Infinite Series: The Abel-Dini Theorem and Convergence Tests

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1 Introduction

In this paper, we will discuss infinite series. In K. Knopp's book [3], a proof that there is no perfect test for convergence is given. To do this, Knopp uses the Abel-Dini Theorem, which is of interest in its own right. The Abel-Dini Theorem is discussed more fully in T. H. Hildebrandt's article [2]. In section 2, we provide a proof of the Abel-Dini Theorem and discuss some applications. In section 3, we will discuss the results in [3] about convergence tests.

Fairly little advanced machinery is required to prove the results on convergence tests. Since this is a topic in real analysis, the reader should expect to see many inequalities. They should also be familiar with sequences of numbers. Familiarity with sequences of functions and uniform convergence is recommended but by no means required for the discussion of convergence tests. We will occasionally use $\limsup$ and $\liminf$, but the reader does not need to know all of the equivalent definitions. We will give the $\limsup$ definition that is most applicable to us below.

1.1 Real Analysis Background

Here we give the definitions and basic theorems on series in real analysis that we will need. We do not prove any of these results; proofs can be found in most introductory real analysis books or (advanced) calculus books.

Before we discuss series, we define $\limsup$ and $\liminf$.

Definition 1 ($\limsup$ and $\liminf$)**.** Given a sequence $\{a_n\}_{n=1}^{\infty}$, we say that $M = \limsup_{n\to\infty} a_n$ if

- for each $u > M$ there are only finitely many indices n for which $a_n > u$, and
- for each $l < M$ there are infinitely many indices n for which $a_n > l$.

Similarly, we say $m = \liminf_{n\to\infty} a_n$ if

- for each $l < m$ there are only finitely many indices n for which $a_n < l$, and

- for each $u > m$ then there are infinitely many indices n for which $a_n < u$.

We allow lim sup and lim inf to be $\pm\infty$, removing the appropriate condition in the standard definition. For example, we say $\limsup_{n\to\infty} a_n = \infty$ if for each $l \in \mathbb{R}$ there are infinitely many indices n for which $a_n > l$. The remaining modifications are left to the reader.

Every sequence has a lim sup and lim inf, and this definition uniquely specifies these values; see Exercise 1.5.9 and the discussion preceeding it in [1]. We will not prove these facts.

Now we turn to infinite series. We begin with series of numbers.

Definition 2 (Convergence of a Series of Numbers)**.** A series $\sum_{n=1}^{\infty} a_n$, often written $\sum_{1}^{\infty} a_n$ when the index of summation is unambiguous, **converges** if the sequence $\{S_n\}_{n=1}^{\infty}$ defined by $S_n = \sum_{k=1}^{n} a_k$ converges (as a sequence of numbers). If $S_n \to S$, we say that the **sum** of the series is S.

If the starting index of summation is j, so that the series is $\sum_{n=j}^{\infty} a_n$, we modify the definition by declaring $S_N = \sum_{n=j}^{N} a_n$. If the starting index is unknown or irrelevant, we will write $\sum a_n$ instead of $\sum_{j}^{\infty} a_n$. It is useful to give a name to the sequence associated with the series.

Definition 3 (Partial Sums)**.** Given a series $\sum_{n=1}^{\infty} a_n$, we say that $S_N = \sum_{1}^{N} a_n$ is the N**th partial sum** of the series. Thus our definition of convergence is that a series converges if the sequence of partial sums converges.

Observe that whenever the series $\sum_{n=1}^{\infty} a_n$ converges, so does $\sum_{n=N}^{\infty} a_n$, since the constant $\sum_{n=1}^{N-1} a_n$ has been subtracted off from every partial sum of the original series. This observation justifies the following definition.

Definition 4 (Tails)**.** If $\sum_{n=1}^{\infty} a_n$ converges, we say that $\sum_{n=N}^{\infty} a_n$ is the N**th tail** of the series.

The indexing in this definition differs from the definition given in some other texts. Some authors call $\sum_{n=N+1}^{\infty} a_n$ the Nth tail. The difference is not significant as long as the theorems are stated correctly, but the reader should be aware of this distinction.

The following result follows immediately from the fact that the sequence of partial sums converges if and only if it is Cauchy.

Theorem 1 (Cauchy Convergence Criterion). *The series* $\sum_{1}^{\infty} a_k$ *converges if and only if for each* $\epsilon > 0$*, there is* N *such that* $\left|\sum_{k=n}^{m} a_k\right| < \epsilon$ *whenever* $m \geq n \geq N$*.*

From this, one can prove the *basic comparison test for series of nonnegative terms.* Also, by applying the triangle inequality, one finds that *absolute convergence implies convergence.* For the statements of these results, see Theorems 6.11 and 6.17 in [1].

On occasion we will have something to say about uniform convergence of a series of functions. The definition of uniform convergence of a sequence of functions can be found in Section 7.1 of [1].

Definition 5 (Uniform Convergence of a Series of Functions). Given a sequence of real-valued functions $\{f_n\}_{n=1}^{\infty}$ defined on a set S, we say that the series of functions $\sum_{n=1}^{\infty} f_n$ **converges uniformly on a set** $W \subseteq S$ if the sequence of partial sums $S_N = \sum_{n=1}^{N} f_n$, which are also real-valued functions defined on S, converges uniformly on W.

We also have an analogous result to the Cauchy Convergence Criterion, which follows from the fact that a sequence of functions converges uniformly if and only if the sequence is uniformly Cauchy (Theorem 7.7 in [1]).

Theorem 2 (Cauchy Convergence Criterion for Functions). *Given a sequence of functions* $f_k : S \to \mathbb{R}$, $\sum_{1}^{\infty} f_k$ *converges uniformly on* S *if and only if for each* $\epsilon > 0$*, there is* N *such that* $\left|\sum_{k=n}^{m} f_k(x)\right| < \epsilon$ *for all* $x \in S$ *whenever* $m \geq n \geq N$*.*

It is possible to prove the *Weierstrass M-Test* (Theorem 7.9 in [1]) with the triangle inequality and the Cauchy Convergence Criterion for numbers and the Cauchy Convergence Criterion for functions.

While the Cauchy Convergence Criterion is important, it is not always easy to apply. There are several tests that can be used to determine whether a series converges. The first is the standard limit comparison test, not stated here (see Theorem 6.12 in [1]). This isn't always easy to apply because we need to come up with a series to compare to. Two of the best known convergence tests which do not require us to find a comparison series are the ratio and root tests.

