Pólya’s Random Walk Theorem

Recurrence of Simple Random Walks

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Abstract

This paper provides a brief, targeted introduction to simple random walks and relevant mathematics in order to present the reader with Pólya’s Random Walk Theorem, discuss a direct counting based proof of it along with Novak’s less direct proof, and lightly discuss the theorem.

Introduction

We start with some definitions to allow for clear discussion of the material, assuming a knowledge of material from Math 334-336. We will be discussing random walks that occur in $\mathbb{Z}^d$ for different d. We define a *simple random walk* by a process where we place a marker at a starting position (which we will define as the origin), then there are discrete increments in time, and for each time increment the marker moves one spatial increment in any direction on the $\mathbb{Z}^d$ lattice, where the marker has an equal probability in moving in any direction and the walk itself is the path taken. We will be especially interested in the cases where $d \leq 2$ and $d > 2$. We note that for we a simple random walk in $\mathbb{Z}^d$ each direction of motions has a probability of being the direction of motion of the marker for the next time increment of $\frac{1}{2d}$ and a given walk of n steps has a probability of occurring of $\frac{1}{(2d)^n}$, in particular for $d = 2$ this means each direction step has a probability of occurring of $\frac{1}{4}$ and a given walk of n steps has a probability of occurring of $\frac{1}{4^n} = \frac{1}{2^{2n}}$.

We must also consider *loops*, which will be defined as walks that starts and ends at the origin. We will call the walk of length zero, where no movement occurs, the *trivial loop*. We immediately note that a loop must have an even number of steps. We will call nontrivial loops that not the concatenation of two nontrivial loops *indecomposable*. We will say that if a random walk returns to the origin with probability one (at some point in time) it is *recurrent*, if it does not we will call it *transient*.

We will be considering multiple proofs of Pólya's Random Walk Theorem, some of which require some peripheral machinery that we will now introduce. The Borel transform is defined (acting on a function $f(z)$) by [3]

$$\beta[f(z)] = \int_0^\infty f(tz)e^{-t}dt$$

We also introduce the modified Bessel function of the first kind, a solution $F(z)$ to the differential equation [3]

$$(z^2\tfrac{d^2}{dz^2} + z\tfrac{d}{dz} - (z^2 + \alpha^2))F(z) = 0$$

We must also note Stirling's Approximation [1]

$$n! \approx \sqrt{2\pi n}e^{-n}n^n$$

We must also introduce the idea of a *generating function*. Generating functions we consider here will be, for an infinite sequence $[a_n] = (a_0, a_1, a_2, ...)$ the power series [2]

$$G(x) = a_0 + a_1x + a_2x^2 + ... = \textstyle\sum_{k=0}^\infty a_k x^k$$

Finally, we will use Abel's power series theorem so we state it here: [3] For some generating function $G(z)$ with radius of convergence 1 and real coefficients for each term. If $G(1)$ converges then

$$\lim_{z\to 1-} G(Z) = G(1)$$

We now state Pólya's Random Walk Theorem: the simple random walk in $\mathbb{Z}^d$ is recurrent for $d \leq 2$ and transient for $d > 2$ [3]. With our definitions in hand this theorem is very briefly and cleanly stated though it is certainly a very interesting, and not very intuitive, result. We will now present and discuss three different proofs. It should be noted that the proofs that will be presented and discussed all rely on some approximation or approximations and the author has seen no method of proof that does not rely on some kind of approximation.

The Direct Counting Proof

This proof is quite straightforward and uses quite an elementary approach, though it relies on Stirling's approximation. It should be noted that the the core of this proof was noted from Kozdron's lecture notes, from part 3, and leaves generalization for $d > 3$ cases to the reader [1].

Assuming all random walks here start at the origin we define u to be the probability that a random walk returns to the origin. We then see that the probability that the marker returns to the origin *exactly* m times is, in general

$$\binom{m}{m-1} u^{m-1}(1-u)^{m-(m-1)} = u^{m-1}(1-u)$$

So with the expression $u^{m-1}(1-u)$ in hand we then look at E which will be the expected number of times the marker is at the origin. We then see that we can express E, for an infinite random walk, as

$$E = \sum_{m=1}^{\infty} m(u^{m-1}(1-u))$$

$$= (1-u)\sum_{m=1}^{\infty} mu^{m-1}$$

$$= (1-u)\sum_{m=1}^{\infty} \frac{d}{du}u^{m}$$

Then we note that limit exchanging is acceptable in this case (of the differentiation and the summation) as we are just looking at a power series

$$\sum_{n=1}^{\infty} x^{n}$$

where we certainly have uniform convergence (assuming $u < 1$) so we obtain

$$= (1-u)\frac{d}{du}\sum_{m=1}^{\infty} u^{m}$$

$$= (1-u)\frac{d}{du}\frac{1}{1-u}$$

$$= \frac{1}{1-u}$$

So we see that if E is finite then the walk is transient ($u < 1$) and if $E = \infty$ then the walk is recurrent.

