Limits, Sequences, and Series Involving Prime

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Math 336

1 Introduction

Prime numbers have long been a fascination of mathematicians. Prime numbers are positive integers whose factors are only one and itself. This paper will introduce the reader to interesting limits, sequences, and series involving prime numbers and the solutions giving insight into this realm of mathematics. At the end there is a discussion about current research and more difficult problems as well as connections to other areas of math. Finally there is a brief appendix covering in introductory detail other areas of math that are often used with these problems.

One of the first properties of this subset of the integers is that there are an infinite amount of prime numbers. This was proven in Euclid's *Elements* around 300 BC. The well known proof takes the form of contradiction, suppose that there exists a finite amount of prime numbers and hence a largest prime. Index each prime number, that is two is the first prime, three is the second, five is the third, and so forth. The notation, p_n will be used to notate the n^{th} prime. Suppose that the largest prime is the j^{th} prime. Multiply all the prime numbers together which is a finite number of elements so its product is also finite which is

$$\prod_{n=1}^{j} p_n = q$$

. Now look at $q + 1$. This new number is also an integer which itself is either prime or it has a prime factor greater than p_j since no prime number less than or equal to p_j can divide $q + 1$. This proves that there is an infinite amount of primes. The Chaitin book in the additional reading gives a radically different proof of the infinity of the primes. While also using contradiction, Chaitin, a computer scientist makes an argument that if there were finitely many primes then it would be "too efficient" for integers larger than some large number to be represented. However the properties of this infinite set of prime numbers has been and still is an extremely active area of mathematical interest. It is helpful to establish some notation for the discussion of prime numbers, $\pi(x)$ is the number of primes less than x. The relation $\sim$ between two expression, $f(n)$

and $g(n)$, $f(n) \sim g(n)$ holds if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

. This means that $f(n)$ is asymptotic to $g(n)$ and that the two functions are approximately equal for large enough n. An example is, $\frac{1}{n^2+n} \sim \frac{1}{n^2}$ which is easily proved by L'Hôpital's rule. Note that $\pi(n)$ is locally irregular, it stays constant for a while and then occasionally bumps up by one. A further discussion of approximation theory is included at the end.

One interesting limit to look at is $\lim_{n\to\infty} \frac{\pi(n)}{n}$ that is number of primes less than or equal to n over n. This is looking at the density of the primes. Suppose the limit existed and was equal to some number k. Take some very large number n and it follows then that $\frac{\pi(n)}{n} = k$ and $\frac{\pi(2n)}{2n} = k$. Note that the number of primes between $n+1$ and $2n$ is equal to $\pi(2n)$ minus $\pi(n)$. Consider $\binom{2n}{n<4^n}$ which holds as the total number of subsets, $2^{2n} = 4^n$, is less than the number of n-subsets. Note that $\binom{2n}{n}$ is an integer and by writing out it in factorial form it is clear that all the prime numbers between $n+1$ and $2n$ are factors of $\binom{2n}{n}$. Since each of these prime numbers is less than n it follows that $n^{\pi(2n)-\pi(n)} < \binom{2n}{n} < 4^n$. Taking the logarithm of both sides gives,

$$\pi(2n) - \pi(n) < \log 4 \cdot \frac{n}{\log n}$$

. This can be expanded to

$$\pi(2^k) - \pi(2^{k-1}) < \log 4 \cdot \frac{2^{k-1}}{(k-1)\log 2} = \frac{2^k}{k-1}$$

. Now sum from $k=2$ to $k=2m$ and noting that the left side telescopes and doing some series and inequality manipulation to the right side gives,

$$\pi(2^{2m}) - \pi(2) < 2^{m+1} + \frac{2^{2m+1}}{m}$$

. For any positive x there is an integer j such that $4^{j-1} < x \leq 4^j$. This implies $j-1 < \log_4(x) \leq j$. Note that $\pi(4^j) < 1 + 2^{j+1} + \frac{2^{2j+1}}{j}$. Combining this gives

$$\pi(x) \leq \pi(4^j) < 1 + 2^{2+\log_4(x)} + \frac{2^{2(1+\log_4(x))+1}}{\log_4(x)} = 1 + 4\sqrt{x} + \frac{8x}{\log_4(x)}$$