Theorem 3 (Ratio Test). *If* $\sum_{1}^{\infty} a_k$ *is a series of positive terms,* $u = \limsup_{k\to\infty} \frac{a_{k+1}}{a_k}$*, and* $l = \liminf_{k\to\infty} \frac{a_{k+1}}{a_k}$*, then the series converges if* $u < 1$ *and diverges if* $l > 1$*. Otherwise no conclusions can be drawn.*

Theorem 4 (Root Test). *If $\sum_1^\infty a_k$ is a series of nonnegative terms and $r = \limsup_{k\to\infty} (a_k)^{\frac{1}{k}}$, then the series converges if $r < 1$ and diverges if $r > 1$. If $r = 1$ then no conclusions can be drawn.*

It is possible to prove that if all the a_n are positive, then

$$\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} \le \liminf_{k\to\infty} (a_k)^{\frac{1}{k}} \le \limsup_{k\to\infty} (a_k)^{\frac{1}{k}} \le \limsup_{k\to\infty} \frac{a_{k+1}}{a_k},$$

which shows that if the ratio test gives the convergence/divergence of a particular series then so does the root test (i.e., the root test is better than the ratio test). However, the root test can also be inconclusive; both the divergent series $\sum_1^\infty 1$ and the convergent series $\sum_1^\infty \frac{1}{n^2}$ are series on which the root test gives $r = 1$. To understand why $r = 1$ is an issue, we need to examine the proof of the root test. For convergence when $r < 1$, the series is compared to a geometric series $\sum x^n$ with $r < x < 1$. For divergence when $r > 1$, one observes that $|a_n| \ge 1$ infinitely often, so $a_n \not\to 0$ and hence the series diverges. However, since the geometric series has radius of convergence 1, when $r = 1$ we can't compare to a geometric series, but we also cannot say that the terms are bigger than 1 in absolute value infinitely often.

Inconclusiveness is not specific to tests relying on comparison to geometric series. As an example, we briefly discuss Raabe's Test.

Theorem 5 (Raabe's Test). *If $\sum_1^\infty a_k$ is a series of positive terms for which $\frac{a_{n+1}}{a_n} \to 1$ and $n\left(1 - \frac{a_{n+1}}{a_n}\right) \to L$ as $n \to \infty$, then the series converges if $L > 1$ and diverges if $L < 1$. If $L = 1$ then no conclusions can be drawn.*

For a more complete discussion of Raabe's Test, along with a proof, see the subsection "Raabe's Test" in Section 6.2 of [1]. The proof involves comparing the series to the p-series for a well-chosen p. As we will see in Section 2.1, the p-series converges if and only if $p > 1$. When $L = 1$, we cannot compare to the p-series and hence the test is inconclusive.

Notice that all of our tests rely on comparison. In Section 3, to prove that there is "no perfect test for convergence", we will show that no series can be used in the comparison test to find the convergence/divergence of all other series.

2 Abel-Dini Theorem

In this section, we prove the Abel-Dini Theorem and discuss some of its corollaries. Unless otherwise stated, **all series have positive terms.**

The proof will be very similar to the proof in [2], but there are some differences. Our first step is to prove a result in the case that the original series converges.

Lemma 1. *If* $\sum_{1}^{\infty} a_k$ *converges, then with* T_n *as the* n*th tail,* $\sum_{1}^{\infty} \frac{a_n}{T_n^{1+\alpha}}$ *converges if and only if* $\alpha < 0$.

The proof of this result given in [2] requires Bernoulli's inequality to show that for $x \geq 0$ and $0 < \beta < 1$, $x^\beta = (1+(x-1))^\beta \leq 1+\beta(x-1)$. The standard proof of Bernoulli's inequality that we are aware of requires taking derivatives; however, the proof of Lemma 1 that we present here only requires an inequality that can be derived from the **elephant-teacup formula**, or the difference of powers formula.

Inequality 1. For any integer $m > 0$ and real number $v > 0$,

$$v^{\frac{1}{m}} \leq \tfrac{1}{m}(v-1)+1.$$

The reader should note that this is Bernoulli's inequality with $\beta = \dfrac{1}{m}$.

Proof of Inequality 1. Let $\mu = v^{\frac{1}{m}}$. Then $\mu > 0$ and the inequality we wish to show is equivalent to $m(\mu - 1) \leq \mu^m - 1$. There are three cases to consider: $\mu = 1$, $\mu < 1$, and $\mu > 1$.

If $\mu = 1$ then we have equality, since $m(\mu-1) = 0 = \mu^m - 1$.

In the other two cases, we apply the elephant-teacup formula. We do the case that $\mu < 1$. Since $\mu < 1$, we have $\mu^k < 1$ for all integers $k \geq 1$ and hence

$$m > \mu^{m-1} + \mu^{m-2} + \cdots + 1 = \frac{\mu^m - 1}{\mu - 1}.$$

Since $\mu - 1 < 0$ it follows that $m(\mu-1) < \mu^m - 1$. The case $\mu > 1$ differs only in that certain inequalities are reversed and hence is left to the reader. □

Before proving the lemma, we state one more fact. For fixed v satisfying $0 < v < 1$, the function $t \mapsto v^{-t}$ is monotonically increasing. For the proof, write $v^{-t} = e^{-t\log v}$ and apply that e^x is monotonically increasing and $-\log v > 0$.

Now we can prove Lemma 1. The proof involves writing the terms in the original series as differences of tails and applying the Cauchy Convergence Criterion.