We then define u_n to be the probability that a given walk is at the origin on the n^{th} step, defining the value 1 for the trivial loop. We also introduce x_n, which takes the value zero if the marker is at the origin on the n^{th} step and zero otherwise. Then

$$T = \sum_{n=0}^{\infty} x_n$$

is the total number of times the marker is at the origin so E is equal to the expectation value of T which is equal to

$$\sum_{n=0}^{\infty} 1(u_n)$$

so

$$E = \sum_{n=0}^{\infty} u_n$$

But we showed previously that if E is finite then the walk is transient and if $E = \infty$ then the walk is recurrent so we see that we have established that if

$$\sum_{n=0}^{\infty} u_n$$

converges then the walk is transient and if the sum diverges then the walk is recurrent.

$\mathbb{Z}^1$ Case

Now that we have built up the necessary tools we will consider cases. We start by considering a simple random walk on $\mathbb{Z}^1$. Since, as mentioned previously, a walk must have an even number of steps to be a loop we only look at u_{2n}, $n \in \mathbb{Z}^+$. Using the probability of a given walk of n steps occurring from the introduction for $d = 1$ we see that, denoting W as the number of possible paths of length $2n$ that end at the origin, $u_{2n} = (\frac{1}{2})^{2n} W$, so

$$u_{2n} = (\tfrac{1}{2})^{2n} \tbinom{2n}{n}$$

$$u_{2n} = \tfrac{(2n)!}{n!(2n-n)!} \tfrac{1}{2^{2n}}$$

So, using Stirling's Approximation we obtain

$$u_{2n} \approx \tfrac{\sqrt{2\pi 2n} e^{-2n} (2n)^{2n}}{(\sqrt{2\pi n} e^{-n} n^n)^2 2^{2n}} = \tfrac{1}{\sqrt{\pi n}}$$

So we see

$$\sum_{n=0}^{\infty} u_{2n} \approx \sum_{n=0}^{\infty} \frac{1}{\sqrt{n\pi}}$$

And the series on the right certainly diverges so we see that a simple random walk in $\mathbb{Z}^1$ is recurrent since E diverges, the first part of the theorem.

$\mathbb{Z}^2$ Case

Again, we first note that to form a loop a walk must have equal steps "forwards" and "backwards" in each direction and each path of $2n$ steps has a probability of occurring of $\frac{1}{4^{2n}}$. We then consider that the number of paths with equal steps left and right (say L steps in each "horizontal" direction) and equal steps up and down (then $n - L$) is

$\binom{2n}{L,L,n-L,n-L}$ will represent the total number of loops of length $2n$,

$= \frac{(2n)!}{L!L!(n-L)!(n-L)!}$

So we get that

$$u_{2n} = (\frac{1}{4})^{2n} \sum_{L=0}^{n} \frac{(2n)!}{L!L!(n-L)!(n-L)!}$$

$$= (\frac{1}{4})^{2n} \sum_{L=0}^{n} \frac{(2n)!n!n!}{L!L!(n-L)!(n-L)!n!n!}$$

$$= (\frac{1}{4})^{2n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{L}^2 = (\frac{1}{4})^{2n} \binom{2n}{n} \binom{2n}{n}$$

So we get

$$u_{2n} = (\tfrac{1}{2^{2n}} \tbinom{2n}{n})^2$$

So we have that it is just the square of the result from $\mathbb{Z}^1$ so we see that in this case

$$\sum_{n=0}^{\infty} u_{2n} \approx \sum_{n=0}^{\infty} \frac{1}{\pi n}$$

And the series on the right certainly diverges so we see that a simple random walk in $\mathbb{Z}^2$ is recurrent since E diverges, the second part of the theorem.

