove the prime number theorem. The proof is really similar to Newman's proof [7], we will follow the outline of Newman's proof, while being slightly more expansive.

As with most proofs of the prime number theorem, the proof is divided into two parts. The first is to show that the zeta function has no zeros on $\mathrm{Re}(s) \geqslant 1$. The second is to establish the "Tauberian theorem". The following proofs were inspired by Sutherland [6] and Gamelin [3] and are essential to prove the prime number theorem.

The prime counting function $\pi(x) : \mathbb{R} \to \mathbb{Z}_{\geqslant 0}$ is defined as

$$\pi(x) = \sum_{p \leqslant x} 1,$$

it counts the number of primes up to x. The prime number theorem says that the number $\pi(x)$ of prime numbers not exceeding x satisfies

$$\pi(x) \sim \frac{x}{\log(x)}$$

as $x \to \infty$.
Here, the notation $f(x) \sim g(x)$ ("f and g are asymptotically equal") means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. In other words, the functions f and g grow at the same rate, asymptotically.

To prove the prime number theorem, the first step is to show that the zeta function has no zeros on $\mathrm{Re}(s) \geqslant 1$. Unless otherwise stated, $\log(z), z \in \mathbb{C}$ stands for the principal branch or the principal value of the logarithm.

Theorem 3.1 (Euler Product). For $\mathrm{Re}(s) > 1$, we have

$$\zeta(s) = \sum_{n\geqslant 1} n^{-s} = \prod_{p} \left(1 - p^{-s}\right)^{-1},$$

where the product converges absolutely. In particular, $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$.

Proof. Since $n \geqslant 1, n \in \mathbb{Z}$, by the Fundamental Theorem of Arithmetic, any positive integer greater than 1 can be factored into a product of prime numbers in a unique way, that is, $n = 2^{e_2}3^{e_3}....$ Since $\sum_{n\geqslant 1} n^{-s}$ converges absolutely for $\mathrm{Re}(s) > 1$, we have $\sum_{n\geqslant 1} n^{-s} = \sum_{e_2,e_3...\geqslant 0} (2^{e_2}3^{e_3}...)^{-s}$.

Now for each $m \geqslant 1$, let S_m be the m-smooth numbers: positive integers with prime factors $p \leqslant m$, and define

$$\zeta_m(s) = \sum_{n\in S_m} n^{-s},$$

which converges absolutely and uniformly on $\mathrm{Re}(s) > 1$. If $p_1...p_k$ are the primes up to m, we can rewrite

$$\zeta_m(s) = \sum_{n\in S_m} n^{-s} = \sum_{e_1...e_k\geqslant 0} ({p_1}^{e_1}...{p_k}^{e_k})^{-s} = (\sum_{e_1\geqslant 0} {p_1}^{-e_1 s})...(\sum_{e_k\geqslant 0} {p_k}^{-e_k s}),$$

since $\zeta_m(s)$ converge absolutely, rearrangement is justified. For $\mathrm{Re}(s) > 1$, we have $|p^{-s}| < 1, \sum_{e\geqslant o} p^{-es} = \sum_{e\geqslant o} \left(p^{-s}\right)^{e} = \frac{1}{1-p^{-s}}$ (geometric series) for any prime

p. If we apply this k times, we obtain

$$\zeta_m(s) = \sum_{n \in S_m} n^{-s} = \prod_p (\sum_{e \geqslant o} p^{-es}) = \prod_{p \leqslant m} (1-p^{-s})^{-1}.$$

Now for any $\epsilon > 0$, $\mathrm{Re}(s) > 1+\delta, \delta > 0$, we have

$$|\zeta_m(s) - \zeta(s)| \leqslant \Big| \sum_{n \geqslant m} n^{-s} \Big| \leqslant \sum_{n \geqslant m} |n^{-s}| = \sum_{n \geqslant m} n^{-Re(s)} \leqslant \int_m^\infty x^{-1-\delta} dx \leqslant \frac{1}{\delta} m^{-\delta} < \epsilon$$

for m sufficiently large, given $n^{-Re(s)}$ is a positive, decreasing function of s on $[m, \infty)$. Then by definition, $\zeta_m(s)$ converge to $\zeta(s)$ uniformly on $\mathrm{Re}(s) > 1$, hence the sequence of functions $P_m(s) = \prod_{p \leqslant m} (1-p^{-s})^{-1}$ converges to $\zeta(s)$ uniformly as well. Therefore we have for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \sum_{n \geqslant 1} n^{-s} = \prod_p (1-p^{-s})^{-1}.$$

Then clearly, the sequence $\log P_m(s)$ converge uniformly to $\log \prod_p (1-p^{-s})^{-1}$ locally for $\mathrm{Re}(s) > 1$ as $m \to \infty$, since $\log(s)$ is a continuous function on that domain. Because $\log(1-z) = -\sum_{n \geqslant 1} \frac{z^n}{n}$ for $|z| < 1$, and $\mathrm{Re}(s) > 1, |p^{-s}| < 1$, we have

$$\log |(1-p^{-s})^{-1}| = \log (1-p^{-s})^{-1} = \sum_{e \geqslant 1} \frac{1}{e} p^{-es}.$$

