Regularity of Divergent Series

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Abstract

In this paper, we focus on investigating summation methods for divergent series that also produce correct results for convergent series. The *Silverman-Toeplitz Theorem* will provide a powerful test for regularity of many different methods of summation. With the development of this theorem we will introduce several techniques of summation and show how they can be used to produce "interesting" results for divergent series.

Introduction

The typical method of summing a series $\sum_0^\infty a_n$ is to define the sequence of partial sums

$$\left\{\sum_0^n a_n\right\}_{n=0}^\infty = \{s_n\}.$$

If the sequence $\{s_n\}$ has a limit s, we say that the series converges to s. If $\{s_n\}$ does not have a limit, the series is said to diverge.

This is a good method for convergent series, but tells us very little about the divergence of a series. In this paper, we will develop a technique for constructing new summation methods that will extend to some divergent series. We will start by stating some rules that we would like to impose on these potential summation methods.

Definition 1. *A method of summation is said to be regular if it sums every convergent series to its ordinary sum.*

Given a method of summation $\mathcal{M}$, we will use the following axioms

a) $\mathcal{M}$ must be regular.

b) if $\sum a_n = s$, then $\sum ka_n = ks$

c) if $\sum a_n = s$ and $\sum b_n = t$, then $\sum(a_n + b_n) = s + t$

d) $a_0 + a_1 + ... = s \iff a_1 + a_2 + ... = s - a_0$

Definition 2. ***(Cesàro Summation)*** *Given the series $\sum a_n$ with partial sums s_n, if the following limit exists*

$$\lim_{n\to\infty} \frac{s_0 + s_1 + ... + s_n}{n+1} = s$$

We will say the Cesàro summation of $\sum a_n = s$ $(C, 1)$.

Cesàro summation is a prototypical example that will be used in this paper. It is important to include the $(C, 1)$ by the sum to distinguish the sum from that of a convergent series. However, if $\sum a_n$ is known to be convergent to s, then we can omit the $(C, 1)$, provided that Cesàro Summation is regular. Now we will show that Cesàro summation satisfies the axioms.

Proof. Suppose the series $\sum a_n$ converges to s. Let $\varepsilon > 0$, then there are N_1, N_2 such that

$$|s_n - s| < \frac{\varepsilon}{2} \text{ for all } n > N_1, \qquad \left|\frac{1}{n+1}\sum_0^{N_1}(s_j - s)\right| \leq \frac{\varepsilon}{2} \text{ for all } n > N_2.$$

$$\left|\frac{s_0 + s_1 + ... + s_n}{n+1} - s\right| = \left|\frac{1}{n+1}\sum_0^{N_1}(s_j - s) + \frac{1}{n+1}\sum_{N_1+1}^{n}(s_j - s)\right|$$

$$\leq \frac{\varepsilon}{2} + \left|\frac{(n - N_1)\varepsilon}{2(n+1)}\right| \leq \varepsilon \text{ for all } n > \max(N_1, N_2).$$

We have now established that Cesàro summation is regular. It is clear that conditions b) and c) are satisfied because of the linearity of the limit and arithmetic mean formula. To show d), we let $t_n = s_{n+1} - a_0$

$$\frac{t_0 + t_1 + ... + t_n}{n+1} = \frac{n+2}{n+1}\left(\frac{s_0 + s_1 + ... + s_{n+1}}{n+2} - \frac{s_0}{n+2}\right) - a_0 \to s - a_0$$

$$\frac{s_0 + s_1 + ... + s_n}{n+1} = a_0 + \frac{s_0 - a_0}{n+1} + \frac{n}{n+1}\left(\frac{t_0 + t_1 + ... + t_{n-1}}{n}\right) \to s.$$

Cesàro summation satisfies all the axioms and our proof is finished. □

1 The Regularity of a Method

Consider the following

$$\begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n} & \cdots \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ a_{n,0} & a_{n,1} & \cdots & a_{n,n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \\ \vdots \end{bmatrix} = \begin{bmatrix} t_0 \\ t_1 \\ \vdots \\ t_n \\ \vdots \end{bmatrix} \sim \mathcal{M}\vec{s} = \vec{t}.$$

