Introduction to Fuzzy Set Theory and The Hyperplane Separation Theorem of Fuzzy Convex Sets

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Abstract

A fuzzy set is a class of mathematical objects in which membership is continuous. Unlike classical set theory, membership is no longer crisp, but is characterized by a membership (characteristic) function which assigns an element to a grade of membership ranging in a finite nonnegative interval. The analogous notion of inclusion, union, intersection, complement, convex sets, etc are defined to fuzzy sets and with these notions, various properties of fuzzy sets are proved. These properties will lead to the showcase of the hyperplane separation theorem for fuzzy convex sets without having the sets be disjoint.

Contents

1 Introduction to the Developments and Ideas in Fuzzy Set Theory 3

2 Basic Definitions and Operations of Fuzzy Sets 5
2.1 The Mathematical Definition of a Fuzzy Set 5
2.2 Inclusion, the Empty Fuzzy Set, and Fuzzy Set Equality . . . 7
2.3 Containment, Union, Intersection, and Complement 8

3 Properties of Union, Intersection, and Complements 10
3.1 Machinery of Fuzzy Set Operations 10

4 Algebraic Operations and Relations on Fuzzy Sets 12
4.1 Fuzzy Algebraic Operations 12
4.2 Fuzzy Relations and Sets Constructed by Mappings 14

5 Convex Fuzzy Sets and Its Properties 16
5.1 Basic Definitions of Convex Fuzzy Sets and Boundeness 16
5.2 Properties of Convex and Bounded Fuzzy Sets 18

6 The Hyperplane Separation Theorem of Fuzzy Convex Sets 20
6.1 Preliminary Ideas and Definitions 20
6.2 Proof of The Hyperplane Separation Theorem of Fuzzy Convex Sets 22

7 Acknowledgements 25

References 26

1 Introduction to the Developments and Ideas in Fuzzy Set Theory

> "As the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics."
>
> -L. Zadeh [5]

Most of our traditional tools of developing formal models, logic, and reasoning are considered crisp. We define crisp as being bivalent, meaning a yes or no relationship. In traditional formal logic, a statement is true or false – and nothing in between. This notion is then extended onto classical set theory where the sets are crisp. An element is included in a set or not – there is no ambiguity. This precision supposes that the model can represent exactly the real system that it is intends to model which further implies that the model is unequivocal. This certainty eventually indicates that we are able to suppose the foundations of a model to be completely known and have no doubts on their occurrence. However, the real world gives evidence that these assumptions are not justified. A complete model of a real system often requires far more detailed data than we could ever recognize simultaneously, process, and understand.

We can see since the introduction of classical axiomatic set theory in the early 20th century, the field has been extensively used to advance many disciplines found in mathematics, formal logic, and computer science. However, it is very evident that mathematical developments and research has reached to a very high standard and are still climbing to this day. In this review, the basic mathematical framework of fuzzy set theory will be established in L.A. Zadeh's, *Fuzzy Sets* [4], will be described. Such frameworks will include the definitions of inclusion, union, intersection, complement, convex sets, relation, etc. We will further state and prove properties of fuzzy sets analogous to classical set theory that will be preliminary to the main result of this paper: The showcase and proof of the hyperplane separation theorem for fuzzy convex sets without the sets being disjoint (the significance of the sets not being disjoint will be explained in its respective section). The foundations for each proof are given by Zadeh but I will provide my own input and style in the execution. If the proofs presented do not make sense, please read Zadeh's paper [4]. Even though there is a lot of machinery behind the nature of fuzzy sets, it has proven to be very applicable in the world of engineering. Recent developments in the 1970s have allowed fuzzy set theory to take rise in solving problems in modern day engineering such as data mining, optimization, systems and control, etc. However, because of the limitations of this paper and my own knowledge, we will not go in-depth into these applications but please read Zimmermann's paper, *Fuzzy Set Theory* [6], to learn more about the applications of fuzzy set theory.

2 Basic Definitions and Operations of Fuzzy Sets

> "All traditional logic habitually assumes that precise symbols are being employed. It is, therefore, not applicable to this terrestrial life but only to an imagined celestial existence."
>
> -Bertrand Russell [2]

2.1 The Mathematical Definition of a Fuzzy Set

We will begin with the basic mathematical definition of a fuzzy set and its variations given by different prespectives:

Definition. (Fuzzy Set [Zimmerman's Definition] [6]) *If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs:*

$$A = \{(x, \mu_A(x)) : x \in X\}$$

where $\mu_A(x)$ is the membership function which maps X to the membership space $M \subseteq \mathbb{R}$. The range of the membership function is a subset of nonnaga-tive real numbers whose supremum is finite.