Given that $P_m(s)$ is never 0, it follows that

$$|\zeta(s)| = \Big| \prod_{n=1}^{\infty} \frac{1}{(1-{p_n}^{-s})} \Big| = \lim_{m \to \infty} \prod_{n=1}^{m} \Big| \frac{1}{(1-{p_n}^{-s})} \Big|$$

$$= \lim_{m \to \infty} \exp(\log \prod_{n=1}^{m} \Big| \frac{1}{(1-{p_n}^{-s})} \Big|) = \lim_{m \to \infty} \exp(\sum_{n=1}^{m} \log \Big| \frac{1}{(1-{p_n}^{-s})} \Big|).$$

Moreover, we have

$$\sum_p |\log (1-p^{-s})^{-1}| = \sum_p \Big| \sum_{e \geqslant 1} \frac{1}{e} p^{-es} \Big| \leqslant \sum_p \sum_{e \geqslant 1} |\frac{1}{e} p^{-es}| \leqslant \sum_p \sum_{e \geqslant 1} |p^{-s}|^e$$

$$= \sum_p (|p^s| - 1)^{-1} < \sum_p |p^s|^{-1} = \sum_p p^{-Re(s)} < \infty.$$

Therefore $\sum_p |\log (1-p^{-s})^{-1}|$ converge absolutely (i.e. finite), thus $\prod_p (1-p^{-s})^{-1}$ converge absolutely, hence nonzero because the exponential function is continuous and never 0. Thus we have shown that $\zeta(s) = \prod_p (1-p^{-s})^{-1} \neq 0$ for $\mathrm{Re}(s) > 1$. □

Theorem 3.2. For $\mathrm{Re}(s) > 1$, we have

$$\zeta(s) = \frac{1}{s-1} + \phi(s),$$

where $\phi(s)$ is analytic on $\mathrm{Re}(s) > 0$. Thus we conclude that $\zeta(s)$ extends to a meromorphic function on $\mathrm{Re}(s) > 0$ and has only one simple pole at $s = 1$ with residue 1 and no other poles.

Proof. For $\mathrm{Re}(s) > 1$, we have

$$\zeta(s) - \frac{1}{s-1} = \sum_{n \geqslant 1} n^{-s} - \int_1^{\infty} x^{-s} dx.$$

Since $\sum_{n \geqslant 1} n^{-s}$ converges absolutely and uniformly for $\mathrm{Re}(s) > 1$, and $\int_1^{\infty} x^{-s} dx = \sum_{n \geqslant 1} \Big(\int_n^{n+1} x^{-s} dx \Big)$ converges absolutely for $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned} \zeta(s) - \frac{1}{s-1} &= \sum_{n \geqslant 1} n^{-s} - \sum_{n \geqslant 1} (\int_n^{n+1} x^{-s} dx) \\ &= \sum_{n \geqslant 1} (n^{-s} - \int_n^{n+1} x^{-s} dx) = \sum_{n \geqslant 1} \int_n^{n+1} (n^{-s} - x^{-s}) dx. \end{aligned}$$

For each $n \geqslant 1$, define $\phi_n(s) = \int_n^{n+1} (n^{-s} - x^{-s}) dx$. Since $(n^{-s} - x^{-s})$ is a continuous complex-valued function for $n \leqslant x \leqslant n+1$, and for each fixed x, $(n^{-s} - x^{-s})$ is an analytic function of s on $\mathrm{Re}(s) > 0, \phi_n(s) = \int_n^{n+1} (n^{-s} - x^{-s}) dx$ is analytic on $\mathrm{Re}(s) > 0$. For each fixed s such that $\mathrm{Re}(s) > 0, x \in [n, n+1]$, we have

$$|n^{-s} - x^{-s}| = \Big| \int_n^x s t^{-s-1} dt \Big| \leqslant \int_n^x \frac{|s|}{|t^{s+1}|} dt = \int_n^x \frac{|s|}{t^{Re(s)+1}} dt \leqslant \frac{|s|}{n^{Re(s)+1}}.$$

Since the interval $[n, n+1]$ is a piecewise smooth curve with length 1, by ML-estimate [3, page 105], we have

$$|\phi_n(s)| = \Big| \int_n^{n+1} (n^{-s} - x^{-s}) dx \Big| \leqslant \int_n^{n+1} |n^{-s} - x^{-s}| |dx| \leqslant \frac{|s|}{n^{Re(s)+1}}.$$

For any s_0 with $\mathrm{Re}(s_0) > 0$, if we take $\epsilon = \dfrac{Re(s_0)}{2}, \epsilon > 0$, and let U be the union of closed disk D_ϵ centered at s_0 with radius less than ϵ, then for each $n \geqslant 1$, we have

$$\sup_{s\in U}|\phi_n(s)| \leqslant \frac{|s_0|+\epsilon}{n^{1+\epsilon}}.$$

Let $M_n = \dfrac{|s_0|+\epsilon}{n^{1+\epsilon}}$, then $\sum_n M_n = (|s_0|+\epsilon)\zeta(1+\epsilon)$ converges by the p-test. Then by Weierstrass M-test, $\sum_n \phi_n$ converges uniformly on each closed disk in $\text{Re}(s) > 0$. Hence $\sum_n \phi_n$ converges normally to a function $\phi(s) = \zeta(s) - \dfrac{1}{s-1}$.

Since $\phi_n(s)$ is analytic on $\text{Re}(s) > 0$, $\phi(s) = \zeta(s) - \dfrac{1}{s-1}$ is analytic on $\text{Re}(s) > 0$. Therefore, $\zeta(s) = \dfrac{1}{s-1} + \phi(s)$ extends to a meromorphic function on $\text{Re}(s) > 0$ and has only one simple pole at $s = 1$ with residue 1 and no other poles. □

We have shown from Theorem 3.1 that $\zeta(s) \neq 0$ for $\text{Re}(s) > 1$. We now want to show that $\zeta(s)$ has no zeros on $\text{Re}(s) = 1$, which is the key to prove the prime number theorem. To prove this, we rely on the following lemma.