Proof of Lemma 1. Since $\sum_{1}^{\infty} a_k$ converges, for n large enough we have $T_n < 1$, so $T_n \in (0,1)$. Hence for $\alpha > 0$ one finds $0 < \dfrac{a_n}{T_n} \leq \dfrac{a_n}{T_n^{1+\alpha}}$. Thus by the basic comparison test it suffices to show that $\sum_{1}^{\infty} \dfrac{b_n}{T_n}$ diverges. For this, we write $b_n = T_n - T_{n+1}$ and apply the Cauchy Convergence Criterion. Since the tails form a decreasing sequence, we have

$$\sum_{k=n}^{m} \frac{b_k}{T_k} = \sum_{k=n}^{m} \frac{T_k - T_{k+1}}{T_k} \geq \frac{1}{T_n} \sum_{k=n}^{m} T_k - T_{k+1} = \frac{1}{T_n}(T_n - T_{m+1}) = 1 - \frac{T_{m+1}}{T_n}.$$

Now for each (fixed) n, there is $m \geq n$, m dependent upon n, such that $\frac{T_{m+1}}{T_n} < \frac{1}{2}$. To see this, observe that the denominator is positive and fixed while the numerator decreases to 0. Now we apply the Cauchy Convergence Criterion with $\epsilon = \frac{1}{2}$. For any N, let $n = N$ and then take $m \geq n$ such that $\frac{T_{m+1}}{T_n} < \frac{1}{2}$. Then we have

$$\sum_{k=n}^{m} \frac{b_k}{T_k} \geq 1 - \frac{T_{m+1}}{T_n} > \frac{1}{2}.$$

In other words, there is no N such that $\sum_{k=n}^{m} \frac{b_k}{T_k} < \frac{1}{2}$ whenever $m \geq n \geq N$. Hence by the Cauchy Convergence Criterion, the series diverges.

We now want to show that $\sum_1^\infty \frac{b_n}{T_n^{1+\alpha}}$ converges for each $\alpha < 0$. We will instead prove that $\sum_1^\infty \frac{b_n}{T_n^{1-\frac{1}{m}}}$ converges for each integer $m \geq 0$. This will give the desired result, because for any $\alpha < 0$ we can take m so large that $\alpha \leq -\frac{1}{m}$. Then we have $0 \leq \frac{b_n}{T_n^{1+\alpha}} \leq \frac{b_n}{T_n^{1-\frac{1}{m}}}$, whence $\sum_1^\infty \frac{b_n}{T_n^{1+\alpha}}$ converges by the basic comparison test.

Let m be fixed. We can rewrite the mth term in the series, $\frac{b_n}{T_n^{1-\frac{1}{m}}}$, as $\frac{T_n - T_{n+1}}{T_n^{1-\frac{1}{m}}} = T_n^{\frac{1}{m}} \frac{T_n - T_{n+1}}{T_n}$. Now we can apply **Inequality 1** in the form $1 - u \leq m(1 - u^{\frac{1}{m}})$, obtaining

$$\frac{b_n}{T_n^{1-\frac{1}{m}}} = T_n^{\frac{1}{m}} \left(1 - \frac{T_{n+1}}{T_n}\right) \leq T_n^{\frac{1}{m}} m \left(1 - \left(\frac{T_{n+1}}{T_n}\right)^{\frac{1}{m}}\right) = mT_n^{\frac{1}{m}} - mT_{n+1}^{\frac{1}{m}}.$$

The convergence of the telescoping series $\sum_1^\infty mT_n^{\frac{1}{m}} - mT_{n+1}^{\frac{1}{m}}$ implies the convergence of $\sum_1^\infty \frac{b_n}{T_n^{1-\frac{1}{m}}}$ by the basic comparison test. □

The reason why we include this proof rather than the proof using Bernoulli's inequality is that this proof only requires basic algebra and somewhat intuitive facts about convergence. The ingredients of the proof are the elephant-teacup formula, the increasing property of e^x, properties of limits, the Cauchy Convergence Criterion, and the basic comparison test. Moreover, the proof also contains a proof that the function f defined by

$$f(\alpha) = \sum_1^\infty \frac{a_n}{T_n^{1+\alpha}}, \qquad \alpha < 0$$

is continuous in α, because each term in the sum is continuous in α and the series converges uniformly on each of the intervals $S_m = \left(-\infty, -\frac{1}{m}\right]$ by the Weierstrass M-Test with $M_n = \frac{a_n}{T_n^{1-\frac{1}{m}}}$. The continuity of f is not particularly important in the material on convergence tests, but it is an interesting result to obtain.

We are now ready to prove the Abel-Dini Theorem.

Theorem (Abel-Dini)**.** *If* $\sum_1^\infty b_k$ *diverges, then with* S_n *as the* n*th partial sum,* $\sum_1^\infty \frac{b_n}{S_n^{1+\alpha}}$ *converges if and only if* $\alpha > 0$*.*

The proof involves taking the divergent series and forming a convergent series with the appropriate tails, then applying Lemma 1. We will want the nth tail of the series to be $\frac{1}{S_n}$, so the nth term of the series is $a_n = \frac{1}{S_n} - \frac{1}{S_{n+1}} = \frac{S_{n+1} - S_n}{S_n S_{n+1}} = \frac{b_{n+1}}{S_n S_{n+1}}$.

Before we begin the proof, we record another fact. For fixed t, the function defined for positive v by $v \mapsto v^{-t}$ is increasing if $t < 0$ and decreasing if $t > 0$. To see this, again write $v^{-t} = e^{-t \log v}$ and notice that $\log v$ is increasing, so $-t \log v$ is increasing if $t < 0$ and decreasing if $t > 0$; since e^x is increasing, we find that $e^{-t \log v}$ increases if $t < 0$ and decreases if $t > 0$. If $t = 0$ then the function is constant.

Proof of the Abel-Dini Theorem. Suppose $\sum_1^\infty b_n$ diverges. Define

$$a_n = \frac{b_{n+1}}{S_n S_{n+1}} = \frac{1}{S_n} - \frac{1}{S_{n+1}}.$$

Then the series $\sum_1^\infty a_k$ converges, because its nth partial sum is $\frac{1}{S_1} - \frac{1}{S_n}$ and $\frac{1}{S_n} \to 0$ as $n \to \infty$. By the same reasoning, the nth tail is $T_n = \sum_n^\infty a_k = \frac{1}{S_n}$. By Lemma 1 the series $\sum_1^\infty \frac{a_n}{T_n^{1+\alpha}}$ converges precisely when $\alpha < 0$. Now we have

$$\frac{a_n}{T_n^{1+\alpha}} = \frac{b_n}{S_n S_{n+1}} S_n^{1+\alpha} = \frac{b_{n+1}}{S_n^{-\alpha} S_{n+1}}.$$

By replacing α with $-\alpha$ and switching the direction of $\alpha < 0$, one finds[1] $\sum_1^\infty \frac{b_{n+1}}{S_n^\alpha S_{n+1}}$ converges if and only if $\alpha > 0$.