Given a sequence of numbers $\{s_n\}$ and a matrix $\mathcal{M}$ as above, we will define the multiplication of an infinite matrix by the sequence in terms of the dot product of the matrix rows with the sequence. More precisely,

$$\{a_{n,k}\}_{k=0}^{\infty} \cdot \{s_k\}_{k=0}^{\infty} = \sum_{k=0}^{\infty} a_{n,k} s_k = t_n$$

where the sequence $\{t_n\}$ is said to exist if the series $\sum a_{n,k} s_k$ converges for any fixed n. From this point forward we will refer to $a_{i,j}$ as the element in the i^{th} row and j^{th} column of the matrix $\mathcal{M}$ which is called our "method of summation." Since we are only working with infinite vectors, we will may use $\vec{s}$ and $\{s_n\}$ interchangeably (likewise for t). It is important to note that $a_{i,j}$ is an element of the matrix while a_n is a term of our starting series, and they should not be confused for one another.
For a given method of summation $\mathcal{M}$, we can multiply it on the right by a sequence $\{s_n\}$ (possibly the sequence of partial sums) to produce a new sequence $\{t_n\}$ that will have the same limit if $\mathcal{M}$ is regular. For example, let $\mathcal{M}$ be defined by $a_{i,j} = 1$ for $i \leq j$ and $a_{i,j} = 0$ otherwise for $i, j \in \mathbb{N}$. Then $\mathcal{M}$ as a method of summation merely takes any sequence $\{s_n\}$ and produces the sequence of partial sums $\{t_n\}$ of $\{s_n\}$. We can also define $\mathcal{M}$ to be a matrix that performs Cesàro summation

$$a_{i,j} = \begin{cases} \frac{1}{n+1} \text{ for } i \leq j \\ 0 \text{ otherwise} \end{cases} \qquad \begin{bmatrix} \frac{1}{1} & 0 & 0 & \cdots & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \frac{1}{n+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_n \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{s_0}{1} \\ \frac{s_0+s_1}{2} \\ \frac{s_0+s_1+s_2}{3} \\ \vdots \\ \frac{s_0+s_1+\ldots+s_n}{n+1} \\ \vdots \end{bmatrix}.$$

Our goal is to determine the necessary and sufficient conditions for a method of summation that is written in the form of an infinite matrix to be regular. To do that we will need some lemmas about sequences that will lead us to proving the *Silverman-Toeplitz Theorem.* We will be using similar techniques as those used in Brian Ruder's 1966 paper on the Silverman-Toeplitz Theorem[2].

Lemma 1. *If for any bounded sequence $\{s_n\}$ the series $\sum a_n s_n$ converges, then $\sum a_n$ converges absolutely.*

Proof. By way of contraposition, we will suppose that $\sum |a_n|$ diverges. Consider the bounded sequence $\{b_n\} = \left\{\frac{|a_n|}{a_n}\right\}$. We have that

$$\sum |a_n| = \sum a_n \frac{|a_n|}{a_n} = \sum a_n b_n \text{ diverges.}$$

□

With this we can now multiply a row of the matrix with the column vector, that is, a bounded sequence, giving us a single entry in our new sequence. Our next step is to extend this to produce the rest of our new sequence. For that we need the following lemma.

Lemma 2. *Let $\{s_n\}$ be any bounded sequence, and*

$$\{R_n\} = \left\{\sum_{k=0}^{\infty} |a_{n,k}|\right\}_{n=0}^{\infty}, \quad \{t_n\} = \left\{\sum_{k=0}^{\infty} a_{n,k}s_k\right\}_{n=0}^{\infty}.$$

The sequence $\{R_n\}$ is bounded if and only if $\{t_n\}$ is bounded.

Proof. Suppose $\sup\{R_n\} = R$, and $\sup\{|s_n|\} = S$. By the triangle inequality,

$$|t_n| \leq \sum_{k=0}^{\infty} |a_{n,k}s_k| \leq S \sum_{k=0}^{\infty} |a_{n,k}| \leq SR_n \leq SR$$

we know that $\{t_n\}$ is bounded. Conversely, if $\sup\{|t_n|\} = T$ then surely for a fixed n, $\sum a_{n,k}s_k$ converges. By Lemma 1, for any fixed n, R_n is finite. We will assume that $\{R_n\}$ is unbounded and construct a bounded sequence $\{s_n\}$ to show that $\{t_n\}$ is unbounded. Without loss of generality, we will assume that $\{R_n\}$ is a monotone increasing sequence because $\{R_n\}$ is non-negative and unbounded, so we can create a monotone increasing sub-sequence.