Lemma 3.3 (Mertens). For all $x, y \in \mathbb{R}, x > 1$, we have $|\zeta(x)^3\zeta(x+iy)^4\zeta(x+2iy)| \geqslant 1$.

Proof. We have from Euler product $\zeta(s) = \prod_p (1-p^{-s})^{-1}$, and we know that $\log|z| = Re(\log z)$, $\log(1-z) = -\sum_{n\geqslant 1}\dfrac{z^n}{n}$ for $|z| < 1$. Since $\text{Re}(s) > 1, |p^{-s}| < 1$, we have

$$\log|\zeta(s)| = -\sum_p \log|1-p^{-s}| = -\sum_p Re(\log(1-p^{-s})) = \sum_p\sum_{n\geqslant 1}\frac{Re(p^{-ns})}{n}.$$

Set $\zeta = x + iy$, since $s^\alpha = e^{\alpha logs}$ for $s \neq 0$, we have

$$\begin{aligned} p^{-ns} &= p^{-n(x+iy)} = e^{-n(x+iy)\log p} = e^{-nx\log p}e^{-i(ny\log p)} \\ &= p^{-nx}[\cos(ny\log p) - i\sin(ny\log p)]. \end{aligned}$$

$$Re(p^{-ns}) = Re(p^{-nx}[\cos(ny\log p) - i\sin(ny\log p)]) = p^{-nx}\cos(ny\log p).$$

Then we have

$$\log|\zeta(x+iy)| = \sum_p\sum_{n\geqslant 1}\frac{\cos(ny\log p)}{np^{nx}},$$

$$\log|\zeta(x+iy)^4| = 4\log|\zeta(x+iy)| = \sum_p \sum_{n\geqslant 1} \frac{4\cos(ny\log p)}{np^{nx}}.$$

Similarly, we have

$$\log|\zeta(x)^3| = \sum_p \sum_{n\geqslant 1} \frac{3}{np^{nx}},$$

$$\log|\zeta(x+2iy)| = \sum_p \sum_{n\geqslant 1} \frac{\cos(2ny\log p)}{np^{nx}}.$$

Now, recall that

$$\log|\zeta(x)^3\zeta(x+iy)^4\zeta(x+2iy)| = \log|\zeta(x)^3| + \log|\zeta(x+iy)^4| + \log|\zeta(x+2iy)|,$$

and $\sum_p |\log{(1-p^{-s})}^{-1}|$ converges absolutely for $\mathrm{Re}(s) > 1$, then we have

$$\log|\zeta(x)^3\zeta(x+iy)^4\zeta(x+2iy)| = \sum_p \sum_{n\geqslant 1} \frac{3+4\cos(ny\log p)+\cos(2ny\log p)}{np^{nx}}.$$

Recall the double angle formula for cosine, $\cos 2\theta = 2{(\cos\theta)}^2 - 1$, let $\theta = ny\log p$, then we have

$$3 + 4\cos\theta + \cos 2\theta = 3 + 4\cos\theta + 2{(\cos\theta)}^2 - 1 = 2{(\cos\theta + 1)}^2 \geqslant 0.$$

Then since every term in the series is positive, we have

$$\log|\zeta(x)^3\zeta(x+iy)^4\zeta(x+2iy)| = \sum_p \sum_{n\geqslant 1} \frac{3+4\cos(ny\log p)+cos(2ny\log p)}{np^{nx}} \geqslant 0.$$

Hence $|\zeta(x)^3\zeta(x+iy)^4\zeta(x+2iy)| \geqslant 1$ as desired. □

Corollary 3.4. $\zeta(s)$ has no zeros on $\mathrm{Re}(s) \geqslant 1$.

Proof. From Theorem 3.1, we know that $\zeta(s)$ has no zeros on $\mathrm{Re}(s) > 1$. From Theorem 3.2, we know that $\zeta(s)$ extends to a meromorphic function on $\mathrm{Re}(s) > 0$.

Let $s = x + iy$, and suppose that $\zeta(1+iy) = 0$ for some $y \in \mathbb{R}$. We know that $y \neq 0$, since by Theorem 3.2, $\zeta(s) = \frac{1}{s-1} + \phi(s)$ has only one simple pole at $s = 1$ with residue 1 and no other poles. Then $\zeta(s)$ does not have a pole at $(1+2iy) \neq 1$. Since $\zeta(1+iy) = 0$ for some $y \in \mathbb{R}, y \neq 0, \zeta{(1+iy)}^4 = 0$, that is, $\zeta{(x+iy)}^4$ has a zero at $(1+iy)$ with order at least 4. Suppose $(1+iy)$ is a zero of $\zeta{(x+iy)}^4$ with order 4. Given $\zeta(s)$ extends to a meromorphic function on $\text{Re}(s) > 0$, $\zeta(s)^4$ extends to a meromorphic function on $\text{Re}(s) > 0$. $\zeta(s)^4 = \zeta(x+iy)^4 = ((x+iy)-(1+iy))^4 f(x+iy) = (x-1)^4 f(x+iy), f(1+iy) \neq 0$.