A **normalized fuzzy set** is a fuzzy set in which $\sup \mu_A(x) = 1$. Zadeh has his own definition of a fuzzy sets in which it is normalized but extends the normality of the fuzzy set by making $\in \mu_A(x) = 0$ as well.

Definition. (Fuzzy Set [Zadeh's Definition] [4]) *If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs:*

$$A = \{(x, \mu_A(x)) : x \in X\} \tag{1}$$

where $\mu_A(x)$ is the membership function which maps X to a real number in the interval $[0, 1]$.

For simplicity sake of arguments presented later on in this paper, we will take Zadeh's definition (1) of a fuzzy set as our definition and work with it exclusively, unless otherwise noted. With this definition, we will proceed to give an example to support this concept.

Example. Let X be the real number line $\mathbb{R}$ and let $A = \{\mathbb{R}, \mu_A(x) : x \in \mathbb{R}\}$ be a fuzzy set of numbers. Then we can give a characterization of A by specifying $\mu_A(x)$ as a function of $\mathbb{R}$. We can give explicit values for μ_A such as: $\mu_A(1000) = 0.2$, $\mu_A(45) = 0.52$, $\mu_A(928) = \frac{1}{\pi}$, etc.

Also with this definition, we can establish the definition of a set found in classical set theory and we will call it a **classic crisp set**.

Definition. (Classic Crisp Set) *A fuzzy set* $A = \{(X, \mu_A(x) : x \in X\}$ *where the membership function outputs only 0 or 1, exclusively. So for all* $x \in X$*,*

$$\mu_A(x) = 1 \quad or \quad \mu_A(x) = 0$$

where the 'or' is an exclusive disjunction.

It should be noted that the careful reader might see that the membership function of a fuzzy set has strong resemblance to a **probabilistic set** which is defined as

Definition. (Probabilistic Set [3]) *A probabilistic set* $A = \{X, \mu_A(x, \omega)\}$ *on* X *is defined by the defining function*

$$\mu_A : X \times \Omega \ni (x, \omega) \to \mu_A(x, \omega) \in \Omega_C$$

where $(\Omega_C, B_C) = [0, 1]$ *are Borel sets* [1]

However, even with these similarities, there are many essential differences between these topics which should be clear once more properties of the membership function has been established. Surprisingly, the concept of a fuzzy set is completely nonstatistical in nature. With that in mind, we will now establish several definitions of fuzzy sets that are analogous to their corresponding definitions found in classical set theory.

[1]a Borel set is any set in a topological space that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and relative complement. For a topological space X, the collection of all Borel sets on X forms a σ-algebra. The σ-algebra on X is the smallest σ-algebra containing all open sets.

2.2 Inclusion, the Empty Fuzzy Set, and Fuzzy Set Equality

From this section and onwards, if A is a fuzzy set, A will always be in X and the membership function will always be denotes $\mu_A(x)$ for $x \in X$. Furthermore, any relation with μ_A will imply that the relation holds for μ_A for all $x \in X$.

Definition. *Let X be a collection of objects and $\mu_A(x) : X \to [0,1]$ be a membership function. Now let $x \in X$ and construct the fuzzy set $A = (X, \mu_A(x))$. Then x is called*
not included *in the fuzzy set A if $\mu_A(x) = 0$*
fully included *in the fuzzy set A if $\mu_A(x) = 1$*
partially included *in the fuzzy set A if $0 < \mu_A(x) < 1$.*

Now the definitions of the **empty fuzzy set** and **fuzzy set equality** should be easily seen as

Definition. (Empty Fuzzy Set) *A fuzzy set A is empty if and only if its membership function is identically zero on X, $\mu_A \equiv 0$.*

Definition. (Fuzzy Set Equality) *Two fuzzy sets A and B are equal if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$. We will denote equality as $A = B$.*

With these definitions, we are now able to establish the notion of containment, union, and intersection.

2.3 Containment, Union, Intersection, and Complement

The notion of **containment** or subset can be defined similarly as fuzzy set equality as

Definition. (Containment) The fuzzy set A is contained in the fuzzy set B (or A is a subset of B) if and only if $\mu_A \leq \mu_B$. We will denote subset between two fuzzy sets as $A \subseteq B$.

From this definition of containment, it can easily be seen that for two fuzzy sets A and B that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. Furthermore, the notion of a **proper subset** will be the same as the definition of subset but with $\mu_A < \mu_B$ and be denoted as $A \subset B$.