First we will prove that there is a bound T_k for any element $a_{n,k}$ independent of n. Fix k, consider $s_n = 1$ if $n = k$ and zero otherwise. Then $|a_{n,k}| = |t_n| \leq T_k$ by assumption.

Let $N_0 = 0$, $x_0 = 0$, and y_i be the smallest integer such that $y_i > x_i + 1$ and $\sum_{y_i}^{\infty} |a_{i,k}| \leq 1$. Now let N_n be the smallest integer larger than N_{n-1} such that there is an integer $\tilde{x} > y_{n-1}$ such that

$$\sum_{k=0}^{\tilde{x}} |a_{n,k}| \leq \frac{1}{3}R_n - 1 \qquad \text{when } n = N_n.$$

We know that such an N_n exists because R_n is monotonically increasing and $|a_{n,k}| \leq T_k$ has an upper bound independent of n. Now let $x_i = y_{i-1} + 1$ so that when $\tilde{x} = x_n$ the above equation is satisfied and

$$x_0 < y_0 < x_1 < y_1 < ... < x_n < y_n < ...$$

Now for the hard part, we will define the sequence $\{s_k\}$ and put this all together while performing miracles of computation. Let

$$s_k = \begin{cases} \text{sgn}(a_{0,k}) & , \text{ if } x_0 \leq k \leq y_0 \\ \text{sgn}(a_{1,k}) & , \text{ if } x_1 \leq k \leq y_1 \\ \vdots & \\ \text{sgn}(a_{n,k}) & , \text{ if } x_n \leq k \leq y_n \\ \vdots & \\ 0 & \quad \text{otherwise.} \end{cases}$$

We can make use of the signum function and the fact that $a_{n,k}\text{sgn}(a_{j,k}) \geq -|a_{n,k}|$ to say that

$$\begin{aligned} |t_n| = \left|\sum_{k=0}^{\infty} a_{n,k}s_k\right| &= \left|\sum_{x_n}^{y_n} |a_{n,k}| + \sum_{j=0}^{n-1}\sum_{k=x_j}^{y_j} a_{n,k}\text{sgn}(a_{j,k}) + \sum_{j=n+1}^{\infty}\sum_{k=x_j}^{y_j} a_{n,k}\text{sgn}(a_{j,k})\right| \\ &\geq \left|R_n - \sum_{k=0}^{x_n-1} |a_{n,k}| - \sum_{k=y_n+1}^{\infty} |a_{n,k}|\right| - \left|\sum_{j\neq n}\sum_{k=x_j}^{y_j} |a_{n,k}|\right| \\ &\geq R_n - 2\sum_{k=0}^{x_n-1} |a_{n,k}| - 2\sum_{k=y_n+1}^{\infty} |a_{n,k}| \geq R_n - \left(\frac{2}{3}R_n + 2 - 2\right) = \frac{1}{3}R_n. \end{aligned}$$

In the last steps we used our previously constructed inequalities and the fact that

$$R_n = \sum_{k=0}^{x_n-1} |a_{n,k}| + \sum_{k=x_n}^{y_n} |a_{n,k}| + \sum_{k=y_n+1}^{\infty} |a_{n,k}|.$$

Since R_n is monotonically increasing, $\{t_n\}$ is unbounded, thus the other direction has been proved. $\square$

We now know that it is both necessary and sufficient that the sum across a row of the matrix are absolutely convergent to produce a bounded sequence from a bounded sequence via matrix multiplication. But our objective is not to just have bounded sequences, we want to know when our new sequence has a limit, but that is still a difficult question to answer. Thus, we will need two more lemmas before we can establish such results.