$\mathbb{Z}^3$ Case

As before in order to have a walk return to the origin we must have equal steps in the positive and negative directions for each direction and each path of $2n$ steps has a probability of occurring of $\frac{1}{6^{2n}}$. Then, extending the idea from the previous subsection, the number of paths (of total length $2n$) with L steps "left" and "right", U steps "up" and "down", and $n-L-U$ steps "forward" and "backward" is

$\binom{2n}{L,L,U,U,n-U-L,n-U-L}$ which is equivalent $\frac{(2n)!}{L!L!U!U!(n-U-L)!(n-U-L)!}$

So we get

$$u_{2n} = (\frac{1}{6})^{2n} \sum_{L,U,L+U\leq n} \frac{(2n)!}{L!L!U!U!(n-U-L)!(n-U-L)!}$$

$$= (\frac{1}{2})^{2n} \binom{2n}{n} \sum_{L,U,L+U\leq n} (\frac{n!}{3^n U!L!(n-U-L)!})^2$$

And we have that

$$\frac{n!}{3^n U!L!(n-U-L)!} = (\frac{1}{3})^n \binom{n}{L,U,n-L-U} \leq \frac{n!}{3^n \lfloor\frac{n}{3}\rfloor!\lfloor\frac{n}{3}\rfloor!\lfloor\frac{n}{3}\rfloor!}$$

The above inequality comes from the idea of treating the statement on the left side of the inequality as a probability of placing n objects in 3 places and recognizing that this probability is then maximized when U, L and $n-U-L$ are all as close to $\frac{n}{3}$ as possible. Note that thinking this way also gives us that

$$\sum_{U,L} \frac{n!}{3^n U!L!(n-U-L)!} = 1$$

, since we must always have the sum over all the probability values of something sum to unity. We then see that

$$u_{2n} \leq (\frac{1}{2})^{2n} \binom{2n}{n} \frac{n!}{3^n \lfloor\frac{n}{3}\rfloor!\lfloor\frac{n}{3}\rfloor!\lfloor\frac{n}{3}\rfloor!} = (\frac{1}{2})^{2n} \binom{2n}{n} \frac{n!}{3^n \lfloor\frac{n}{3}\rfloor!^3}$$

Then, considering the powers of factorials and that $n > 0$ (so we can obtain a larger upper bound on u_{2n} by removing the fractions in the above inequality) and using Stirling's approximation we get the large upper bound

$$u_{2n} \leq \frac{M}{n^{\frac{3}{2}}}$$

Where M is some positive constant. We then see

$$\sum_n u_{2n} \leq M \sum_n \frac{1}{n^{\frac{3}{2}}}$$

But the sum on the right of the inequality converges (we see this by recognizing it as a p-series with $p > 1$) so we have that a simple random walk in three dimensions is transient, as we claimed. The ideas used here for ^{3} can be extended for $n > 3$ and in doing so we see that for $n \geq 3$ the simple random walk in ^{n} is transient, as claimed, and this proof of Pólya's Random Walk Theorem is complete.

Discussion of the Direct Counting Proof

This proof is valuable because it provides a very direct method of verification. This method can also be a bit frustrating, however, as it utilizes a decent amount of probability, that being its focus, potentially making the proof less accessible for those of different backgrounds. This focus on probabilistic mathematics in the method of proof also limits the insight techniques, the approaches of the proof, to mostly mathematics more focused on probability. While one may expect this, we will consider the next proof which takes a decently different approach.

Novak's Proof

Finding a Limit which Determines Recurrence

It should be noted that this proof comes from Novak [3]. We start by defining E to be the event that the walk returns to the origin and p to be the probability of the event E occurring. We also define E_n to be the event that the walk returns to the origin for the first time specifically after $n > 0$ steps. Since we do not consider the trivial loop from the start of the random walk as recurrence we will define $E_0 = 0$. We immediately see that E is the union of all of the E_ns and so, defining p_n to be the probability of the event E_n occurring,

$$p = \sum_{n \geq 0} p_n$$

Now, in $\mathbb{Z}^d$, recalling that our random walks all start at the origin, call l_n the number of loops at the origin of length n and i_n the number of indecomposable loops at the origin of length n. Because the only possibilities we have for "decomposable" loops (those that are not indecomposable) are those where an indecomposable loop is followed by (possibly) another loop we see that we have the relation, for all $n \geq 1$,