. Then dividing by x gives,

$$\frac{\pi(x)}{x} = \frac{1}{x} + \frac{4}{\sqrt{x}} + \frac{8}{\log_4(x)}$$

which shows that

$$\lim_{x\to\infty} \frac{\pi(x)}{x} = 0$$

proving the density of primes is zero. This naturally leaves the reader to wonder the behavior of $\pi(n)$ which leads to the prime number theorem.[2]

Perhaps the most famous limit involving prime numbers is the prime number theorem. As this theorem will be covered in this course through the methods of complex analysis the proof and understanding is left to the classroom. Briefly stated, $\pi(n) \sim \frac{n}{\log(n)}$. An elegant proof of the theorem is in the additional reading section. A very brief but insightful sketch was given by Terence Tao. Create a "sound wave" (von Mangoldt function) which is noisy at primes but quiet elsewhere. Take Fourier transforms, the recorded "notes" are the zeros of the Riemann zeta function. Further work using Fourier series and other mathematical manipulations gives the prime number theorem. Below shows the very slow convergence.

n	p_n	$n \ln n$	Error
10^3	7,919	6,907	−13%
10^6	15,485,863	13,815,510	−10%
10^9	22,801,763,489	20,723,265,836	−9%
10^{12}	29,996,224,275,833	27,631,021,115,928	−8%

2 Analogies Between Integers and the Set of Primes

With the knowledge that the set of primes is infinite and its density in the integers goes to zero, it is interesting to look at previous problems and just looking over the set of primes. The first example that might come to mind is the harmonic series, which diverges when summed over all the integers but converges for certain subsets like integers that are powers of two. This section will focus on further defining properties of the set of prime numbers through familiar problems.

2.1 Reciprocal of the Primes (Harmonic Series)

The famous harmonic series, $\sum \frac{1}{n}$ diverges when summed over the integers and it can easily be shown from this summing the harmonic series over subsets that do not have zero density in the integers will also diverge. However, certain subsets will converge such as the powers of two. Consider $\sum \frac{1}{p}$ that is the sum of the reciprocals of the primes. This proof is from Euler in 1731. Let

$$A_1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$$

and

$$A_n = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \cdots$$

. Euler proves early in the paper that $\frac{2\cdot3\cdot5\cdot7\cdot11\cdots}{1\cdot2\cdot4\cdot6\cdot10\cdots}$ where the numerators are the prime numbers and the denominators are one less than the prime number is the same as the sum of the harmonic series and diverges. A corollary of this is that if ∞ is absolute infinity, then the value of the above expression is $\ell\infty$ which is the minimum among all powers of infinity. This is theorem seven for the interested reader. Clearly A_n converges for $n > 1$ and since $\sum \frac{1}{k^n} > A_n$ where the sum $\frac{1}{k^n}$ is over all the integers greater than one. It follows since $\sum_2^\infty \frac{1}{k^2} < 1$ and $A_n < (A_2)^{n-1}$ that $\sum_2^\infty \frac{1}{n} A_n$ converges. Note that

$$\sum_1^\infty \frac{1}{n} A_n = \ell\frac{2}{1} + \ell\frac{3}{2} + \ell\frac{4}{3} + \cdots$$

and so $e^{\sum_1^\infty \frac{1}{n} A_n} = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdots} = \sum_1^{\infty)\frac{1}{n}}$ by theorem seven. Since $\sum_2^\infty \frac{1}{n} A_n$ converges it follows that A_1, the sum of the reciprocal of the primes diverges. It is easy to share by comparison to $\int_1^\infty \frac{1}{x} dx$ that the harmonic series diverges logarithmically. It is left as an exercise to the reader to show that the sum of the reciprocal of the primes diverges as $\ln \ln(n)$. Hint: theorem seven establishes an important relationship and since $\sum_2^\infty \frac{1}{n} A_n$ converges try to make an argument for just an expression of e^{A_1}. [4]

2.2 Prime Number Equidistribution Theorem

An interesting math problem is consider an irrational α and the sequence αn. In the early twentieth century a number of mathematicians separately proved that the sequence mod 1 is uniformly distributed on the interval $[0, 1]$. Uniformly distributed means that amount of points of the sequence in a sub-interval divided by the total number of points in the sequence is equal to the length of the sub-interval divided by the length of the entire interval. It follows then that a uniformly distributed sequence is dense in the interval.