[1] This is refered to in [2] as the "Pringsheim modification" of the result.

From here, to obtain the result we apply the basic comparison test. Since the S_n are a monotonically increasing sequence of positive numbers, we have $S_n^{-\alpha} \geq S_{n+1}^{-\alpha}$ if $\alpha > 0$ and $S_n^{-\alpha} \leq S_{n+1}^{-\alpha}$ if $\alpha \leq 0$. Hence we have

$$\frac{b_{n+1}}{S_n^\alpha S_{n+1}} \geq \frac{b_{n+1}}{S_{n+1}^\alpha S_{n+1}}, \qquad \alpha > 0$$

so by the basic comparison test $\displaystyle\sum_1^\infty \frac{b_{n+1}}{S_{n+1}^{1+\alpha}}$ converges if $\alpha > 0$. On the other hand, we have

$$\frac{b_{n+1}}{S_n^\alpha S_{n+1}} \leq \frac{b_{n+1}}{S_{n+1}^\alpha S_{n+1}}, \qquad \alpha \leq 0$$

so $\displaystyle\sum_1^\infty \frac{b_{n+1}}{S_{n+1}^{1+\alpha}}$ diverges if $\alpha \leq 0$. Hence $\displaystyle\sum_2^\infty \frac{b_n}{S_n^{1+\alpha}}$ converges if and only if $\alpha > 0$, and since adding $\dfrac{b_1}{S_1^{1+\alpha}}$ does not affect the convergence or divergence of the series, $\displaystyle\sum_1^\infty \frac{b_n}{S_n^{1+\alpha}}$ converges if and only if $\alpha > 0$. □

Once again, the function $g(\alpha) = \displaystyle\sum_1^\infty \frac{b_n}{S_n^{1+\alpha}}$ defined for $\alpha > 0$ is continuous in α. This can be shown by comparison with the series defining f in Lemma 1.

Lemma 2. *If f_n and g_n are sequences of functions defined on a set S, $0 \leq g_n \leq f_n$ for all $n \geq 1$, and $\displaystyle\sum_1^\infty f_n$ converges uniformly on S, then $\displaystyle\sum_1^\infty g_n$ converges uniformly on S.*

Proof. We will apply the Cauchy Convergence Criterion for functions. Let $\epsilon > 0$ be arbitrary; since $\displaystyle\sum_1^\infty f_n$ converges uniformly on S, there is K such that for all $k \geq j \geq K$ we have $\displaystyle\sum_{n=j}^k f_n(x) = \left|\sum_{n=j}^k f_n(x)\right| < \epsilon$ for all $x \in S$. Hence for all $k \geq j \geq K$ we have $\displaystyle\left|\sum_{n=j}^k g_n(x)\right| = \sum_{n=j}^k g_n(x) \leq \sum_{n=j}^k f_n(x) < \epsilon$ for all $x \in S$, so $\displaystyle\sum_1^\infty g_n$ converges uniformly on S. □

From this it is possible to deduce that g is continuous. Define sequences of functions $f_n, g_n : (0, \infty) \to \mathbb{R}$ by $f_n(\alpha) = \dfrac{b_{n+1}}{S_n^\alpha S_{n+1}} = \dfrac{a_n}{T_n^{1-\alpha}}$, with a_n as defined in the proof of the Abel-Dini Theorem, and $g_n(\alpha) = \dfrac{b_{n+1}}{S_{n+1}^{1+\alpha}}$. These are positive and satisfy $0 \leq g_n \leq f_n$ on $(0, \infty)$, and in the proof of Lemma 1 we proved that $\displaystyle\sum_1^\infty f_n$ converges uniformly on $\left[\dfrac{1}{m}, \infty\right)$ for each positive integer m, so $\displaystyle\sum_1^\infty g_n$ converges uniformly on $\left[\dfrac{1}{m}, \infty\right)$ for each positive integer

m. Since the g_n are all continuous, the series defines a continuous function on $(0, \infty)$. Hence $\sum_2^\infty \frac{b_n}{S_n^{1+\alpha}}$ defines a continuous function of α. Adding in the first term, which is continuous in α on $(0, \infty)$, we find that g is continuous on $(0, \infty)$. As was the case with f, the continuity of g is also not used in our discussion of comparison tests, but it is interesting to obtain.

2.1 Applications

We now discuss some applications of the Abel-Dini Theorem. The results about convergence tests do not depend on the results in this section, so the reader can skip to Section 3.

The Abel-Dini Theorem yields the convergence properties of the p-series, which is often examined via the integral test (see the proof of Theorem 6.9 in [1]). Indeed, let $b_k = 1$. The Abel-Dini theorem applies and gives that $\sum_1^\infty \frac{1}{n^{1+\alpha}}$ converges if and only if $\alpha > 0$.

The convergence properties of $\sum_1^\infty \frac{1}{n(\log n)^{1+\alpha}}$ can also be proven from the Abel-Dini Theorem. The previous paragraph shows that if $b'_k = \frac{1}{k}$, the series $\sum_1^\infty b'_k$ diverges. Setting $S_n = \sum_{k=1}^n \frac{1}{k}$, the Abel-Dini Theorem gives that $\sum_1^\infty \frac{1}{nS_n^{1+\alpha}}$ converges if and only if $\alpha > 0$. To get the result with logarithms we will apply the limit comparison test.

It is possible to show (Exercise 6.2.24 of [1]) that the sequence $\{c_n\}_{n=1}^\infty$ defined by $S_n = c_n + \log n$ is positive and decreasing and hence convergent; its limit γ is the Euler-Mascheroni constant. So $\frac{\log n}{S_n} = 1 - \frac{c_n}{S_n} \to 1$ as $n \to \infty$. Hence

$$\frac{\frac{1}{nS_n^{1+\alpha}}}{\frac{1}{n(\log n)^{1+\alpha}}} = \left(\frac{\log n}{S_n}\right)^{1+\alpha} \to 1 \text{ as } n \to \infty,$$

so the limit comparison test gives that $\sum_2^\infty \frac{1}{n(\log n)^{1+\alpha}}$ converges if and only if $\alpha > 0$. Notice that we have to replace the lower index of summation with a 2 because when $n = 1$ the denominator is zero.