Lemma 3. *Let $\{s_n\}$ be any sequence with limit zero. The sequence $\{t_n\}$ will have a limit if and only if*

a) $\{R_n\}$ *is bounded, and*

b) $\lim_{n\to\infty} a_{n,k} = l_k$ *exists for each* k.

Then we get the new formula for the sequence

$$\lim_{n\to\infty} t_n = \sum_{k=0}^{\infty} l_k s_k.$$

Proof. Both directions for part (a) are immediate from Lemma 2. Suppose $\{t_n\}$ has a limit, then $\{t_n\}$ and $\{s_n\}$ are both bounded, which implies $\{R_n\}$ is bounded by Lemma 2. If we fix k, then by the same argument first discussed in the proof of Lemma 2, if we let $s_n = 1$ for $n = k$ and zero otherwise, then we still have $\{s_n\} \to 0$ and $t_n = a_{n,k}$. But since t_n has a limit l_k, we get that

$$\lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} t_n = l_k.$$

Conversely, let $R = \sup\{R_n\}$ and use the triangle and inequality to get

$$\left| t_n - \sum_{k=0}^{K} a_{n,k} s_k \right| \leq \sum_{K+1}^{\infty} |a_{n,k} s_k| \leq R L_K \quad \text{where } L_K = \sup_{k>K} |s_k|.$$

Since L_K monotonically decreases to zero as $K \to \infty$, all we need to do is some manipulations of the inequalities and we can use the squeeze theorem on t_n.

$$-RL_K \leq t_n - \sum_{k=0}^{K} a_{n,k} s_k \leq RL_K \qquad \text{add the finite sum to each side.}$$

$$-RL_K + \sum_{k=0}^{K} a_{n,k} s_k \leq t_n \leq RL_K + \sum_{k=0}^{K} a_{n,k} s_k \qquad \text{let } n \to \infty.$$

$$-RL_K + \sum_{k=0}^{K} l_k s_k \leq \liminf t_n \leq \limsup t_n \leq RL_K + \sum_{k=0}^{K} l_k s_k \qquad \text{let } K \to \infty.$$

$$\sum_{k=0}^{\infty} l_k s_k \leq \liminf t_n \leq \limsup t_n \leq \sum_{k=0}^{\infty} l_k s_k \qquad t_n \text{ is bounded by Lemma 2.}$$

We have simultaneously shown that $\sum l_k s_k$ is bounded above and below by the liminf, and limsup of t_n proving the convergence of the series, which in turn implies the existence of the limit of t_n, and finally that

$$\lim_{t\to\infty} t_n = \sum_{k=0}^{\infty} l_k s_k.$$

□

Let us take a moment to reflect on what just happened. We just used some estimates on the tails of infinite series so that we could take limits of the finite sums that were left over. We also used suprema, limit suprema, and limit infima in the whole mix for a grand finale of proving three things at once with the help of a lemma. That was cool. Now that we have had our time for appreciation, let us move on to the next lemma.

Lemma 4. *Let $\{s_n\}$ be any sequence with limit zero. The sequence $\{t_n\}$ has a limit zero if and only if*

(a) *$\{R_n\}$ is bounded, and*

(b) *for any k, $\lim_{n\to\infty} a_{n,k} = 0$.*

Proof. Suppose

$$\lim_{n\to\infty} a_{n,k} = l_k = 0 \Longrightarrow \lim_{n\to\infty} t_n = \sum_{k=0}^{\infty} l_k s_k = 0$$

by Lemma 3. Conversely, let $s_n = 0$ for $n \neq k$ and 1 for $n = k$ where $k \in \mathbb{N}$ so that we have a sequence $\{s_n\}$ with limit zero and

$$\lim_{n\to\infty} t_n = 0 \Longrightarrow \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,j} s_j = \lim_{n\to\infty} a_{n,k} = 0.$$

We have established part (b), and (a) follows from the previous lemma. □

Notice that we were only able to choose a specific sequence for s_n on the second part of the proof because $\{t_n\}$ must have a limit of zero for every sequence $\{s_n\}$ with limit zero, in particular the one we chose. Thus, with our choice of s_n,

$$\lim_{n\to\infty} t_n = 0 \Rightarrow \lim_{n\to\infty} a_{n,k} = 0 \qquad \lim_{n\to\infty} a_{n,k} = 0 \not\Rightarrow \lim_{n\to\infty} t_n = 0.$$

In other words, Lemma 3 was necessary for the first part of the proof.