Note that unlike classical set theory, the intuitive idea of "belonging" to a set does not play a fundamental role in fuzzy sets. It is not meaningful to talk about a point x "belonging" to a fuzzy set A, but depends on its membership function. The idea of a point "belonging" to a fuzzy set can be defined by introducing levels and level sets (introduced in Section 3). With that in mind, we see the pattern that operations defined on fuzzy sets depend entirely on the membership function. With this idea in mind, we are able to construct analogous definitions for **unions**, **intersection**, and **complement** of fuzzy sets using the membership function.

Definition. (Union) *The union of two fuzzy sets A and B with membership functions μ_A and μ_B, respectively, is a fuzzy set C denoted as $C = A \cup B$, whose membership function μ_C is defined as*

$$\mu_C(x) = \max(\mu_A(x), \mu_B(x)). \tag{2}$$

We will shorthand this notation by

$$\mu_C = \mu_A \vee \mu_B.$$

Definition. (Intersection) *The intersection of two fuzzy sets A and B with membership functions μ_A and μ_B, respsectively, is a fuzzy set C denoted as $C = A \cup B$, whose membership function μ_C is defined as*

$$\mu_C(x) = \min(\mu_A(x), \mu_B(x)).$$

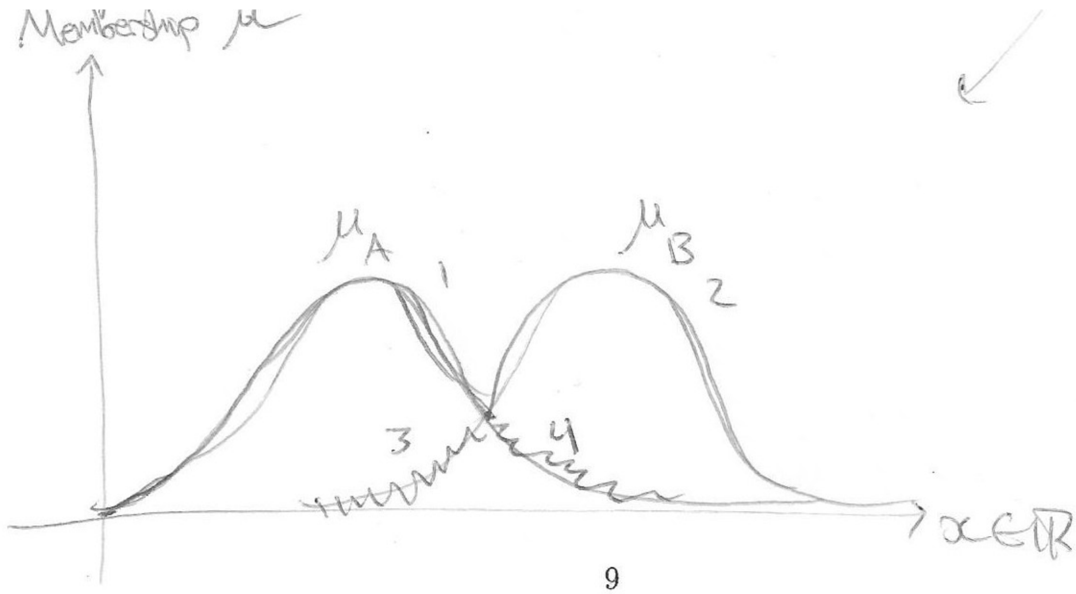

Figure 1: Graphical interpretation of fuzzy union and intersection.

We will shorthand this notation by

$$\mu_C = \mu_A \wedge \mu_B.$$

By the definition, fuzzy sets are closed by union and intersection. Furthermore, it is easily seen that the union and intersection of fuzzy sets is commutative and associative. To get a better understanding of fuzzy union and intersection, we can look at **Figure 1.** where the graph represents the union and intersection of fuzzy sets in $\mathbb{R}$. The curves represent the value of the membership function at the respective $x \in \mathbb{R}$. Segments 1 and 2 compromises the union while segments 3 and 4 compromises the intersection. Finally, we define the **complement** of a fuzzy set as

Definition. (Complement) *The complement of a fuzzy set A with membership function μ_A is a fuzzy set denoted A^c whose membership function μ_{A^c} is defined as*

$$\mu_{A^c} = 1 - \mu_A.$$

3 Properties of Union, Intersection, and Complements

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

-Albert Einstein

3.1 Machinery of Fuzzy Set Operations

Just like classical set theory, De Morgan's laws and the distributive laws hold. With the definitions for the operations, it is easy enough to extend these notions to fuzzy set theory by working with the membership function.