Since $\zeta(s)$ has a simple pole at $s = 1$ with residue 1, $\zeta(x)$ has a simple pole at $x = 1$. Moreover, $\zeta(x)^3$ has a pole at $x = 1$ with order 3, that is, $\zeta(x)^3 = \dfrac{g(x)}{(x-1)^3}, g(1) \neq 0$, since $\zeta(x)^3$ extends to a meromorphic function on $\text{Re}(s) > 0$.

Then we have

$$\lim_{x\to 1^+} |\zeta(x)^3\zeta(x+iy)^4\zeta(x+2iy)| = \lim_{x\to 1^+} |\frac{g(x)(x-1)^4 f(x+iy)\zeta(x+2iy)}{(x-1)^3}| = 0,$$

since $\zeta(s)$ has a simple pole at $s = 1$ and a zero at $(1 + iy)$ and no pole at $(1 + 2iy)$, but this contradicts Lemma 3.3. Therefore $\zeta(s)$ doesn't have a zero at $(1+iy)$, hence $\zeta(s)$ has no zeros on $\text{Re}(s) \geqslant 1$. □

Notice that none of the proofs of the prime number theorem treats the function $\pi(x)$ directly. They treat the function

$$\vartheta(x) = \sum_{p\leqslant x} \log p$$

instead. We will now introduce a useful property of $\vartheta(x)$.

Lemma 3.5 (Chebyshev). For $x \geqslant 1, \vartheta(x) = O(x)$, where $O(f)$ denotes a quantity bounded in absolute value by a fixed multiple of f. $\vartheta(x) = O(x)$ means that there is a constant $C > 0$ such that $|\vartheta(x)| \leqslant C|x|$ for all $x \geqslant 1$.

Proof. For any integer $n \geqslant 1$, the binomial theorem gives us

$$2^{2n} = (1+1)^{2n} = \sum_{m=0}^{2n} \binom{2n}{m}(1^{2n})(1^{2n-m}) = \sum_{m=0}^{2n} \binom{2n}{m} \geqslant \binom{2n}{n} = \frac{2n!}{n!n!}.$$

Since $2n!$ is divisible by each prime number p between n and $2n$ and $n!$ is not divisible by such p, we have

$$2^{2n} \geqslant \frac{2n!}{n!n!} \geqslant \prod_{n<p\leqslant 2n} p = e^{(\vartheta(2n)-\vartheta(n))},$$

since $\vartheta(2n) - \vartheta(n) = \sum_{n<p\leqslant 2n} \log p, e^{(x+y)} = e^x e^y$. Take the logarithm on both sides, we have

$$\log(2^{2n}) \geqslant \log(e^{(\vartheta(2n)-\vartheta(n))}) \Rightarrow \vartheta(2n) - \vartheta(n) \leqslant 2n \log 2.$$

Now for any integer $m \geqslant 1$, we have

$$\begin{aligned}\vartheta(2^m) = \sum_{p \leqslant 2^m} \log p &= \sum_{n=1}^{m} \Big(\sum_{p \leqslant 2^n} \log p - \sum_{p \leqslant 2^{n-1}} \log p \Big) \\ &= \sum_{n=1}^{m} (\vartheta(2^n) - \vartheta(2^{n-1})) \leqslant \sum_{n=1}^{m} 2^n \log 2 \leqslant 2^{m+1} \log 2,\end{aligned}$$

since $\vartheta(2^n) - \vartheta(2^{n-1}) \leqslant 2^n \log 2$, and the elephant tea-cup formula gives us $1 + 2 + ... + 2^m = \dfrac{2^{m+1} - 1}{2 - 1}$; $2 + ... + 2^m = 2^{m+1} - 2$.

Then for any real $x \geqslant 1$, we can choose $m \geqslant 1$ such that $2^{m-1} \leqslant x < 2^m$, then we have

$$\vartheta(x) \leqslant \vartheta(2^m) \leqslant 2^{m+1} \log 2 = (4 \log 2) 2^{m-1} \leqslant (4 \log 2) x.$$

Hence $\vartheta(x) = O(x)$ as desired. □

Lemma 3.6. The function $\Phi(s) - \dfrac{1}{s-1}$ extends to a meromorphic function on $\mathrm{Re}(s) > \frac{1}{2}$, and is analytic on $\mathrm{Re}(s) \geqslant 1$.

Proof. By Theorem 3.2, we know that $\zeta(s)$ extends to a meromorephic function on $\mathrm{Re}(s) > 0$ and has one simple pole at $s = 1$ with residue 1. Given the absolute convergence of the Euler Product (Theorem 3.1), if we differentiate logarithmically the product formula for the zeta function, we obtain

$$\begin{aligned}-\frac{\zeta'(s)}{\zeta(s)} &= \frac{d}{ds}(-\log(\zeta(s))) = \frac{d}{ds}(-\log \prod_p (1 - p^{-s})^{-1}) = \frac{d}{ds}(\sum_p \log(1 - p^{-s})) \\ &= \sum_p \frac{p^{-s} \log p}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s - 1} = \sum_p \Big(\frac{\log p}{p^s} + \frac{\log p}{p^s(p^s - 1)} \Big) \\ &= \sum_p \frac{\log p}{p^s} + \sum_p \frac{\log p}{p^s(p^s - 1)} = \Phi(s) + \sum_p \frac{\log p}{p^s(p^s - 1)}.\end{aligned}$$

The last series converges absolutely and locally uniformly to an analytic function by Weiersrass M-test on $\mathrm{Re}(s) > \frac{1}{2}$. Since $-\dfrac{\zeta'(s)}{\zeta(s)}$ is meromorphic on $\mathrm{Re}(s) > 0$, it follows that $\Phi(s)$ extends to be meromorphic for $\mathrm{Re}(s) > \frac{1}{2}$, with simple poles and zeros of the zeta function.