Perhaps surprisingly, this property also holds for the sequence $\alpha p_n \mod 1$, where irrational number is only multiplied by the primes. This was proven by I.M. Vinogradov in 1935 but in Russian. While it is easy to verify that αn mod 1 is dense in $[0, 1]$ using the pigeonhole principle the prime number sequence is more nuanced and left to the reader to seek out. This paper will look at an easier result to understand instead looking at $\log p_n \mod 1$ and showing that it is not uniformly distributed. Suppose it is uniformly distributed.

$$N_k = \inf\{\ n : p_n > e^k\}\ M_k = \inf\{\ n : p_n > e^{k-\frac{1}{2}}\}$$

$$F(x) = 1\ x \in [0, \frac{1}{2}),\ F(x) = 0\ x \in [\frac{1}{2}, 1)$$

$F(x)$ is one periodic. Since it is assumed that it is uniformly distributed it follows that,

$$\frac{1}{M_k} \sum_{n \leq M_k} F(\log p_n) \quad \frac{1}{N_k} \sum_{n \leq N_k} F(\log p_n)$$

have the same non-zero limit. By the prime number theorem

$$N_k = \pi(e^k) \sim \frac{e^k}{k} \sim \frac{e^k}{k - \frac{1}{2}} \sim \sqrt{e} M_k$$

This is a contradiction about them being equidistant and so $\log p_n \mod 1$ is not uniformly distributed. [1]

3 Familiar Mathematical Constants Come Up Again

Perhaps the most famous mathematical constants, π, the ration between the circumference of a circle and its diameter, comes up in seemingly unrelated areas of math. One example of this is the Basel problem, whose solution shows that $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. Consider,

$$\frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdots}{4 \cdot 4 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdots}$$

where the numerators are the product of the prime numbers greater than two and the denominators are the closest factor of four (Euler calls these even-even numbers) to the prime number. Using Leibniz's formula for π, using the Taylor series of arctan gives, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$. It follows that, $\frac{1}{3} \cdot \frac{\pi}{4} = \frac{1}{3} - \frac{1}{9} + \frac{1}{15} - \frac{1}{21} + \cdots$ and then summing gives, $\frac{4}{3} \cdot \frac{\pi}{4} = 1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \cdots$. Multiplying by $\frac{1}{5}$ gives $\frac{1}{5} \cdot \frac{4}{3} \cdot \frac{\pi}{4} = \frac{1}{5} + \frac{1}{25} - \frac{1}{35} - \frac{1}{55} \cdots +$ and then with subtraction gives, $\frac{4}{5} \cdot \frac{4}{3} \cdot \frac{\pi}{4} = 1 - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \cdots$ Note that this series has no denominators that are divisible by either three or five. This same process can be continued by multiplying by $\frac{1}{7}$ and doing the same process to get $\frac{8}{7} \cdot \frac{4}{5} \cdot \frac{4}{3} \cdot \frac{\pi}{4} = 1 - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} \cdots$. Note that this process which are divisible by a prime number of from $4n - 1$ are removed through addition and a new factor is added, $\frac{4n}{4n-1}$. The subtraction part gets rid of prime numbers of the form $4n + 1$ in the denominators while adding a factor of $\frac{4n}{4n+1}$. This process continued gives denominators that are prime numbers and numerators that are the closest multiple of four. Since the original series for $\frac{\pi}{4}$ converges and each process removes a term to the right of the 1 it follows that

$$\frac{4 \cdot 4 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdots}{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdots} \cdot \frac{\pi}{4} = 1 \cdots$$

but as each process removes a term to the left of the 1 the sum of the remaining terms go to zero as it convergent and hence Cauchy. It follows after inverting the fraction that

$$\frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdots}{4 \cdot 4 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdots} = \frac{\pi}{4}$$

.