Theorem. *(De Morgan's Laws) Let A and B be fuzzy sets. Then $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.*

Proof. To show the first equality, it suffices to show that

$$1 - \mu_A \vee \mu_B = \mu_A \wedge \mu_B$$

which can easily be shown by proving it for the two possible cases: $\mu_A > \mu_B$ or $\mu_B > \mu_A$. The second equality is shown using a similar argument. The rest of the proof is left for the reader. □

Theorem. *(Distributive Laws) Let A, B, and C be fuzzy sets. Then $C \cup (A \cap B) = (C \cup A) \cap (C \cup B)$ and $C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$.*

Proof. To show the first equality, it suffices to show that

$$\mu_C \vee (\mu_A \wedge \mu_B) = (\mu_C \vee \mu_A) \wedge (\mu_C \vee \mu_B)$$

which can be easily (but also tediously) shown by considering the six cases:

$$\begin{array}{lll} \mu_A > \mu_B > \mu_C, & \mu_A > \mu_C > \mu_B, & \mu_B > \mu_A > \mu_C, \\ \mu_B > \mu_C > \mu_A, & \mu_C > \mu_A > \mu_B, & \mu_C > \mu_B > \mu_A. \end{array}$$

The second equality is shown using a similar argument. The rest of the proof is left for the reader. □

A more interesting property that is similar to classical set theory is presenting a more appealing way of understanding the union.

Theorem. *Let A and B be fuzzy sets. Then the union of A and B is the smallest fuzzy set containing both A and B.*

Proof. It suffices to show that if D is a fuzzy set that contains both A and B, then $C = A \cup B$ is also contained in D. So let $C = A \cup B$ and suppose D is a fuzzy set that contains both A and B. By the definition of union, it follows that

$$\mu_A \vee \mu_B \geq \mu_A \tag{3}$$

and

$$\mu_A \vee \mu_B \geq \mu_B. \tag{4}$$

Since D contains both A and B, by the definition of containment, it follows that

$$\mu_D \geq \mu_A \text{ and } \mu_D \geq \mu_B.$$

Then from (3) and (4), we see that

$$\mu_D \geq \mu_A \vee \mu_B.$$

Therefore, by the definition of containment, $C = A \cup B \subseteq D$. Hence the desired result is obtained. $\square$

4 Algebraic Operations and Relations on Fuzzy Sets

"Either mathematics is too big for the human mind or the human mind is more than a machine."

-Kurt Godel

4.1 Fuzzy Algebraic Operations

Just like classical set theory, we are able to define other ways of combining fuzzy sets and relating them to one another. In this section, we shall define some of the important and interesting algebraic operations.

Definition. (Algebraic Product) *The **algebraic product** of fuzzy sets A and B, denoted AB, is defined by the membership functions of A and B as follows:*

$$\mu_{AB} = \mu_A \mu_B.$$

With the definition above, it follows easily for fuzzy sets A and B that

$$AB \subseteq A \cap B.$$

Definition. (Algebraic Sum) *The **algebraic sum** of fuzzy sets A and B, denoted $A+B$, is defined by the membership functions of A and B as follows:*

$$\mu_{A+B} = \mu_A + \mu_B$$

provided that $\mu_A + \mu_B \leq 1$.

Unlike the algebraic product, the algebraic sum of two fuzzy sets is meaningful only when $\mu_A + \mu_B \leq 1$ is satisfied. Furthermore, it can easily be seen that both the algebraic sum and product are commutative, associative, and distributive.

Definition. (Absolute Difference) *The **Absolute difference** of fuzzy sets A and B, denoted $|A - B|$, is defined by*

$$\mu_{|A-B|} = |\mu_A - \mu_B|.$$

Now if we have two arbitrary vectors x and y, a *convex combination* is a linear combination of x and y of the form

$$xt + (1-t)y \quad 0 \leq t \leq 1.$$

Now the same notion can be applied to fuzzy sets with our definitions of algebraic sum and product.

Definition. (Convex Combination) *Let $A, B,$ and Γ be fuzzy sets. The **Convex combination** of A, B , and Γ, denoted $(A, B; \Gamma)$, is defined by*

$$(A, B; \Gamma) = A\Gamma + \Gamma^C B$$

so the membership function of $(A, B; \Gamma)$ is

$$\mu_{(A,B;\Gamma)} = \mu_A \mu_\Gamma + (1 - \mu_\Gamma)\mu_B.$$

4.2 Fuzzy Relations and Sets Constructed by Mappings

Just as in classical set theory, the idea of a relation plays an important role in the theory and their applications. We will see that the notion of a relation in classical set theory has a very natural extension to fuzzy sets, but first we will establish what a relation is classic set theory.