Since by Corollary 3.4, $\zeta(s)$ has no zeros on $\mathrm{Re}(s) \geqslant 1$, and by Theorem 3.2, $\zeta(s)$ has only one simple pole at $s = 1$ with residue 1, $\Phi(s)$ has only one simple pole at $s = 1$ with residue 1. Then it follows that $\Phi(s) - \dfrac{1}{s-1}$ extends to a meromorphic function on $\mathrm{Re}(s) > \frac{1}{2}$, and it is analytic on $\mathrm{Re}(s) \geqslant 1$ as desired. □

The next step is to show that $\vartheta(x) \sim x$. To prove this, we rely on a general analytic criterion that is applicable to any non-decreasing real function $f(x)$.

Lemma 3.7. Let $f : \mathbb{R}_{\geqslant 1} \to \mathbb{R}$ be a nondecreasing function. If $\displaystyle\int_1^\infty \frac{f(t)-t}{t^2}dt$ converges, then $f(x) \sim x$.

Proof. Let $F(x) = \displaystyle\int_1^\infty \frac{f(t)-t}{t^2}dt$. Since the integral converges, $\lim_{x\to\infty} F(x)$ exists. Then by the definition of limit, for all $\lambda > 1$, and all $\epsilon > 0$, we have $|F(\lambda x) - F(x)| < \epsilon$ for all sufficiently large x. Now fix $\lambda > 1$, and suppose there is an arbitrary large x such that $f(x) \geqslant \lambda x$. Then for such x, since f is nondecreasing, we have

$$F(\lambda x) - F(x) = \int_x^{\lambda x} \frac{f(t)-t}{t^2}dt \geqslant \int_x^{\lambda x} \frac{\lambda x - t}{t^2}dt = \int_1^{\lambda} \frac{\lambda - t}{t^2}dt = c > 0,$$

since the integrand is nonnegative on $[1, \lambda]$, so we have $|F(\lambda x) - F(x)| \geqslant c > 0$. Now if we take $\epsilon < c, x$ large enough, we have $|F(\lambda x) - F(x)| \geqslant c > \epsilon$, which is a contradiction. Thus $f(x) < \lambda x$ for all sufficiently large x.

Similarly, fix $\lambda > 1$, and suppose there is an arbitrary large x such that $f(x) \leqslant \dfrac{1}{\lambda}x$. For such x, we have

$$F(x) - F\Big(\frac{1}{\lambda}x\Big) = \int_{\frac{1}{\lambda}}^{x} \frac{f(t)-t}{t^2}dt \leqslant \int_{\frac{1}{\lambda}x}^{x} \frac{\frac{1}{\lambda}x - t}{t^2}dt = \int_{\frac{1}{\lambda}}^{1} \frac{\frac{1}{\lambda} - t}{t^2}dt < 0,$$

since the integrand is nonpositive on $[\frac{1}{\lambda}, 1]$. But this again contradicts our hypothesis. Thus $f(x) > \dfrac{1}{\lambda}x$ for sufficiently large x. Since those inequalities hold for all $\lambda > 1$, we must have $\displaystyle\lim_{x\to\infty} \frac{f(x)}{x} = 1$, that is $f(x) \sim x$ as desired. □

In order to show that $f(x) = \vartheta(x)$ satisfies the hypothesis of Lemma 3.7, given $\vartheta(x)$ is a nondecreasing function, we need to show that $\displaystyle\int_1^\infty \frac{\vartheta(t)-t}{t^2}dt$ converges. Instead of showing that this integral converges directly, we will work with the function $H(t) = \vartheta(e^t)e^{-t} - 1$. We first want to connect $\Phi(s)$ and $\vartheta(x)$ via the Laplace transform.

Recall the Laplace transform of a real-valued piecewise continuous function $h(t)$, that is defined for all $t \geqslant 0$, $(\mathscr{L}h)(s)$ is the complex function defined by

$$(\mathscr{L}h)(s) = \int_0^\infty h(t)e^{-st}dt,$$

which is analytic and converges absolutely on $\mathrm{Re}(s) > c$ for any $c \in \mathbb{R}$ for which $h(t) = O(e^{ct})$.

Lemma 3.8. The Laplace transform of $\vartheta(e^t)$ is analytic and converges absolutely for $\mathrm{Re}(s) > 1$, *and*

$$\mathscr{L}(\vartheta(e^t))(s) = \frac{\Phi(s)}{s},$$

$\mathrm{Re}(s) > 1$.