For our last step before proving the Silverman-Toeplitz Theorem we need to know when our sequences $\{t_n\}$ and $\{s_n\}$ both have limits that are not necessarily equal or zero. Since this property is interesting in its own right (outside the context of regularity), we will write it as a theorem instead of a lemma.

Theorem 1. *Suppose $\{s_n\}$ is any sequence with a limit. Then $\{t_n\}$ has a limit if and only if* *(a)* $\{R_n\}$ *is bounded,*

(b) for any k, $\lim_{n\to\infty} a_{n,k} = l_k$*, and*

(c) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}$ *converges.*

Proof. First suppose that conditions (a),(b), and (c) hold and that $\{s_n\}$ and limit s. Since the sequence $\{s_n - s\}$ has limit zero, we can use Lemma 3 to say that $\{t'_n\}$ has a limit where

$$\begin{aligned}\lim_{n\to\infty} t'_n &= \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}(s_k - s) = \lim_{n\to\infty} \left(t_n - s \sum_{k=0}^{\infty} a_{n,k} \right) = t' \\ \lim_{n\to\infty} t_n &= t' + s \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = t.\end{aligned}$$

and thus $\{t_n\}$ also has a limit. On the other hand, if we suppose that $\{t_n\}$ has a limit, then we can define $\{s_n\}$ where $s_n = 1$ for all n. Since

$$t_n = \sum_{k=0}^{\infty} a_{n,k}$$

part (c) follows immediately. Parts (b) and (a) follow from the previous lemmas. □

Theorem 2. ***(Silverman-Toeplitz Theorem)*** *Let* $\{s_n\}$ *be any sequence with limit* s*. Then a necessary and sufficient condition for the sequence*

$$\{t_n\} = \left\{ \sum_{k=0}^{\infty} a_{n,k} s_k \right\}_{n=0}^{\infty}$$

to have limit also equal to s *is that*

i) $\sup_n \sum_{k=0}^{\infty} |a_{n,k}| < \infty$,

ii) for any k, $\lim_{n\to\infty} a_{n,k} = 0$*, and*

iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1$.

Proof. Suppose that conditions (i), (ii), and (iii) hold. Then using the formula derived in the proof of Theorem 1, we can say

$$\begin{aligned}\lim_{n\to\infty} t'_n &= \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}(s_k - s) = \lim_{n\to\infty} \left(t_n - s \sum_{k=0}^{\infty} a_{n,k} \right) = s - s \\ \lim_{n\to\infty} t_n &= s \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = s.\end{aligned}$$

For the other direction, if $s = 0$ then we are done by Lemma 4. If $s \neq 0$, then when $\{t_n\}$ has limit s, we can use a similar method as before and let $s_n = s$ for all n so that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} s_k = s \sum_{k=0}^{\infty} a_{n,k} = s$$

which implies part (iii). For part (ii), we can define a sequence where $s_n = s$ for $n \neq k$ and $s_n = s + 1$ for $n = k$, now

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} s_k = \lim_{n\to\infty} \left(a_{n,k} + s \sum_{k=0}^{\infty} a_{n,k} \right) = s.$$

Using part (iii), we can see that $\lim_{n\to\infty} a_{n,k} = 0$. And as before, part (i) comes from the previous lemmas. Thus, our proof is complete. □

This theorem will be extremely useful when investigating new summation techniques and we will continue to use it throughout the rest of this paper. This theorem not only gives us a tool to find regular summation methods, but it also satisfies the axioms given in the introduction as a bonus.
One might say that, for elementary research on divergent series, the Silverman-Toeplitz Theorem is a necessary and sufficient tool for the job.