Classically, a relation is defined as a set of ordered pairs in which the first and second component of the ordered pair satisfies a certain condition. For example, there is the relation R composed of all ordered pairs of real numbers x and y such that $x < y$. Using mathematical notation, we can define the relation R as

$$R = \{(x, y) : x \in \mathbb{R} \wedge y \in \mathbb{R} \wedge x < y\}$$

In the context of fuzzy sets, we can define Γ as a fuzzy relation in a fuzzy set Λ that is contained in the product space $\Lambda \times \Lambda$. Defined more generally,

Definition. (n-ary fuzzy relation) *A* ***n-ary fuzzy relation****, Γ in a fuzzy set Λ is in the product space $\Lambda \times \Lambda \times \cdots$ where the membership function is in the form $\mu_\Gamma(x_1, x_2, \ldots, x_n)$ for $x_i \in \Lambda$; $1 \leq i \leq n$.*

A relation can be thought of as a generalized notion of a mapping from one set to another. Now one can wonder what happens to fuzzy sets from mappings.

Definition. (Fuzzy Sets Constructed by Mappings 1) *Let $T : \Gamma \to \Omega$ be a mapping from some space Γ to some other space Ω. Now let $B \subseteq \Omega$ be a fuzzy set contained in Ω with a membership function $\mu_B(\omega)$. Then the inverse mapping T^{-1} constructs a fuzzy set $A \subseteq \Gamma$ contained in Γ whose membership function is defined as*

$$\mu_A(\gamma) = \mu_B(\omega) \qquad \omega \in \Omega$$

for all $\gamma \in \Gamma$ which are mapped by T into ω.

Conversely, we can consider what happens if $A \subseteq \Gamma$ is a fuzzy set contained in Γ and we have the same $T : \Gamma \to \Omega$ from the definition map from Γ to Ω. We need to determine the membership function for the fuzzy set $B \subseteq \Omega$ induced by the mapping T. One problem is that if T is not one-to-one, then there is an ambiguity when two or more distinct points, $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, with different values of membership in Γ is mapped to the same point $\omega \in \Omega$. In this case, we need to determine what the membership value for ω in Ω has to be. In this case, we shall agree to assign the bigger of the two values of membership to ω which gives us the definition:

Definition. (Fuzzy Sets Constructed by Mappings 2) *Let $T : \Gamma \to \Omega$ be a mapping from some space Γ to some other space Ω. Now let $A \subseteq \Gamma$ be a fuzzy set contained in Ω with a membership function $\mu_A(\gamma)$. Then the mapping T constructs a fuzzy set $B \subseteq \Omega$ contained in Ω whose membership function is defined as*

$$\mu_B(\omega) = \max_{\gamma \in range(T^{-1})} \mu_A(\gamma).$$

5 Convex Fuzzy Sets and Its Properties

"A set is a Many that allows itself to be thought of as a One."

-Georg Cantor

As we will define, it turns out that the analogous notion of a convex set can readily be extended to the context of fuzzy sets in such a way to preserve many of the properties that classical convex sets hold.

5.1 Basic Definitions of Convex Fuzzy Sets and Boundeness

Throughout this section, to deal with ambiguity, we will assume that the fuzzy set X is a real Euclidean space $\mathbb{E}^n$ [2]. With that in mind, we will now introduce the definition of a convex fuzzy set.

Definition. (Convex Fuzzy Set) *A fuzzy set A is **convex** if and only if the sets Γ_α defined by*

$$\Gamma_\alpha = \{x : \mu_A(x) \geq \alpha\}$$

are convex (in classical set theory) for all α in the interval $[0, 1]$.

Definition. (Strictly Convex Fuzzy Set) *A fuzzy set A is **strictly convex** if the sets Γ_α for $0 < \alpha \leq 1$ are strictly convex (the midpoint of any two distinct points in Γ_α lies in the inferior of Γ_α).*

Definition. (Strongly Convex Fuzzy Set) *A fuzzy set A is **strongly convex** if for any two distinct points x_1 and x_2 and $\lambda \in (0, 1)$, we have*

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) > \min(\mu_A(x_1), \mu_A(x_2)).$$

We note that the last two definitions are independent from each other. That is, a fuzzy set begin strictly convex does not imply that it is strongly convex or vice-versa. Now we will introduce the core of fuzzy sets.

[2] Euclidean n-space, sometimes called Cartesian space or simply n-space, is the space of all n-tuples of real numbers, $(x_1, x_2, ..., x_n)$. Such n-tuples are sometimes called points

Definition. (Essentially Attained Point) *Let A be a fuzzy convex set and let $M = \sup_{x \in A} \mu_A(x)$. If A is bounded, there is at least point x_0 at which M is* ***essentially attained*** *where for each $\epsilon > 0$, any ball about x_0 contains points in the set $Q(\epsilon) = \{x : M - \mu_A(x) \leq \epsilon\}$.*

Definition. (Core of a Fuzzy Set) *The* ***core*** *of a fuzzy set A is the set of all points X at which M is essentially attained and is denoted $C(A)$.*

As we will see in the next definition, the set Γ_α constructed in the definition above plays the foundational role in determining what property a fuzzy set has. With that, we will introduce the definition of a bounded fuzzy set.