The usual proof of both of these results uses the integral test. The benefit to the Abel-Dini method is that we didn't have to show that $f(x) = \frac{1}{x(\log x)^{1+\alpha}}$ is decreasing for large x. For series of the form

$$\sum \frac{1}{n(\log n)(\log \log n) \cdots (\log \log \cdots \log n)^{1+\alpha}},$$

where the starting index is chosen so that all terms are well defined, the the Abel-Dini Theorem does not apply so easily, as it's not immediately clear how to get terms of the form $\log \cdots \log n$. The following way of extending this result is due to [2].

We begin by attempting to generalize the result that $\sum_1^\infty \frac{1}{n(\log n)^{1+\alpha}}$ converges if and only if $\alpha > 0$, just as the Abel-Dini Theorem generalizes the p-test. A nice generalization would be that if $\sum_1^\infty b_k$ diverges, then with S_n as the nth partial sum, $\sum_1^\infty \frac{b_n}{S_n(\log S_n)^{1+\alpha}}$ converges if and only if $\alpha > 0$, but this is unfortunately not true. The "if" direction is true (that is, if $\alpha > 0$ then the series converges), but in some cases the "only if" direction fails. An example given in [2] is a series whose nth partial sum is $S_n = n^{n^n}$; the details are not discussed here.

To shed light on the issue, we attempt to prove the result to understand what additional assumptions need to be made. Consider the series whose kth term is $b'_k = \log S_k - \log S_{k-1}$, where we declare $S_0 = 1$. Then $\sum_1^\infty b'_k$ diverges because the nth partial sum is $\log S_n$. Hence by the Abel-Dini Theorem, $\sum \frac{b'_n}{(\log S_n)^{1+\alpha}} = \sum \frac{\log S_n - \log S_{n-1}}{(\log S_n)^{1+\alpha}}$ converges if and only if $\alpha > 0$, where we take the starting index large enough that the denominator is never zero. Recalling that $b_n = S_n - S_{n-1}$, the quotient of the nth terms in the original and new series is

$$\frac{\dfrac{b_n}{S_n(\log S_n)^{1+\alpha}}}{\dfrac{\log S_n - \log S_{n-1}}{(\log S_n)^{1+\alpha}}} = \frac{S_n - S_{n-1}}{S_n \log \dfrac{S_n}{S_{n-1}}} = \frac{1 - \dfrac{S_{n-1}}{S_n}}{\log \dfrac{S_n}{S_{n-1}}} = \frac{\dfrac{S_{n-1}}{S_n} - 1}{\log \dfrac{S_{n-1}}{S_n}}.$$

It is possible to show that $\frac{x-1}{\log x}$ is strictly increasing on $(0,1)$, that $\lim_{x\to 0^+} \frac{x-1}{\log x} = 0$, and that $\lim_{x\to 1} \frac{x-1}{\log x} = 1$; to prove these facts, differentiate or use L'Hôpital's rule. Since the terms b_n in the original series are all positive, $\frac{S_{n-1}}{S_n} \in (0,1)$ so the ratio satisfies

$$\frac{\dfrac{S_{n-1}}{S_n} - 1}{\log \dfrac{S_{n-1}}{S_n}} \le 1 \text{ and hence } \frac{b_n}{S_n(\log S_n)^{1+\alpha}} \le \frac{\log S_n - \log S_{n-1}}{(\log S_n)^{1+\alpha}};$$

thus $\sum_1^\infty \frac{b_n}{S_n(\log S_n)^{1+\alpha}}$ converges if $\alpha > 0$ by the basic comparison test.

The issue in the "only if" direction is that $\frac{S_{n-1}}{S_n}$ could tend to 0. If it does, then the limit of the sequence $\frac{\frac{S_{n-1}}{S_n} - 1}{\log \frac{S_{n-1}}{S_n}}$ is also 0, so the limit comparison test (which would show that either both series converge or both series diverge) does not apply. On the other hand, if there is a $\delta > 0$ for which $\frac{S_{n-1}}{S_n} > \delta$ for n large, then setting $\epsilon = \frac{\delta - 1}{\log \delta}$, we have $\epsilon > 0$ and

(whenever n is large)

$$\epsilon \leq \frac{\frac{S_{n-1}}{S_n} - 1}{\log \frac{S_{n-1}}{S_n}} \leq 1 \text{ and hence } \epsilon \cdot \frac{\log S_n - \log S_{n-1}}{(\log S_n)^{1+\alpha}} \leq \frac{b_n}{S_n(\log S_n)^{1+\alpha}} \leq \frac{\log S_n - \log S_{n-1}}{(\log S_n)^{1+\alpha}},$$

so $\sum_{1}^{\infty} \frac{b_n}{S_n(\log S_n)^{1+\alpha}}$ converges if and only if $\alpha > 0$. The condition that $\frac{S_{n-1}}{S_n} > \delta > 0$ can be rewritten as $1 - \frac{b_n}{S_n} > \delta$ or as $\frac{b_n}{S_n} < 1 - \delta < 1$. Hence we have shown the following:

Lemma 3. *If $\sum_{1}^{\infty} b_n$ diverges and S_n is the nth partial sum, and if there is $\delta > 0$ so that $\frac{b_n}{S_n} < 1 - \delta$ for n large (that is, if the ratio $\frac{b_n}{S_n}$ is bounded away from 1), then $\sum_{1}^{\infty} \frac{b_n}{S_n(\log S_n)^{1+\alpha}}$ converges if and only if $\alpha > 0$.*

With $b_n = 1$ the hypotheses are satisfied (take $r = \frac{3}{4}$, and then the inequality is satisfied for $n \geq 2$). This yields the logarithm result which we proved earlier by using the Euler-Mascheroni constant.

In [2] the author states the result for $\sum_{1}^{\infty} \frac{b_n}{S_n(\log S_n)(\log \log S_n)^{1+\alpha}}$ and says that it is possible to continue this reasoning to obtain the general result. The proof of the general result is not conceptually different from the proof of this result, so we will give the general result.