Another interesting ratio is

$$\frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdots}{2 \cdot 6 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdots}$$

where the numerators are the odd primes and the denominators are the nearest "odd-even" number to the prime. That is the closest even number when divided by two is an odd number. Through previous work in Euler's 1744 paper,

$$\frac{\pi^2}{8} = \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 11 \cdot 11 \cdots}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdots}$$

where the numerators are twice the odd prime numbers and the denominators are the numbers one greater and one less than the prime. Coupled with the expression for $\frac{pi}{4}$ gives

$$\frac{\frac{\pi^2}{8}}{\frac{pi}{4}} = \frac{\pi}{2} = \frac{\frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 11 \cdot 11 \cdots}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdots}}{\frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdots}{4 \cdot 4 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdots}} = \frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdots}{2 \cdot 6 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdots}$$

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One of Euler's most famous formulas involves a generalization of the Basel problem.

$$\frac{2^n \cdot 3^n \cdot 5^n \cdot 7^n \cdot 11^n \cdots}{(2^n - 1)(3^n - 1)(5^n - 1)(7^n - 1)(11^n - 1) \cdots} = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

Let $x = \sum_{k=1}^{\infty} \frac{1}{k^n}$ hence $\frac{1}{2^n}x = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \cdots$ hence, $\frac{2^n-1}{2^n}x = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \cdots$ hence, $\frac{2^n-1}{2^n} \cdot \frac{1}{3^n}x = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \cdots$ hence, $\frac{(2^n-1)(3^n-1)}{2^n \cdot 3^n}x = 1 + \frac{1}{5^n} + \frac{1}{7^n} \ldots$. This procedure can be continued for each prime number and the sum of the remaining terms after 1 on the right side are eliminated as it extends to all primes giving,

$$\frac{(2^n - 1)(3^n - 1)(5^n - 1)(7^n - 1)(11^n - 1) \cdots}{2^n \cdot 3^n \cdot 5^n \cdot 7^n \cdot 11^n \cdots} x = 1$$

Finally solving for x and writing x out gives the desired result.
These above solutions and problems are from Euler's seminary work published in 1737. An enjoying aspect of this paper is the language of Euler's time such as multiples of four being called even-even numbers. It was the first proof that reciprocal of the primes diverges however there was some ambiguity in his remarks about the rate of divergence and if he meant that it grew as $\ln \ln n$. It would not be until 1874 when Franz Mertens gave a more formal proof that it was considered to be justified. While Euler's problems are tricky, his paper is highly accessible to undergraduates and as seen in the worked out proofs often the same sort of tricks are used. It is helpful to remember that a convergent series is also Cauchy. While not the focus of this paper, many of the problems in this paper have pushed the boundaries of approximation theory and have led to a cross over into probability. One area of current research is the distribution of the last digit of prime numbers. For large primes it must be either one, three, seven, or nine. It is a mostly open question as to this overall distribution and recently in 2016 progress was made on the distribution of the last digit given the last digit of the previous prime. This paper is located in additionally reading section. Just as complex analysis does not have an obvious immediate connection to prime numbers but holds as the backbone for a common proof

of the prime number theorem perhaps there are more connections to be found between the primes and other fields of math. [4]

Another famous constant is e. First define

$$\theta(x) = \sum_{p \leq x} \log p$$

This is known as Chebyshev's function. In a proof given by Dusart in 1999 proved $p_n = n(\log n + \log\log n - 1) + n \cdot \theta(n)$ to an order of magnitude. Let $f(n) = \log n + \log\log n - 1$.

$$\log p_n - \frac{p_n}{n} = \log\left(nf(n) + n\theta(n)\right) - f(n) - \theta(n) =$$

$$\log\left(1 + \frac{\log\log n - 1 + \theta(n)}{\log n}\right) + 1 - \theta(n) \to 1$$

Note that relies on a previously proven limit of $\frac{\log\log n - 1 + \theta(n)}{\log n}) \to 0$ Importantly though it shows that

$$\log p_n - \frac{p_n}{n} \to 1$$

It was proven in a 1962 paper that there is a constant $c > 0$ that for all positive x,

$$|\theta(x) - x| < c\frac{x}{\log^2 x}$$

Now consider

$$A_n = \frac{p_n}{\sqrt[n]{p_1 \cdots p_n}}$$

$$\log A_n = \log p_n - \frac{1}{n}\theta(p_n) = \log p_n - \frac{p_n}{n} + \frac{1}{n}(p_n - \theta(p_n))$$