Definition. (Bounded Fuzzy Set) *A fuzzy set A is bounded if and only if the set $\Gamma_\alpha = \{x : \mu_A(x) \geq \alpha\}$ are bounded (in classical set theory) for all $\alpha > 0$; that is, for every $\alpha > 0$ there exists a finite bound $R(a)$ such that $|x| \leq R(a)$ for all $x \in \Gamma_\alpha$.*

With these definitions, we will now begin to prove basic properties of convex/bounded fuzzy sets and see the strong resemblance to classical set theory.

5.2 Properties of Convex and Bounded Fuzzy Sets

We start by introducing a more intuitive definition of a convex fuzzy set and showing that these two definitions are equivalent.

Theorem. *A fuzzy set A is **convex** if and only if*

$$\mu_A(x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$.

Proof. Let A be a fuzzy set.
($\Longrightarrow$) If A is convex and use the first definition and let $\alpha = \mu_A(x_1) \leq \mu_(x_2)$, then $x_2 \in \Gamma_\alpha$ and $\lambda x_2 + (1-\lambda)x_2 \in \Gamma_\alpha$ since Γ_α is convex. Therefore, we have that

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha = mu_A(x_1) = \min(\mu_A(x_1), \mu_A(x_2)).$$

($\Longleftarrow$) If $\mu_A(x_1 + (1-)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$ and $\alpha = mu_A(x_1)$, then Γ_α is the set of all points x_2 such that $\mu_A(x_2) \geq \mu_A(x_1)$. Then we see that $\lambda x_1 + (1-\lambda)x_2$ where $0 \leq \lambda \leq 1$ is also in Γ_α. Therefore, Γ_α is convex (in notion of classical set theory). □

Now we will prove a theorem that analogously the same in the notion of classical set theory.

Theorem. *If A and B are convex, then $A \cap B$ is convex.*

Proof. Let $C = A \cap B$. Then we have that for $\lambda \in [0,1]$

$$\mu_C(\lambda x_1 + (1-\lambda)x_2) = \min(\mu_A(\lambda x_1 + (1-\lambda)x_2), \mu_B(\lambda x_1 + (1-\lambda)x_2)).$$

By assumption, we have that A and B are convex so by the previous theorem, we have that

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$
$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_B(x_1), \mu_B(x_2))$$

so by the definition of fuzzy intersection, we have

$$\mu_C(\lambda x_1 + (1-\lambda)x_2) \geq \min(\min(\mu_A(x_1), \mu_A(x_2)), \min(\mu_B(x_1), \mu_B(x_2))).$$

Rearranging the right hand side of the equation, we get that

$$\mu_C(\lambda x_1 + (1-\lambda)x_2) \geq \min(\min(\mu_A(x_1), \mu_B(x_1)), \min(\mu_A(x_2), \mu_B(x_2)))$$

so

$$\mu_C(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_C(x_1), \mu_c(x_2)).$$

Hence by the previous theorem, C is fuzzy convex. □

Theorem. *If A and B are bounded fuzzy sets, then so is their intersection.*

Proof. The proof is left for the reader. □

Theorem. *If A and B are strictly or strongly convex, then so is their intersection.*

Proof. The proof is left for the reader. □

Theorem. *If A is a convex fuzzy set, then its core $C(A)$ is a convex set*

Proof. It suffices to show that if M is essentially attained at x_0 and x_1 where $x_0 \neq x_1$, then it is also essentially attained at all x where $x = \lambda x_0 + (1-)x_1$ for $\lambda \in [0, 1]$. Now we will construct a cylinder, S, with a radius of ϵ with the line crossing x_0 and x_1 as its axis. Now let x_0' be a point in a ball of radius ϵ about x_0 and x_1' be a point in a ball of radius ϵ about x_1 such that

$$\mu_A(x_0') \geq M - \epsilon$$
$$\mu_A(x_1') \geq M - \epsilon.$$

Then since A is a fuzzy convex set, for any point u on the line $\overline{x_0' x_1'}$, we have that $\mu_A(u) \geq M - \epsilon$. Furthermore, since S is a convex cylinder, all points on the line $\overline{x_0' x_1'}$ will lie in S. Now let x be any point on the line $\overline{x_0' x_1'}$. The distance of this point from the line $\overline{x_0' x_1'}$ must be less than or equal to ϵ since the line lies in S. Furthermore, it follows that a ball with radius ϵ about x will contain at least one point of the line $\overline{x_0' x_1'}$ and hence, contain at least one point, h, such that $\mu_A(h) \geq M - \epsilon$. Therefore, it follows that M is essentially attained at x, and consequently, the core $C(A)$ is convex. □

Now that we have proved these many properties of convex fuzzy sets, we can see that there are a lot of similarities and differences in the idea of a set being convex in classical and fuzzy set theory. Even though the proofs are different, the foundation that the proof lies on is essentially the same and the methodology of each proof can be related.