Theorem (p-test for Logarithms)**.** *If $\sum_{1}^{\infty} b_n$ diverges and S_n is the nth partial sum, and if there is $r < 1$ so that $\frac{b_n}{S_n} < r$ for large n, then*

$$\sum \frac{b_n}{S_n(\log S_n)(\log \log S_n) \cdots ((\log)^{k-1} S_n)((\log)^k S_n)^{1+\alpha}}$$

converges if and only if $\alpha > 0$. Here $(\log)^j$ denotes the logarithm applied j times, so $(\log)^0(x) = x$ and $(\log)^{j+1}(x) = \log(\log^j(x))$ for $j \geq 1$.

Proof. We compare the terms in the series to those in $\sum_{1}^{\infty} \frac{(\log)^k S_n - (\log)^k S_{n-1}}{((\log)^k S_n)^{1+\alpha}}$, declaring $S_0 = 1$; the Abel-Dini Theorem shows that this series converges if and only if $\alpha > 0$, since $(\log)^k S_n - (\log)^k S_{n-1}$ are the terms of a divergent series with nth partial sum $(\log)^k S_n$. The quotient of the nth terms in the series is

$$\frac{\dfrac{b_n}{S_n(\log S_n)\cdots((\log)^{k-1}S_n)((\log)^k S_n)^{1+\alpha}}}{\dfrac{(\log)^k S_n-(\log)^k S_{n-1}}{((\log)^k S_n)^{1+\alpha}}} = \frac{\dfrac{b_n}{S_n(\log S_n)\cdots((\log)^{k-1}S_n)}}{(\log)^k S_n-(\log)^k S_{n-1}}$$

$$= \frac{S_n-S_{n-1}}{S_n(\log S_n)\cdots((\log)^{k-1}S_n)[(\log)^k S_n-(\log)^k S_{n-1}]}.$$

Now we can rewrite this quotient as a product (henceforth referred to as **the product of index** n) by multiplying by 1 several times:

$$\frac{S_n-S_{n-1}}{S_n[\log S_n-\log S_{n-1}]}\frac{\log S_n-\log S_{n-1}}{\log S_n[\log\log S_n-\log\log S_{n-1}]}\cdots\frac{(\log)^{k-1}S_n-(\log)^{k-1}S_{n-1}}{(\log)^{k-1}S_n[(\log)^k S_n-(\log)^k S_{n-1}]}.$$

Now we examine each factor to obtain bounds on the entire product when n is large. The jth factor in the product of index n is

$$\frac{(\log)^{j-1}S_n-(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n[(\log)^j S_n-(\log)^j S_{n-1}]} = \frac{1-\dfrac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n}}{\log\left(\dfrac{(\log)^{j-1}S_n}{(\log)^{j-1}S_{n-1}}\right)} = \frac{\dfrac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n}-1}{\log\left(\dfrac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n}\right)}.$$

In the discussion before Lemma 3, we saw that there is N_1 such that if $n>N_1$ then the first factor in the product of index n,

$$\frac{\dfrac{S_{n-1}}{S_n}-1}{\log\dfrac{S_{n-1}}{S_n}},$$

lies in the interval $[\epsilon,1]$, where ϵ is a fixed positive real number dependent only upon the sequence $\{S_n\}_{n=1}^{\infty}$ (that is, ϵ is not dependent upon the index n). Now we will show that for each j such that $2\le j\le k$, there is N_j so that if $n>N_j$ then the jth factor in the product of index n lies between $\dfrac{1}{2}$ and 1:

$$\frac{1}{2} < \frac{\dfrac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n}-1}{\log\left(\dfrac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n}\right)} < 1.$$

Once this is known, it follows that if $n>\max\{N_1,\ldots,N_k\}$ then the quotient of the nth terms of the series, which is exactly the product of index n, lies in $(2^{-k+1}\epsilon,1)$, whence

$$2^{-k+1}\epsilon\frac{(\log)^k S_n-(\log)^k S_{n-1}}{((\log)^k S_n)^{1+\alpha}} < \frac{b_n}{S_n(\log S_n)(\log\log S_n)\cdots((\log)^{k-1}S_n)((\log)^k S_n)^{1+\alpha}}$$

$$< \frac{(\log)^k S_n-(\log)^k S_{n-1}}{((\log)^k S_n)^{1+\alpha}},$$

so $\sum_{1}^{\infty} \frac{b_n}{S_n(\log S_n)\cdots((\log)^{k-1}S_n)((\log)^k S_n)^{1+\alpha}}$ converges if and only if $\alpha > 0$ by comparison.

We prove the bound on the jth factor of index n, where $2 \leq j \leq k$. First, notice that for n large we have $(\log)^{j-1}S_n > (\log)^{j-1}S_{n-1} > 0$ and hence $\frac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n} \in (0,1)$, so

$$\frac{\frac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n} - 1}{\log\left(\frac{(\log)^{j}S_{n-1}}{(\log)^{j}S_n}\right)} < 1.$$

This proves part of the result. Now I claim that $\frac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n} \to 1$ as $n \to \infty$. If we can show this, then by sequential continuity we would have

$$\frac{\frac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n} - 1}{\log\left(\frac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n}\right)} \to 1$$

as $n \to \infty$, so $\frac{1}{2} < \frac{\frac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n} - 1}{\log\left(\frac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n}\right)}$ whenever n is sufficiently large.

We will show that $\frac{(\log)^{j-1}S_{n-1}}{(\log)^{j-1}S_n} \to 1$ as $n \to \infty$ by induction on j. For the base case, we write

$$\frac{\log S_{n-1}}{\log S_n} = \frac{\log S_n}{\log S_n} + \frac{\log S_{n-1} - \log S_n}{\log S_n} = 1 + \frac{\log \frac{S_{n-1}}{S_n}}{\log S_n}.$$

For n sufficiently large, we have $0 < 1 - r < \frac{S_{n-1}}{S_n} < 1$, so $\log \frac{S_{n-1}}{S_n}$ is bounded. On the other hand, $\log S_n \to \infty$ as $n \to \infty$, so $\frac{\log \frac{S_{n-1}}{S_n}}{\log S_n} \to 0$. Thus it follows that $1 + \frac{\log \frac{S_{n-1}}{S_n}}{\log S_n} \to 1$ as $n \to \infty$, which proves the base case. For the inductive step, we assume that $\frac{(\log)^{j_0-1}S_{n-1}}{(\log)^{j_0-1}S_n} \to 1$ as $n \to \infty$. We need to show that $\frac{(\log)^{j_0}S_{n-1}}{(\log)^{j_0}S_n} \to 1$. We proceed as was done in the base case, writing

$$\frac{(\log)^{j_0}S_{n-1}}{(\log)^{j_0}S_n} = 1 + \frac{(\log)^{j_0}S_{n-1} - (\log)^{j_0}S_n}{(\log)^{j_0}S_n} = 1 + \frac{\log\left(\frac{(\log)^{j_0-1}S_{n-1}}{(\log)^{j_0-1}S_n}\right)}{(\log)^{j_0}S_n}.$$

By the inductive hypothesis and by sequential continuity, we have $\log\left(\frac{(\log)^{j_0-1}S_{n-1}}{(\log)^{j_0-1}S_n}\right) \to 0$, and since $(\log)^{j_0}S_n \to \infty$ we have $\frac{(\log)^{j_0+1}S_{n-1}}{(\log)^{j_0+1}S_n} \to 1$, completing the inductive step.