Corollary 2.1. *If $\mathcal{M}$ is a method of summation that can be represented as an infinite matrix satisfying the conditions of Theorem* 2*, then $\mathcal{M}$ satisfies the axioms a)-d) given in the introduction.*

Proof. Conditions a) is immediate from Theorem 2. By linearity,

$$\mathcal{M} \cdot \vec{s} = \vec{t} \Longrightarrow \mathcal{M} \cdot k\vec{s} = k\vec{t} \qquad \mathcal{M} \cdot (\vec{s_1} + \vec{s_2}) = \mathcal{M}\vec{s_1} + \mathcal{M}\vec{s_2}$$

conditions b) and c) are met. Since for $k = 0$,

$$\lim_{n\to\infty} a_{n,k} = 0$$

condition d) is trivially true. □

Though we will not make use of iterative methods in this paper, the following corollary is immediate from Theorem 2.

Corollary 2.2. *Suppose $\mathcal{M}_0$ is regular. Let $\{s_n\}$ be any sequence where*

$$\mathcal{M}_k \cdot \vec{s} = \mathcal{M}_{k-1} \cdot \mathcal{M}_0 \cdot \vec{s} = \vec{t}$$

defines a limiting sequence $\{t_n\} \to t$ for some $k > 0$. Then the sequence obtained by

$$\mathcal{M}_n \cdot \vec{s} = \vec{t^*} \sim \{t^*\}$$

has limit t for all $n \geq k$.

Summation Methods

In this section, we will introduce various summation methods where regularity is easily demonstrated. The discussion of Nörlund and Euler means is focused on their definition and proving regularity, for more depth see chapter IV from Hardy's *Divergent Series*[1]. The last method is an original to demonstrate the ease of creating regular methods using Theorem 2.

Definition 3. ***(Nörlund Means)*** *We say that a series $\sum a_n$ converges to s (N, p_n) if*

$$\frac{p_k s_0 + p_{k-1} s_1 + ... + p_0 s_k}{p_0 + p_1 + ... + p_k} = t_k \longrightarrow s \text{ as } k \to \infty$$

for $p_0 > 0$ and $p_k \geq 0$ for $k > 0$.

Notice that if $p_n = 1$ for all n, then we get precisely the definition of Cesàro summation. In fact, the reason for the notation $\sum a_n \to s$ $(C, 1)$ is that Cesàro summation can be strengthened by calculating the Nörlund mean with

$$p_n = \binom{n+k-1}{k-1} \Longrightarrow s\ (N, p_n) \equiv s\ (C, k).$$

The connection between these two means is much deeper than the intent of this paper. For the interested reader, more can be found in pages 64 and 109 of *Divergent Series*[1]. Now we will prove that the Nörlund mean is regular precisely when $p_n/P_n \to 0$ where $P_n = p_0 + ... + p_n$.

Proof. As a visual aid, we will consider the matrix defining the Nörlund mean

$$a_{i,j} = \begin{cases} \frac{p_{i-j}}{P_i} \text{ for } i \leq j \\ 0 \text{ otherwise} \end{cases} \qquad \begin{bmatrix} \frac{p_0}{P_0} & 0 & 0 & \cdots & 0 & \cdots \\ \frac{p_1}{P_1} & \frac{p_0}{P_1} & 0 & \cdots & 0 & \cdots \\ \frac{p_2}{P_2} & \frac{p_1}{P_2} & \frac{p_0}{P_2} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ \frac{p_n}{P_n} & \frac{p_{n-1}}{P_n} & \frac{p_{n-2}}{P_n} & \cdots & \frac{p_0}{P_n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_n \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{p_0 s_0}{p_0} \\ \frac{p_1 s_0 + p_0 s_1}{p_0 + p_1} \\ \frac{p_2 s_0 + p_1 s_1 + p_0 s_2}{p_0 + p_1 + p_2} \\ \vdots \\ \frac{p_n s_0 + ... + p_0 s_n}{P_n} \\ \vdots \end{bmatrix}.$$

It is now clear that the sum of each element in a particular row of the matrix will be 1 satisfying condition *iii*) of Theorem 2. Since all of these elements are non-negative we have also satisfied condition *i*). For the second condition, we will make some rough estimates of the elements in the n^{th} row. For a fixed column k

$$\frac{p_{n-k}}{P_{n-k}} \geq \frac{p_{n-k}}{P_n}$$

because each p_n is non-negative. So if $p_n/P_n \to 0$ in the column $k = 0$, then the rest of the columns will follow suit, and if $p_n/P_n \not\to 0$ then condition *ii*) will not be satisfied. Therefore the Nörlund mean is regular precisely when $p_n/P_n \to 0$ by Theorem 2. □