Proof. By Lemma 3.5, $\vartheta(e^t) = O(e^t)$ since $e^t \geqslant 1$ for all $t \geqslant 0$, then the Laplace transform of $\vartheta(e^t)$ is analytic and converges absolutely on $Re(s) > 1$. Let p_n denote the nth prime and $p_0 = 0$, then $\vartheta(e^t)$ is constant for $t \in (\log p_n, \log p_{n+1})$, that is $\vartheta(e^t) = \sum\limits_{p_n < p < p_{n+1}} \log p = \vartheta(p_n)$. We then have

$$\int_{\log p_n}^{\log p_{n+1}} e^{-st}\vartheta(e^t)dt = \vartheta(p_n)\int_{\log p_n}^{\log p_{n+1}} e^{-st}dt = \frac{1}{s}\vartheta(p_n)({p_n}^{-s} - {p_{n+1}}^{-s}).$$

If we sum over the primes and use $\vartheta(p_n) - \vartheta(p_{n-1}) = \log p_n$, we obtain for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}\mathscr{L}(\vartheta(e^t))(s) = \int_0^\infty e^{-st}\vartheta(e^t)dt &= \frac{1}{s}\sum_{n=1}^{\infty}\vartheta(p_n)({p_n}^{-s} - {p_{n+1}}^{-s}) \\ &= \frac{1}{s}\sum_{n=1}^{\infty}(\vartheta(p_n) - \vartheta(p_{n-1})){p_n}^{-s} = \frac{1}{s}\sum_{n=1}^{\infty}{p_n}^{-s}\log p_n = \frac{\Phi(s)}{s},\end{aligned}$$

since $\mathscr{L}(\vartheta(e^t))(s)$ converges absolutely for $\mathrm{Re}(s) > 1$, rearrangement is justified. □

Now let's look at the function $H(t) = \vartheta(e^t)e^{-t} - 1$. It follows from Lemma 3.8 and the standard property of Laplace transform that on $\mathrm{Re}(s) > 0$, we have

$$\mathscr{L}(H(t))(s) = \mathscr{L}(\vartheta(e^t)e^{-t})(s) - \mathscr{L}(1)(s) = \mathscr{L}(\vartheta(e^t))(s+1) - \frac{1}{s} = \frac{\Phi(s+1)}{s+1} - \frac{1}{s}.$$

Now we know the effect of multiplying $\vartheta(e^{-t})$ by e^{-t} is to translate the variable s of the Laplace transform by 1, and the effect of subtracting 1 is to eliminate the pole of $\Phi(s)$ at $s = 1$.

Corollary 3.9. The function $\Phi(s+1) - \frac{1}{s}$ and $\mathscr{L}(H(t))(s) = \frac{\Phi(s+1)}{s+1} - \frac{1}{s}$ both extends to meromorphic functions on $\mathrm{Re}(s) > -\frac{1}{2}$ that are analytic on $\mathrm{Re}(s) \geqslant 0$.

Proof. The first statement follows immediately from Lemma 3.6, since $\Phi(s) - \frac{1}{s-1}$ extends to a meromorphic function on $\mathrm{Re}(s) > \frac{1}{2}$, and is analytic on $\mathrm{Re}(s) \geqslant 1, \Phi(s+1) - \frac{1}{s}$ extends to a meromorphic function on $\mathrm{Re}(s) > -\frac{1}{2}$ that is analytic on $\mathrm{Re}(s) \geqslant 0$. For the second statement,

$$\frac{\Phi(s+1)}{s+1} - \frac{1}{s} = \frac{1}{s+1}(\Phi(s+1) - \frac{1}{s}) - \frac{1}{s+1}.$$

Since $\Phi(s+1) - \frac{1}{s}$ extends to a meromorphic function on $\mathrm{Re}(s) > -\frac{1}{2}$, that is analytic on $\mathrm{Re}(s) \geqslant 0, \frac{1}{s+1}$ is analytic for $\mathrm{Re}(s) > -1, \frac{\Phi(s+1)}{s+1} - \frac{1}{s}$ is meromorphic on $\mathrm{Re}(s) > -\frac{1}{2}$ and is analytic on $\mathrm{Re}(s) \geqslant 0$ as desired. □

The final theorem we need for the proof of the prime number theorem is known as the "Tauberian Theorem". It contains the link between the analyticity of $\Phi(s)$ and the asymptotic behavior of $H(t)$.

Theorem 3.10 (Tauberian Theorem). Let $f(t)$ $(t \geqslant 0)$ be a bounded piecewise continuous function (locally integrable) and suppose its Laplace transform $\int_0^\infty f(t)e^{-st}dt$ $(\mathrm{Re}(s) > 0)$ extends analytically to g(s) on $\mathrm{Re}(s) \geqslant 0$, then $\int_0^\infty f(t)dt$ converges and equals to g(0).

Proof. For fixed $T > 0$, define

$$g_T(s) = \int_0^T f(t)e^{-st}dt.$$

Since $f(t)$ is piecewise continuous, e^{-st} is analytic on $\mathbb{C}$ for each fixed t, $g_T(s)$ is an entire function. By definition, $\int_0^\infty f(t)e^{-st}dt = \lim_{T\to\infty} g_T(0)$, thus it suffices to show that

$$\lim_{T\to\infty} g_T(0) = g(0).$$

We will show that $|g(0) - g_T(0)| < \epsilon$ for large $T, \epsilon > 0$ small. Since $f(t)$ is bounded, without loss of generality, assume $|f(t)| \leqslant 1$ for all $t \geqslant 0$. Now choose R large enough such that $\frac{1}{R} < \frac{\epsilon}{4}$, then choose $\delta > 0$ small enough such that $g(s)$ is defined and analytic in and across the boundary of the domain D_δ consisting of s such that $|s| < R$ and $Re(s) > -\delta$. We know such δ exist since $g(s)$ is analytic on $\mathrm{Re}(s) \geqslant 0$, hence on some open ball $B(iy)$ with radius less than or equal to $2\delta_y$ for each $y \in [-R, R]$ and we can take $\delta = \inf \delta_y, y \in [-R, R]$.