As we described before, this shows that the jth factor in the product of index n lies between $\frac{1}{2}$ and 1 when n is sufficiently large, yielding the bound on the product of index n, which shows that the series in the theorem and $\sum_1^\infty \frac{(\log)^k S_n - (\log)^k S_{n-1}}{((\log)^k S_n)^{1+\alpha}}$ either both converge or both diverge. Therefore,

$$\sum \frac{b_n}{S_n(\log S_n)(\log\log S_n)\cdots((\log)^{k-1}S_n)((\log)^k S_n)^{1+\alpha}}$$

converges if and only if $\alpha > 0$, as claimed. □

By taking $b_n = 1$, the hypotheses are satisfied with $r = \frac{3}{4}$ and $n \geq 2$. Although this proof is possibly not as clear as the one using the integral test, the result obtained is more general. Unfortunately the proof requires more than just "elementary" facts; we needed L'Hôpital's rule, and sequential continuity played a major role in the estimates. However, this doesn't relate directly to the material on convergence, so if the reader is lost in this material they can still continue.

3 Convergence Tests

We are now ready to discuss convergence tests. These results are all due to [3], in which the results are attributed to A. Pringsheim. Once again, **all series have positive terms**.

The first of our results has to do with comparison tests. We have so far used only the basic comparison test. A generalized version can be stated with lim sup and lim inf.

Theorem 6 (Limit Comparison Test, lim sup and lim inf version)**.** *Let* $\sum_1^\infty a_n$ *and* $\sum_1^\infty b_n$ *be series of positive terms. Then:*

- *if* $0 < \liminf_{n\to\infty} \frac{a_n}{b_n}$ *then the convergence of* $\sum_1^\infty a_n$ *implies the convergence of* $\sum_1^\infty b_n$*, and*
- *if* $\limsup_{n\to\infty} \frac{a_n}{b_n} < \infty$*, then the divergence of* $\sum_1^\infty a_n$ *implies the divergence of* $\sum_1^\infty b_n$*.*

Proof. We use the characterization of lim sup given in the section on real analysis, along with the basic comparison test.

For the first statement, choose $l \in \mathbb{R}$ so that $0 < l < \liminf_{n\to\infty} \frac{a_n}{b_n}$. Then there is N so that if $n > N$, then $\frac{a_n}{b_n} > l$. Hence $a_n > lb_n > 0$, so if $\sum_1^\infty a_n$ converges then so does $\sum_1^\infty lb_n$ and hence so does $\sum_1^\infty b_n$.

For the second statement, choose $u \in \mathbb{R}$ so that $\limsup_{n\to\infty} \frac{a_n}{b_n} < u$; then since all terms in both series are positive, $u > 0$. Now there is N so that for $n > N$, we have $\frac{a_n}{b_n} < u$, and hence $a_n < ub_n$. If $\sum_1^\infty a_n$ diverges, then $\sum_1^\infty ub_n$ diverges, whence $\sum_1^\infty b_n$ diverges. □

This is more general, but we still need a convergent or divergent series to compare to. The question is whether there is a convergent series that works with the comparison test to give the convergence of *all* other convergent series, or a divergent series that works with the comparison test to give the divergence of *all* other divergent series.

To restate the question, is there a convergent series $\sum_1^\infty c_n$ of positive terms with the property that whenever $\sum_1^\infty \widetilde{c}_n$ is a convergent series of positive terms, $\liminf_{n\to\infty} \frac{c_n}{\widetilde{c}_n} > 0$? If so, then a series $\sum_1^\infty a_n$ converges if and only if $\liminf_{n\to\infty} \frac{c_n}{a_n} > 0$; the "if" direction is guaranteed by the limit comparison test, and the "only if" direction would be guaranteed by the assumption on $\sum_1^\infty c_n$.

The Abel-Dini Theorem shows that there is no such series. Indeed, if $\sum_1^\infty c_n$ has the desired properties, then it converges. Consider

$$\sum_1^\infty \widetilde{c}_n = \sum_1^\infty \frac{c_n}{\sqrt{T_n}}$$

where T_n is the nth tail of $\sum_1^\infty c_k$; the series $\sum_1^\infty \widetilde{c}_n$ converges by Lemma 1, but the ratio of terms is $\frac{c_n}{\frac{c_n}{\sqrt{T_n}}} = \sqrt{T_n} \to 0$ as $n \to \infty$, so $\liminf_{n\to\infty} \frac{c_n}{\widetilde{c}_n} = 0$. Hence $\sum_1^\infty c_n$ does not satisfy the conditions.

For divergent series, the question is whether there is a divergent series $\sum_1^\infty d_n$ with the property that whenever $\sum_1^\infty \widetilde{d}_n$ diverges, $\limsup_{n\to\infty} \frac{d_n}{\widetilde{d}_n}$ is real. The Abel-Dini Theorem once again shows that there is no such series. Indeed, if $\sum_1^\infty d_n$ has the desired properties, then it diverges. Consider

$$\sum_1^\infty \widetilde{d}_n = \sum_1^\infty \frac{d_n}{S_n}$$

where S_n is the nth partial sum of $\sum_1^\infty d_n$; the series $\sum_1^\infty \widetilde{d}_n$ diverges by the Abel-Dini Theorem, but the ratio of terms is $\frac{d_n}{\frac{d_n}{S_n}} = S_n \to \infty$ as $n \to \infty$, so $\sum_1^\infty d_n$ does not satisfy the conditions.