$$l_n = \sum_{k=0}^{n} i_k l_{n-k}$$

Recalling that the total number of length n walks is $(2d)^n$ we divide both sides of the above equality by $(2d)^n$ and, defining q_n to be the probability that the random walk reaches the origin (not necessarily for the first time) after n steps, obtain the relation

$$q_n = \sum_{k=0}^{n} p_k q_{n-k}$$

We now define the generating functions $P(z)$ and $Q(z)$

$$P(z) = \sum_{n=0}^{\infty} p_n z^n$$

$$Q(z) = \sum_{n=0}^{\infty} q_n z^n$$

The relation between p_n and q_n then gives us, in the algebra of formal power series [3]

$$P(z)Q(z) = Q(z) - 1$$

Then, since we have $Q(z)$ not being zero for $z \in [0, 1)$ we have [3]

$$P(z) = 1 - \frac{1}{Q(z)}, z \in [0, 1)$$

Then, because we have

$$P(1) = \sum_{n=0}^{\infty} p_n = p$$

and $P(z)$ has radius of convergence one, we get, by Abel's power series theorem,

$$p = \lim_{z \to 1-} P(z) = 1 - \frac{1}{\lim_{z \to 1-} Q(z)}, z \in [0, 1)$$

We then note that the limit in the denominator at right either goes to positive infinity (in which case we have $p = 1$ so we have recurrence of the random walk) or to some positive real number (in which case we have $p < 1$ so we have transience of the random walk). Analysis of this limit will then be our guide for whether we have recurrence or transience of a simple random walk on a d dimensional lattice.

Analyzing $Q(z)$

We of course now want to analyze the limit from the previous section so we start by attempting to find a usable expression for $Q(z)$. This can become a problem of finding an expression for the generating function $L(z)$ defined by

$$L(z) = \sum_{n=0}^{\infty} l_n z^n$$

Since we then see that $Q(z) = L(\frac{z}{2d})$.

We now, in an attempt to better understand $L(z)$, we consider the exponential loop generating function $E(z)$ defined by [3]

$$E(z) = \sum_{n=0}^{\infty} l_n \frac{z^n}{n!}$$

Now, considering the random walk on the $\mathbb{Z}^2$ lattice, we recall that a loop of length n on this lattice is made up of k "horizontal" steps and $n-k$ "vertical" steps where the k "horizontal" steps form a loop of length k on $\mathbb{Z}$ while the $n-k$ "vertical" steps form a loop of length $n-k$ on $\mathbb{Z}$. In light of this we see that the number of loops of length n on the $\mathbb{Z}^2$ lattice with the above stated "horizontal" and "vertical" steps is given by

$$\binom{n}{k} l_k l_{n-k}$$

where l_k and l_{n-k} are the number of loops of lengths k and $n-k$ respectively (in a one dimensional lattice). This then gives us an expression for the total number of loops of length n on the $\mathbb{Z}^2$ lattice (where the l_n at left is the number of loops of length n on the lattice of dimension 2 while the l_k and l_{n-k} at right are the number of loops of their respective lengths on the lattice of dimension 1):

$$l_n = \sum_{k=0}^{\infty} \binom{n}{k} l_k l_{n-k}$$

Using $E_1(z)$ to denote the exponential generating function from above specifically when we look at loops on the lattice of dimension one while using $E_2(z)$ to denote the exponential generating function from above specifically when we look at loops on the lattice of dimension two we then see that the above relation gives us [3]

$$E_2(z) = E_1(z)^2$$

And, in general (from using the same logic) that, using $E_d(z)$ to denote the exponential generating function from above when we look at loops on the lattice of dimension d

$$E_d(z) = E_1(z)^d$$

Now, when $d = 1$, we see that any loop must have m steps in the positive direction and m in the negative direction (for some $m > 0$) so the number of loops of length $n = 2m$ in one dimension is $\binom{2m}{m}$ so we have that

$$E_1(z) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!k!}$$

Now, with our expression for $E_1(z)$ in hand, if we stared at it for a while, we may (or may not) notice that $E_1(z)$ is a modified Bessel function of the first kind [3]. The modified Bessel function of the first kind, $I_\alpha(z)$ has a series representation given by

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{2k+\alpha}}{k!\Gamma(k+\alpha+1)}$$