6 The Hyperplane Separation Theorem of Fuzzy Convex Sets

"The sentence 'snow is white' is true if, and only if, snow is white."

-Alfred Tarski

Once again, we will assume that the fuzzy set X to be a real Euclidean space $\mathbb{E}^n$.

6.1 Preliminary Ideas and Definitions

Before we talk about the Hyperplane Separation Theorem, we will introduce new definitions and ideas related to fuzzy sets and hyperplanes.

Definition. (Shadow[Projection] of a Fuzzy Set on a Hyperplane) *Let A be a fuzzy set in X with membership function $\mu_A(x) = \mu_A(x_1, x_2, \ldots, x_n)$. For notational simplicity, the shadow (projection) of A on a hyperplane will be defined for the special case where H is a coordinate hyperplane, $H = \{x : x_1 = 0\}$. We define the shadow of A on $H = \{x : x_1 = 0\}$ as a fuzzy set $S_H(A)$ in E^{n-1} with membership $\mu_{S_H(A)}(x)$ given by*

$$\mu_{S_H(A)}(x) = \mu_{S_H(A)}(x_1, x_2, \ldots, x_n) = \sup_{x_1} \mu_A(x_1, x_2, \ldots, x_n).$$

We will now prove some interesting properties related to the shdaow of a fuzzy set on hyperplanes.

Theorem. *If A is a convex fuzzy set, then its shadow on any hyperplane is also a fuzzy convex set.*

Proof. Result follows directly from the definition of a convex fuzzy set and the shadow. □

Theorem. *Let H be any hyperplane and A and B be fuzzy sets. If $S_H(A) = S_H(B)$, then $A = B$.*

Proof. We will prove this assertion by proving its contrapositive. So suppose that $A \neq B$ for fuzzy sets A and B. Then there exists a point x_0 such that $\mu_A(x_0) \neq \mu_B(x_0)$. We wish to show that there exists a hyperplane H such that $\mu_{S_H(A)}(x_0*) \neq \mu_{S_H(B)}(x_0*)$ for x_0* being the projection of x_0 on H. Suppose without loss of generality that $\mu_A(x_0) > \mu_B(x_0)$. Since B is a convex set, it follows that $\Gamma_\beta = \{x : \mu_B(x) > \beta\}$ is convex for $\beta = \mu_B(x_0)$. Therefore, there exists a hyperplane G such that it supports Γ_β and passing through x_0. Now let H be a hyperplane that is orthogonal to G and x_0* be the projection of x_0 on H. Then since $\mu_B(x) \leq \beta$ for all $x \in G$, it follows that $\mu_{S_H(B)}(x_0*) \leq \beta$. Furthermore, by the same argument, we have that $\mu_{S_H(A)}(x_0*) \leq \alpha$ for $\alpha = \mu_A(x_0)$. Since $\alpha \neq \beta$, it follows that $\mu_{S_H(A)}(x_0*) \neq \mu_{S_H(B)}(x_0*)$. Hence $S_H(A) \neq S_H(B)$ and the desried result is obtained. $\square$

Definition. (Degree of Separation of Two Bounded Fuzzy Sets by a Hypersurface) *Let A and B be bounded fuzzy sets and let H be a hypersurface in X being defined by the equation $h(x) = 0$ with all points for which $h(x) \geq 0$ being on one side of H and all points where $h(x) \leq 0$ being on the other side of H. Now let K_H be a constant that is dependent on H such that $\mu_A(x) \leq K_H$ on one side of H and $\mu_B(x) \geq K_H$ on the other side. Let $M_H = \inf K_H$. Then the constant*

$$D_H = 1 - M_H$$

is the degree of separation of A and B by H.

6.2 Proof of The Hyperplane Separation Theorem of Fuzzy Convex Sets

In classical set theory, the hyperplane separation theorem for convex sets states that that if A and B are *disjoint* convex sets, then there exists a separating hyperplane H such that A is on one side of H and B is on the other side. The study of this theorem can be extensive by itself and has many application in the field of convex optimization. If the reader is curious, please refer to Byod and Vandenberghe's book, *Convex Optimization* [1] that covers the topic in much better detail.