Examining the second part, $\frac{1}{n}(p_n - \theta(p_n)) \to 0$ since $\frac{|\theta(n) - n|}{n} < c\frac{x}{n\log^2 x}$ and so by the prime number theorem it goes to zero. This means,

$$\log A_n \to 1$$

which implies,

$$\lim_{n\to\infty} \frac{p_n}{\sqrt[n]{p_1 \cdots p_n}} = e$$

[3]

4 Current Research and Fields of Exploration

4.1 Theorems

Below are a few theorems with reference to dates give a scale of the development of the field as well as why these particular theorems are interesting or important.

Bertrand's postulate: For all $n > 3$ there is a prime number in between n and $2n - 2$. Or for all integers there is a prime in between n and $2n$. This was first conjectured in 1845 by Joseph Bertrand and then proven by Chebyshev in 1852. Some consequences of this is that every integer can written as the sum of primes and one using each number at most once. It also follows from this that one is the only harmonic integer. Suppose $\sum_{k=1}^{n} \frac{1}{k}$ is an integer for some $n > 1$. There is a prime, p between n and $\frac{n}{2}$. Apart from $\frac{1}{p}$ every term in the sum $\frac{1}{k}$ where k is divisible only by primes smaller than p, hence, $\sum_{k=1}^{n} \frac{1}{k} = \frac{1}{p} + \frac{a}{b}$ and b is not divisible by p and so the sum is not an integer since then $\frac{b}{p} + a$ would be an integer.

The twin prime conjecture is that there are infinitely many prime such that $p_{n+1} = p_n + 2$. This is an extremely famous open problem first formally stated by de Polignac in 1849. Recent progress on this problem includes Yitang Zhang's work in 2013 which proved, $\lim_{n\to\infty} \inf(p_{n+1} - p_n) < 7 * 10^7$. [5]

While the twin prime conjecture is open, it has been shown that the reciprocal of the sum of the twin primes converges by Viggo Brun in 1919. If the twin prime conjecture is true this constant is irrational and if false then rational. The convergence is extremely slow. This result helped advance the construction of sieve methods.

Seemingly connected to probability theory, it has been shown that all large primes have digits $1, 3, 7, 9$ with equal probability showing the randomness of primes. This was proven in a paper by Carl Ludwig Siegel and Arnold Walfisz in 1963.

4.2 Further Exploration

There are two main areas of further exploration that would be suitable for the intended audience. The first option for which math has done an exceptionally good job at over the past centuries of understanding the macroscopic properties of the prime numbers with theorems such as the prime number theorem. Just as the primes are a subset of the integers, there might and can be value in looking at a subset of the primes. For example, consider the "primes of the primes" that is where the prime index is a prime number. This set would be $\{\ 3, 5, 11, 17, 31 \cdots \}$. One could continue to define subject of primes further refining them with each iteration. For example, the primes would be a_1, the above set would be a_2 and so forth. In each iteration the terms get larger and larger which leads back to the classic harmonic series problem. Is there an a_n for which the reciprocals do converge? This is similar to Brun's constant.

As this paper is not focused on number theory or encryption, the fields of probable primes or prime generating functions will be omitted but clearly those fields have a strong connection with the topics. While the large scale behavior of $\pi(x)$ is relatively unknown, its local behavior is unknown and of extreme interest. Of course there are deep connections to the Riemann Hypothesis and Twin prime conjecture. What would make a good launching point is trying to uncover a relationship between the estimated number of primes by the prime number theorem and actual number of primes. Perhaps it is more likely for twin primes to occur when $\pi(n)$ is behind the estimate. If this paper were to be expanded, the next launching point would be to understand more of asymptotic theory and attempt to try to come up with a formula both in magnitude and sign between $\pi(n)$ and the prime number theorem. This might lead to a weaker conjecture than the twin primes but along the lines of given an N where you know $\pi(N)$ and N very large one can make an estimate for the range in the number of primes between N and $N+m$ where m is large but much smaller than N. This might be a lost cause though as there is a strong amount of literature written about the local randomness of primes but perhaps there are connections between the percent error of expected primes and when primes show up. As Euclid proved thousands of years about, not only are the primes infinite, but perhaps they hold an infinite amount of deep math connections waiting to be discovered.