For each $T > 0, h(s) = (g(s) - g_T(s))e^{sT}(1 + \frac{s^2}{R^2})$ is analytic on D_δ since it's a combination of analytic functions on D_δ. Given D_δ is a bounded domain with piecewise smooth boundry, we have by Cauchy's Integral Formula [3, page 113],

$$\begin{aligned} h(0) = g(0) - g_T(0) &= \frac{1}{2\pi i}\int_{\partial D_\delta} (g(s) - g_T(s))e^{sT}(1 + \frac{s^2}{R^2})\frac{ds}{s} \\ &= \frac{1}{2\pi i}\int_{\partial D_\delta} (g(s) - g_T(s))e^{sT}(\frac{1}{s} + \frac{s}{R^2})ds. \end{aligned}$$

Now we will break ∂D_δ into pieces.

Let γ_+ be the semicircle $\{|s| = R, Re(s) > 0\}$, let α_δ be the vertical interval along ∂D_δ where $\mathrm{Re}(s) = -\delta$, and let β_δ be the two short arcs of the circle $\{|s| = R\}$ connecting α_δ to $\pm iR$ all oriented according to the orientation of ∂D_δ.

For $s \in \gamma_+$, we have since $|f(t)| \leqslant 1$,

$$\begin{aligned} \Big|(g(s) - g_T(s))e^{sT}(\frac{1}{s} + \frac{s}{R^2})\Big| &= \Big|\int_T^\infty f(t)e^{-st}dt\Big| \cdot e^{Re(s)T} \cdot \Big|\frac{1}{s} + \frac{s}{R^2}\Big| \\ &= \Big|\int_T^\infty f(t)e^{-st}dt\Big| \cdot e^{Re(s)T} \cdot \Big|\frac{\bar{s}}{|s|^2} + \frac{s}{R^2}\Big| \leqslant \int_T^\infty |f(t)e^{-st}|dt \cdot e^{Re(s)T} \cdot \frac{2Re(s)}{R^2} \\ &\leqslant \int_T^\infty e^{-Re(s)t}dt \cdot e^{Re(s)T} \cdot \frac{2Re(s)}{R^2} \\ &= \frac{1}{Re(s)}e^{-Re(s)T} \cdot e^{Re(s)T} \cdot \frac{2Re(s)}{R^2} = \frac{2}{R^2}. \end{aligned}$$

Hence we have on γ_+

$$\Big|(g(s) - g_T(s))e^{sT}(\frac{1}{s} + \frac{s}{R^2})\Big| \leqslant \frac{2}{R^2}.$$

Then by ML-estimate, we have

$$\Big|\frac{1}{2\pi i}\int_{\gamma_+}(g(s)-g_T(s))e^{sT}(\frac{1}{s}+\frac{s}{R^2})ds\Big| \leqslant \frac{1}{2\pi}\cdot\pi R\cdot\frac{2}{R^2}=\frac{1}{R}<\frac{\epsilon}{4}.$$

Now for the integral of $g(s)-g_T(s)$ over the remainder of ∂D_δ, we will treat the functions separately. Since $g_T(s)$ is entire, the path for the integral involving $g_T(s)$ can be replaced by the semicircle of radius R, $\mathrm{Re}(s)<0$. Let γ_- be the semicircle $\{|s|=R, Re(s)<0\}$, then

$$\frac{1}{2\pi i}\int_{\alpha_\delta\cup\beta_\delta} g_T(s)e^{sT}(\frac{1}{s}+\frac{s}{R^2})ds=\frac{1}{2\pi i}\int_{\gamma_-} g_T(s)e^{sT}(\frac{1}{s}+\frac{s}{R^2})ds.$$

For $s\in\gamma_-$, we have

$$\begin{aligned}\Big|g_T(s)e^{sT}(\frac{1}{s}+\frac{s}{R^2})\Big| &= \Big|\int_0^T f(t)e^{-st}dt\Big|\cdot e^{Re(s)T}\cdot\Big|\frac{1}{s}+\frac{s}{R^2}\Big| \\ &= \Big|\int_0^T f(t)e^{-st}dt\Big|\cdot e^{Re(s)T}\cdot\frac{(-2Re(s))}{R^2} \\ &\leqslant \int_0^T |f(t)e^{-st}|dt\cdot e^{Re(s)T}\cdot\frac{(-2Re(s))}{R^2} \\ &\leqslant \int_0^T e^{-Re(s)t}dt\cdot e^{Re(s)T}\cdot\frac{(-2Re(s))}{R^2} \\ &= (1-\frac{1}{Re(s)}e^{-Re(s)T})\cdot e^{Re(s)T}\cdot\frac{(-2Re(s))}{R^2} \\ &= \frac{2}{R^2}(1-Re(s)e^{Re(s)T}).\end{aligned}$$

As $T\to\infty$, since $\mathrm{Re}(s)<0$, $(1-Re(s)e^{Re(s)T})\to 1$. Hence we have

$$\Big|g_T(s)e^{sT}(\frac{1}{s}+\frac{s}{R^2})\Big|\leqslant\frac{2}{R^2}$$

for T sufficiently large. Then by ML-estimate, we have

$$\Big|\frac{1}{2\pi i}\int_{\gamma_-} g_T(s)e^{sT}(\frac{1}{s}+\frac{s}{R^2})ds\Big|\leqslant\frac{1}{2\pi}\cdot\pi R\cdot\frac{2}{R^2}=\frac{1}{R}<\frac{\epsilon}{4}.$$