Definition 4. ***(Euler Means)*** *With notation as before, we say that a series* $\sum a_n$ *converges to* $s\ (E,q)$ *if*

$$a_{i,j} = \begin{cases} \binom{i+1}{j+1}\frac{q^{i-j}}{(q+1)^{i+1}} \text{ for } i \leq j \\ 0 \text{ otherwise} \end{cases} \qquad \mathcal{M} \cdot \vec{s} = \vec{t} \sim \{t_n\} \to s$$

for some positive real q*. This method can be proved to be regular.*

Proof. For any fixed row, there are only a finite number of non-zero elements, so we can use the binomial theorem

$$\sum_{k=0}^{\infty} a_{n,k} = \sum_{k=0}^{n} \binom{n+1}{k+1} \frac{q^{n-k}}{(q+1)^{n+1}} = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{q^{n-k+1}}{(q+1)^{n+1}} - \frac{q^{n+1}}{(q+1)^{n+1}}$$
$$= \left(\frac{q^{n+1}}{(q+1)^{n+1}}\right)\left((1+q^{-1})^{n+1} - 1\right) = 1 - \left(\frac{q}{q+1}\right)^{n+1}$$

since $q > 0$, this approaches 1 as $n \to \infty$. Additionally, since each $a_{n,k}$ is non-negative, the row sums are absolutely convergent and bounded by 1 independent of the row, thus conditions i) and iii) of Theorem 2 are met. For fixed k,

$$a_{n,k} = \binom{n+1}{k+1} \frac{q^{n-k}}{(q+1)^{n+1}} \leq \left(\frac{q^{-k-1}}{(k+1)!}\right) \frac{(n+1)^{k+1}}{\left(\frac{q+1}{q}\right)^{n+1}} \longrightarrow 0$$

as $n \to \infty$ by L'Hôpital's rule. With condition ii) met, Theorem 2 states that the Euler method of summation is regular for $q > 0$. □

Now that we have gained some familiarity with some regular summation methods, it is time to create a new one. We will need to come up with a formula for $a_{i,j}$ for our matrix $\mathcal{M}$. To make things simple lets make $a_{i,j}$ be rational and be zero for $i > j$. It is important that the denominator decrease as the index of the rows increase. As long as we are having fun, lets do something random for the denominator and define x_n to be a random integer in $[0, n]$. While we are at it, lets put a random variable in the numerator too. Let y_n be a random integer in $[0, p]$ (it is important that this is bounded for hope of regularity). Now with some adjusting we have a new summation method, and all that is left is to check that it is regular.

Definition 5. ***(Shea Means)*** *We will say that a series* $\sum a_n$ *converges to* $s\ (S,p)$ *if*

$$a_{i,j} = \begin{cases} \left(\frac{3y_j}{p(x_i+i+1)}\right) \text{ for } i \leq j \\ 0 \text{ otherwise} \end{cases} \qquad \mathcal{M} \cdot \vec{s} = \vec{t} \sim \{t_n\} \to s$$

where x_i *is the arithmetic mean of* $i+1$ *random integers in* $[0, i]$*, and* y_j *is a random integer in* $[0, p]$*.*

Proof. First we will check the rows

$$\sum_{k=0}^{\infty} a_{n,k} = \sum_{k=0}^{n} \frac{3y_k}{p(x_n+n+1)} \leq \frac{3p(n+1)}{p(n+1)}.$$

The bound on the right is clear for $y_k = p$ and $x_n = 0$. The Central Limit Theorem† tells us that $x_n \sim n/2$ for large n, and since the random variable is uniform over integers the Law of Large Numbers† tells us that $\sum_{k=0}^{n} y_k \sim pn/2$. So as $n \to \infty$ we get

$$\sum_{k=0}^{n} \frac{3y_k}{p(x_n+n+1)} \longrightarrow \frac{3\frac{pn}{2}}{p(\frac{3}{2}n+1)} \longrightarrow \frac{n}{n+\frac{2}{3}} \longrightarrow 1.$$