As this paper has shown, prime numbers connect to many areas of math, sometimes converging to familiar constants or providing a new solution to a known problem. This paper serves as a launching point to think about prime numbers beyond just isolating oneself to number theory problems. In particular, this paper has shied away from most complex analysis to look at this in a different light from class but certainly there are strong connections between the prime numbers and complex analysis.

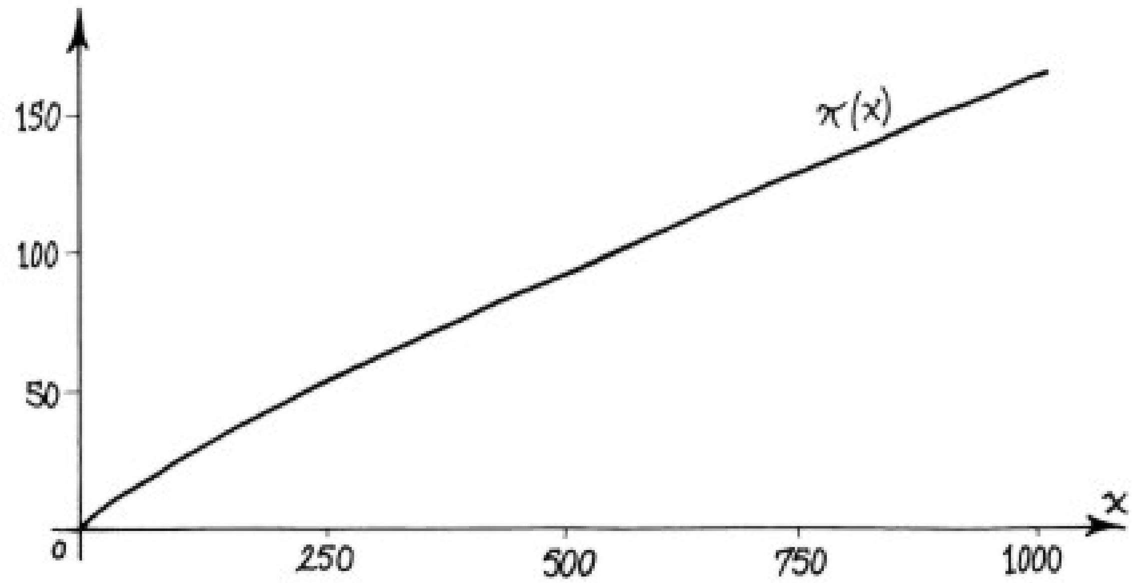
5 Index

5.1 Approximation Theory

As mentioned, $\pi(x)$ is locally irregular and highly unpredictable. Famously, Erdos said, "God may not play dice with the universe, but something strange is going on with the prime numbers." The local randomness of the prime is a subject with deep connections to probability theory. Another way to think about this asymptotic convergence is that for every number k there is an N such that $|\frac{f(x)-g(x)}{g(x)}| < k$ for x greater than k is bounded. This means that the percent error is bounded.

One real world example of this is consider N radioactivity isotopes with N super large, that is you have many moles and the half life is t. Define $g(t)$ to be the expected amount of particles left. $g(t)$ is the classic example of exponential decay. At $t = 0$ the rate of radiation is very close to expected by the large number theorem. Consider $g^{'}(t)$ which is the amount of radiation. For t such that $\frac{N}{2^t}$ is no longer sufficiently large the actual number of particles left will be extremely close the expected value in comparison to N yet the difference between the actual radiation level and the expected radiation level will be much higher. This is an intuitive example where a function is locally irregular yet still has a convergent function. The logarithmic integral is an extension of finding a more accurate fit for $\pi(x)$ and a good launching point to learn more about fitting locally irregular asymptotic functions.

5.2 $\pi(x)$



5.3 Additional Readings

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5.4 References

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