Now since $g(s)(\frac{1}{s}+\frac{s}{R^2})$ is independent of T, $|e^{sT}|\leqslant 1$ on β_δ and the length of β_δ tends to 0 with δ, we can choose $\delta>0$ so small such that for any $T>0$,

$$\Big|\frac{1}{2\pi i}\int_{\beta_\delta} g(s)e^{sT}(\frac{1}{s}+\frac{s}{R^2})ds\Big|<\frac{\epsilon}{4}.$$

Finally, $|e^{sT}|=e^{-\delta T}$ on α_δ, so for T large enough, the integral of g(s) over α_δ is also bounded by $\frac{\epsilon}{4}$. Adding all four estimates, we obtain $|g(0)-g_T(0)|<\epsilon$ for T large. Hence we have shown that $\int_0^\infty f(t)dt$ converges and equals to g(0). □

Theorem 3.11 (Prime Number Theorem). $\pi(x) \sim \dfrac{x}{\log(x)}$.

Proof. By Lemma 3.5, since $e^t \geqslant 1$ for $t \geqslant 0$, we know that $\vartheta(e^t) = O(e^t)$, that is $\vartheta(e^t)$ is bounded in absolute value by some fixed multiple of e^t. Then we have $\mathscr{L}(t) = \vartheta(e^t)e^{-t} - 1$ is also bounded on $t \geqslant 0$. Since $H(t)$ is piecewise continuous, and by Corollary 3.9, the Laplace transform of $H(t), \mathscr{L}(H(t))(s)$ extends to an analytic function on $Re(s) \geqslant 0$, Theorem 3.10 implies that $\int_0^\infty H(t)dt = \int_0^\infty (\vartheta(e^t)e^{-t} - 1)dt$ converges.

Let $t = \log x$, then we have $e^{-t} = e^{-\log x} = \dfrac{1}{x}$, $t \in [0, \infty), x \in [1, \infty), dt = \dfrac{1}{x}dx$. Then

$$\int_0^\infty (\vartheta(e^t)e^{-t} - 1)dt = \int_1^\infty \frac{\vartheta(x) - x}{x^2}dx$$

converges. Given $\vartheta(x)$ is a nondecreasing function for $x \geqslant 1$, Lemma 3.7 implies that $\vartheta(x) \sim x$. Since

$$\vartheta(x) = \sum_{p \leqslant x} \log p \leqslant \sum_{p \leqslant x} \log x = \pi(x) \log x,$$

then

$$0 \leqslant \frac{\vartheta(x)}{x} \leqslant \frac{\pi(x)\log x}{x} \tag{1}$$

for $x \geqslant 1$.

For any $\epsilon > 0$,

$$\begin{aligned} \vartheta(x) = \sum_{p \leqslant x} \log p \geqslant \sum_{x^{1-\epsilon} < p \leqslant x} \log p &\geqslant (1-\epsilon)(\log x)(\pi(x) - \pi(x^{1-\epsilon})) \\ &\geqslant (1-\epsilon)(\log x)(\pi(x) - x^{1-\epsilon}), \end{aligned}$$

then we have

$$\pi(x) \leqslant \frac{\vartheta(x)}{(1-\epsilon)\log(x)} + x^{1-\epsilon}, \tag{2}$$

for $0 < \epsilon < 1$.

Combine inequality (1) and (2), we obtain, for all $0 < \epsilon < 1$,

$$\frac{\vartheta(x)}{x} \leqslant \frac{\pi(x)\log x}{x} \leqslant \frac{\vartheta(x)\log x}{(1-\epsilon)x\log(x)} + \frac{x^{1-\epsilon}\log x}{x} = \frac{1}{1-\epsilon}\frac{\vartheta(x)}{x}\frac{\log x}{x^\epsilon}.$$

As $x \to \infty$, $\dfrac{\log x}{x^\epsilon} \to 0$, so by choosing ϵ sufficiently small, we can make the ratios of $\vartheta(x)$ to x, and $\pi(x)$ to $\dfrac{x}{\log x}$ arbitrarily close together, and if one of them tends to 1, the other must too. Therefore $\pi(x) \sim \dfrac{x}{\log x}$ as $x \to \infty$. □

4 Conclusion

The prime number theorem allows one to predict how the primes are distributed. In this paper, we've provided one of the many proofs of the prime number theorem that requires only careful use of inequalities and complex analysis. One disadvantage of our proof is that it does not give us an error term. How close is our estimate? We will use x = 10,000,000 as an example to show how accurate is our estimate.

For x = 10,000,000, we have

$$\pi(x) = 664,579$$

$$\frac{x}{\log x} \approx 620,420.688433.$$

This estimate is not the best estimate we have for the prime number distribution. In fact, this formula $\pi(x)$ for the asymptotic distribution of primes dates back to Gauss in the late 18th century. Gauss believed that the function (also called the logarithmic integral) :

$$Li(x) = \int_2^x \frac{dx}{\log x},$$

provides a much more accurate numerical approximation to $\pi(x)$ than $\dfrac{x}{\log x}$. Gauss calculated all primes up to about 3,000,000 and compared the number of primes found with the above integral [4]. The result he found is fascinating. For example, between 2,600,000 and 2,700,000, Gauss found 6762 primes and

$$\int_{2{,}600{,}000}^{2{,}700{,}000} \frac{dx}{\log x} = 6761.332.$$

However Gauss never published his investigations on the distribution of primes.

Because of the connection of Riemann Zeta function and $\pi(x)$, many mathematicians have shown if the Riemann hypothesis is true, it will yield a better estimate of the error involved in the prime number theorem compared to any other estimates that is available today.

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