For fixed k we also have

$$\frac{3y_k}{p(x_n+n+1)} \leq \frac{3p}{p(n+1)} \longrightarrow 0 \text{ as } n \to \infty.$$

By Theorem 2, this new summation method is regular. □

Grandi's Series

Now that we have established some methods that produce correct answers for convergent series, lets see what happens when we apply them to some well known divergent series. Consider Grandi's series

$$1 - 1 + 1 - 1 + 1 - 1 + ...$$

Lets use our new summation method with $\{x_n\} = \left\{0, \frac{1}{2}, \frac{5}{3}, \frac{2}{4}, \frac{7}{5}, \frac{24}{6}, ...\right\}$, and $p = 10$ so that $\{y_n\} = \{7, 5, 6, 4, 8, 1, ...\}$ (generated by random.org).

$$a_{i,j} = \begin{cases} \left(\frac{3y_j}{10(x_i+i+1)}\right) & \text{for } i \leq j \\ 0 \text{ otherwise} \end{cases} \qquad \begin{bmatrix} \frac{21}{10} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{21}{25} & \frac{15}{25} & 0 & 0 & 0 & 0 & \cdots \\ \frac{63}{140} & \frac{45}{140} & \frac{54}{140} & 0 & 0 & 0 & \cdots \\ \frac{21}{45} & \frac{15}{45} & \frac{18}{45} & \frac{12}{45} & 0 & 0 & \cdots \\ \frac{21}{66} & \frac{15}{66} & \frac{18}{66} & \frac{12}{66} & \frac{24}{66} & 0 & \cdots \\ \frac{21}{100} & \frac{15}{100} & \frac{18}{100} & \frac{12}{100} & \frac{24}{100} & \frac{3}{100} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 21/10 \\ 21/25 \\ 117/140 \\ 13/15 \\ 21/22 \\ 63/100 \\ \vdots \end{bmatrix}.$$

At this point it is not obvious what this new sequence is converging to, so we will use the approximation of $\frac{n}{2}$ for x_n and $\frac{p}{2} = 5$ for y_n. So as $n \to \infty$ we have

$$t_n = \sum_{k \text{ odd}}^{n} \frac{3y_k}{10(x_n+n+1)} \longrightarrow \frac{3\frac{5n}{2}}{10(\frac{n}{2}+n+1)} = \frac{3n}{6n+4} \longrightarrow \frac{1}{2}.$$

[1]†The Central Limit Theorem and The Law of Large Numbers can be found in most elementary statistics and probability textbooks.

Thus $\sum(-1)^n = \frac{1}{2}$ $(S, 10)$. This is good because if $s = 1 - 1 + 1 - 1 + ...$, then $s = 1 - s$ which implies $s = \frac{1}{2}$. It will be left as an exercise to verify that the other summation methods in this paper also produce the same result.
It should be noted that the consistency of these methods is not guaranteed from Theorem 2. Consistency is when two different methods produce the same results for a divergent series, while regularity is when one method produces the same results as the method of partial sums for convergent series.
One obstacle that is difficult to overcome is being able to find sums of unbounded series. There are more powerful summation methods that can be used to sum $1-2+3-4+5-... = s$, but we can also manipulate the sum as before using our axioms.

$$s = 1-(2-3+4-5+...) = 1-(1-1+1-1+...)-(1-2+3-4+...) = \frac{1}{2}-s \Longrightarrow s = \frac{1}{4}.$$

Now with several summation techniques and the Silverman-Toeplitz Theorem, one can produce new summation techniques, new sequences, and find interesting values to represent the "sum" of a divergent series. But the user should be cautioned, because the sum s of a divergent series holds a different meaning than that of a convergent series. However, if you already know that a series is convergent, using a regular method of summation can be used to find what the series converges to in a new and more interesting way. With the "sum" of a divergent series, paired with the method of summation, we can learn about the behavior and rate of divergence of that given series.

References

[1] Hardy, Godfrey Harold. *Divergent Series*. Oxford: Clarendon, 1949.

[2] Ruder, Brian. *Silverman-Toeplitz Theorem*. Kansas State University, 1966