Now back to fuzzy sets. It is natural to question if we can extend this theroem to fuzzy set theory as we were successful in all the previous notions. It follows that the theorem can easily be extended to convex fuzzy sets with slight modifications to the proof related to classical sets. However, the main question is whether or not we can extend this theorem to convex fuzzy sets without the sets being disjoint. The notion of two fuzzy sets being *disjoint* seems to be too restrictive in the case of fuzzy sets. As it turns out, it is possible (and not too difficult) to prove this assertion with the help of the separation theorem of classical sets. The following theorem can be interpreted as the hyperplane separation theorem of fuzzy convex sets. It states that the highest degree of separation of two fuzzy convex sets A and B that can be achieved with a hyperplane in X is the difference between one and the maximal value of the membership of the intersection $A \cap B$.

Theorem. *(The Hyperplane Separation Theorem of Fuzzy Convex Sets)*

Let A and B be bounded fuzzy convex sets in X, with maximal membership values $M_A = \sup_x \mu_A(x)$ and $M_B = \sup_x \mu_B(x)$, respectively. Let M be the maximal membership value for the intersection $A \cap B$ where $M = \sup_x \min(\mu_A(x), \mu_B(x))$. Then the highest degree of separation is $D = 1 - M$. (Notice how this theorem does not require A and B to be disjoint).

Proof. [4] For simplicity sake, we will split the argument into two different cases: (1) $M = \min(M_A, M_B)$ and (2) $M < \min(M_A, M_B)$. We note that the second case takes care of the case for $A \subset B$ and $B \subset A$.

Case 1. $M = \min(M_A, M_B)$. Suppose without loss of generality that $M_A < M_B$. Then we have that $M = M_A$. Then by the property of bounded sets stated in the previous subsection, we know there exists a hyperplane H such that $\mu_B(x) \leq M$ for all x on one side of H and $\mu_A(x) \leq M$ since $\mu_A(x) \leq M_A = M$ for all x on the other side of H. It suffices to show that there does not exist a $M' < M$ and a hyperplane H' such that $\mu_A(x) \leq M'$ on one side of H' and $\mu_B(x) \leq M'$ on the other side. So suppose for contradiction that such a H' and M' exists and assume without loss of generality that the core of A, $C(A)$, is on the plus side of H'. Recall that $C(A)$ is the set of points at which $M_A = M$ is **essentially attained**. Then we see that this rules out the possibility $\mu_A(x) \leq M'$ for all x on the plus side of H'. Hence it must be that $\mu_A(x) \leq M'$ for all x on the minus side of H' and so $\mu_B(x) \leq M'$ for all x on the plus side of H'. It follows as a consequence that for all x on the plus side of H' that

$$\sup_x \min(\mu_A(x), \mu_B(x)) \leq M'$$

and similar on the minus side of H'. Therefore, for all $x \in X$, it follows that $\sup_x \min(\mu_A(x), \mu_B(x)) \leq M'$ which contradicts the assumption that $\sup_x \min(\mu_A(x), \mu_B(x)) = M > M'$. Hence there does not exist a $M' < M$ and a hyperpline H' such that $\mu_A(x) \leq M'$ on one side of H" and $\mu_B(x) \leq M'$ on the other side.

Case 2. $M < \min(M_A, M_B)$. For this case, we will consider the convex sets (since A and B are convex) $\Gamma_A = \{x : \mu_A(x) > M\}$ and $\Gamma_B = \{x : \mu_B(x) > M\}$. We will first show that these sets are nonempty and disjoint. The sets are trivially nonempty. Now suppose that Γ_A and Γ_B are not disjoint. Then there exists a point u such that $\mu_A(u) > M$ and $\mu_B(u) > M$ so that $\mu_{A\cap B}(u) > M$. However, that contradicts the assumption that $M = \sup \mu_{A\cap B}(x)$. Therefore, the sets Γ_A and Γ_B are nonempty and disjoint. Now since Γ_A and Γ_B are disjoint and convex, by the separation theorem of classical convex sets, there exists a hyperplane H such that Γ_A is on one side of H and Γ_B is on the other side. Suppose without loss of generality that Γ_A is on the plus side and Γ_B is on the minus side. Furthremore, by the definitions of Γ_A and Γ_B, for all points on the minus side of H, we have that $\mu_A(x) \leq M$, and for all points on the plus side of H, $\mu_B(x) \leq M$. Hence we have shown that there exists a hyperplane H with $D = 1 - M$ as the degree of separation of A and B. If a higher degree of separation of A

and B, $D' > D$, was obtained, it would contradict the argument presented in *Case 1.* of this proof.

We have shown both cases, and hence, the desired result is obtained. $\square$

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