With very little effort, the Abel-Dini Theorem has allowed us to show that there is no series which is "perfect" for the comparison test. Less formally, there is no series that converges (resp. diverges) slower than any other convergent (resp. divergent) series.

What about other tests? One conceivable test is to have a sequence $\{p_n\}_{n=1}^\infty$ of positive numbers such that for any series $\sum_1^\infty a_n$ of positive terms, if $a_n p_n \to 0$, then the series converges, and if there is $\alpha > 0$ such that for all n, $a_n p_n \geq \alpha$, then the series diverges. We give two examples to show that there are sequences that satisfy the first condition and sequences that satisfy the second condition, so that the conditions are not untenable. If $p_n = n^2$, then the first condition is satisfied because if $n^2 a_n \to 0$ then there is C such that $n^2 a_n \leq C$ for all n, whence $a_n \leq \frac{C}{n^2}$, so that $\sum_{n=1}^\infty a_n$ converges by comparison to $\sum_{n=1}^\infty \frac{C}{n^2}$. If $p_n = 1$, then the second condition is satisfied because if $a_n \geq \alpha > 0$, then the series diverges by comparison to $\sum_{n=1}^\infty \alpha$.

Since in both cases we used comparison, we should expect that the sequences fail one of the properties. In the case that $p_n = 1$, we have the harmonic series as a counterexample, since $a_n p_n = \frac{1}{n} \to 0$ but $\sum_1^\infty \frac{1}{n}$ diverges. In the case that $p_n = n^2$, we have $a_n = \frac{1}{n^{\frac{3}{2}}}$ as a counterexample; for all $n \geq 1$ we have $a_n p_n = n^{\frac{1}{2}} \geq 1 > 0$, but $\sum_1^\infty \frac{1}{n^{\frac{3}{2}}}$ converges.

Now the question is whether there is a sequence that satisfies both conditions. Notice that even if can we find such a sequence, it will not give the convergence/divergence of *every* series. In particular, given the sequence $\{p_n\}_{n=1}^\infty$, define the terms of a series by

$$a_{2k} = \frac{1}{2kp_{2k}}, a_{2k+1} = \frac{1}{p_{2k+1}}.$$

Then $a_{2k}p_{2k} = \frac{1}{2k} \to 0$ but $a_{2k+1}p_{2k+1} = 1 > 0$. Hence $a_n p_n \not\to 0$ but also there is no $\alpha > 0$ such that for all n, $a_n p_n \geq \alpha$, so the test would be inconclusive.

In any case, we do not have to worry about the test being inconclusive because there is no such sequence. We will prove this by using the Abel-Dini Theorem. Indeed, suppose that the sequence $\{p_n\}_{n=1}^\infty$ satisfies the desired conditions. Take any $\alpha > 0$ and consider the series whose nth term is $a_n = \frac{\alpha}{p_n}$. Since $a_n p_n = \alpha > 0$, the series must diverge if the test works. With S_n as the nth partial sum, the Abel-Dini Theorem shows that $\sum_1^\infty \frac{a_n}{S_n}$ diverges. But

$\dfrac{a_n}{S_n}p_n = \dfrac{\alpha}{S_n} \to 0$ since $S_n \to \infty$, so if our test worked then $\sum_1^\infty \dfrac{a_n}{S_n}$ would converge. Hence $\{p_n\}_{n=1}^\infty$ does not satisfy the desired conditions.

We have now seen that there is no perfect comparison test for convergence. We close by discussing a result in [3] relating convergent and divergent series.

Theorem 7. *For any monotonely decreasing sequence $\{\epsilon_n\}_{n=1}^\infty$ tending to 0, there is a convergent series $\sum_1^\infty c_n$ and divergent series $\sum_1^\infty d_n$ with the property that $c_n = \epsilon_n d_n$.*

The reader should not be surprised to learn that we will apply the Abel-Dini Theorem.

Proof. Let $p_n = \dfrac{1}{\epsilon_n}$ if $n \geq 1$, $p_0 = 0$. Then p_n increases monotonically to ∞. Now the series $\sum_{n=1}^\infty p_n - p_{n-1}$ has nth partial sum p_n, and since this increases monotonically to ∞, the series diverges. The Abel-Dini Theorem applies, and we find that $\sum_{n=1}^\infty \dfrac{p_n - p_{n-1}}{p_n}$ diverges. Reindexing gives that $\sum_{n=0}^\infty \dfrac{p_{n+1} - p_n}{p_{n+1}}$ diverges, and hence $\sum_{n=1}^\infty \dfrac{p_{n+1} - p_n}{p_{n+1}}$ diverges. Now $\dfrac{p_{n+1} - p_n}{p_{n+1}} = 1 - \dfrac{p_n}{p_{n+1}} = 1 - \dfrac{\epsilon_{n+1}}{\epsilon_n}$. Therefore, with $d_n = \dfrac{p_{n+1} - p_n}{p_{n+1}}$, we have that $\sum_1^\infty d_n$ diverges. But $\epsilon_n d_n = \left(1 - \dfrac{\epsilon_{n+1}}{\epsilon_n}\right)\epsilon_n = \epsilon_n - \epsilon_{n+1}$, so with $c_n = \epsilon_n d_n$, the series $\sum_1^\infty c_n$ is telescoping and hence convergent, since the nth partial sum is $\epsilon_1 - \epsilon_{n+1} \to 0$ as $n \to \infty$. □

4 Conclusion

The Abel-Dini Theorem offers a generalization of the p-test. While the p-test is usually proved via the integral test and hence requires improper integrals, the proof of the Abel-Dini Theorem presented here requires almost no results in advanced calculus. We only needed to carefully use inequalities. The theorem is also widely applicable. We saw that it could be used to generalize the p-test for logarithms. The advantage to this proof is that we did not have to verify the hypotheses of the integral test, which requires that the function involved is increasing, and we also did not need the theory of improper integrals. Moreover, once we started examining convergence tests, the Abel-Dini Theorem gave immediately that no series could be totally effective with the comparison test. Convergence tests, in general, are proved by comparison to a convergent or divergent series. Indeed, the ratio, root, and Raabe's tests are all ways of using the comparison test indirectly, since they compare a given series to a geometric series or p-series. Since there is no perfect series for comparison, the work done here strongly suggests that there cannot be a perfect convergence test.

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