From this representation we see then that $E_1(z) = I_0(2z)$ so we have, referring to our above relation of exponential generating loop functions in different dimensions

$$E(z) = I_0(2z)^d$$

We now wish to return with our gained tools and representations to better understand $L(z)$. We can do this using the Borel transformation β (which converts exponential generating functions into normal generating functions), defined by [3]

$$\beta f(z) = \int_0^\infty f(tz)e^{-t}dt$$

We can then obtain a representation for $L(z)$ as desired:

$$L(z) = \beta E(z) = \beta I_0(2z)^d = \int_0^\infty I_0(2tz)^d e^{-t}dt$$

Then, recalling that $Q(z) = L(\frac{z}{2d})$ we have obtained a new representation of $Q(z)$:

$$Q(z) = \int_0^\infty I_0(\frac{tz}{d})^d e^{-t}dt$$

Determining Transience or Recurrence

As a brief reminder: we have transience when $\lim_{z\to 1-} Q(z) < \infty$ and recurrence when $\lim_{z\to 1-} Q(z) = \infty$. We will now look at the integral representation of $Q(z)$ we obtained above to see when it converges and when it does not (which tells us when we have recurrence versus when we have transience). We need only look at the tail integral of the integral representation of $Q(z)$ to find when it converges or diverges, which is (for K very large)

$$\int_K^\infty I_0(\frac{tz}{d})^d e^{-t}dt$$

Looking at the integrand and defining $f(\theta) = \frac{z}{d}cos(\theta)$ we have [3]

$$I_0(\frac{tz}{d}) = \frac{1}{\pi}\int_0^{\pi} e^{tf(\theta)}d\theta$$

Considering the Taylor approximation of $f(\theta)$, we have $f(\theta) \approx f(0) - |f''(0)|\frac{\theta^2}{2}$ so

$$\int_0^{\pi} e^{tf(\theta)}d\theta \approx e^{tf(0)}\int_0^{\pi} e^{\frac{-t|f''(0)|\theta^2}{2}}$$

Extending the integral on the right above over the positive reals we get half of a Gaussian integral which we can compute exactly as one would expect, so we should have

$$\int_0^{\pi} e^{tf(\theta)}d\theta \approx e^{tf(0)}\sqrt{\frac{\pi}{2t|f''(0)|}}$$

Where the accuracy of the above approximation increases as $t \to \infty$ since the error of extending the integral from earlier to a half Gaussian integral is decaying [3]. We then have that (for some constant C), putting pieces together [3]

$$I_0(\frac{tz}{d})^d \approx Ce^{tz-t}(tz)^{\frac{-d}{2}}$$

And, using the monotone convergence theorem and considering that the integral of the tail of the above approximation converges uniformly (where it does converge) we have, for $z \in [0, 1)$

$$lim_{z\to 1-}\int_K^{\infty} e^{tz-t}(tz)^{\frac{-d}{2}}dt = \int_K^{\infty} lim_{z\to 1-}e^{tz-t}(tz)^{\frac{-d}{2}}dt = \int_K^{\infty} t^{\frac{-d}{2}}dt$$

So we see that we have recurrence of the simple random walk when the above integral diverges and transience when the above integral converges. We then see, by the equivalent of the p test for integrals, that the above integral diverges for $d = 1, 2$ and converges for $d \geq 3$, so Pólya's Random Walk Theorem is proven.

Discussion of Novak's Proof

This proof is valuable because it utilizes much less direct, potentially less obvious, techniques than the first proof presented. This proof in a sense translates a probabilistic problem into another mathematical dialect, something which can prove very useful in more general settings. Specifically the ideas of using generating functions and different transforms may prove useful for other random walk problems. It should be noted that there are certainly some techniques utilized that would most likely not be useful for other problems, such as recognizing a function as a modified Bessel function of the first kind to obtain an integral representation.

Conclusion

Pólya's Random Walk Theorem is a particularly interesting, and some may say beautiful, result but it also opens the door for much more discussion of random walks. Since this result is such a basic one of random walks any techniques for its proof certainly have great potential for use in probing further into the study of random walks, which is, again, what makes Novak's proof so valuable as it demonstrates how, and in some instances potentially what, different, less direct tools can be used in pursuit of the subject. It should be noted that just how different Novak's proof is becomes especially evident when compared with the direct counting proof discussed above.

References

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3. J. Novak. Pólya's Random Walk Theorem. *American Mathematical Monthly,* 121:711-716, October 2014.