Deriving the Black-Scholes Equation and Basic Mathematical Finance

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June, 2017

1 Introduction

In the 1970's Fischer Black and Myron Scholes published a model which would attempt to tackle the issue of pricing a common financial asset. The introduction of this model into finance gave rise to a new era of mathematical finance and changed the way economists would work in the following decades. In this paper, I'll describe the background necessary to understand and derive the Black-Scholes equation (central to the aforementioned model).

2 Financial Background

To get started, I'll introduce some basic finance background so as to help make sense of the significance of the Black-Scholes Equation (B.S.Eq):

a. *Option*: An option is a contract between a buyer and a seller. We'll refer to the two types: **Call options** and **Put options**. Such options are part of a broader idea known as **derivatives trading** in which an asset has a price that is based on a seperate underlying asset. The **underlying asset** is exactly what it sounds like, in this case, the commodity for which an option is to be agreed upon by a buyer and seller.

   Upon buying a call option, the buyer gets the right to buy an agreed amount of an underlying asset from the seller - at a specified time (the **expiration date**) and for a specified price (the **strike price**). Thus, if the buyer decides to exercise the call option (follow through), the seller is obligated to sell them the underlying asset according to the terms.

   Upon buying a put option, the buyer gets the right to sell a specified amount of an underlying asset at a specified time and for a specified price. Thus, if the buyer decides to exercise the put option, the seller is obligated to buy the underlying asset according to the terms.

There are two main variants, the American and the European options. The American option allows the buyer to exercise the option at the strike price at any time between the purchase and the expiration date while the European option allows the buyer to exercise the option only on the expiration date.
For the purposes of the this paper, we'll focus on European options.

b. *Risk-Free Rate of Return* (Risk-Free Interest Rate): The **risk-free rate of return** is the rate of return of an investment over a given period of time with zero risk of financial loss. As even the safest investments in real life have some sort of risk, the interest rate on a three-month U.S. Treasury bill is often used as the risk free-rate for U.S. investors.
So this rate serves as a sort of baseline for investors: As this rate could theoretically be achieved with no risk, any other investment with some risk will have to have a higher rate of return than this baseline to be worth considering.

c. *Volatility* (Stock volatility): Simply put, **volatility** refers to the amount of uncertainty in the changes in an asset's value. In finance, volatility is primarily seen as a variable in option pricing formulas (like the one we'll be examining) and it varies with how it is measured. In general, it's a measured as the variance bewteen returns for a certain asset, and the higher the volatility, the riskier the asset.

With the existence of call options, naturally, the question of figuring out how much a call option is actually worth arises. Before Black-Scholes, options prices were decided roughly by human judgement. Obviously, certain refined strategies existed (most of which are still certainly used today), but it was an inexact science and many thought that it would continue like this with some even doubting that . As stated in the introduction, in the 1970s, economists Fischer Black and Myron Scholes published a derivatives model which would tackle just this problem - with certain limits and simplifications, of course. The introduction of their model and the thought behind it basically heralded a new era in finance, as derivatives trading exploded and mathematicians became the new hot thing in finance.

3 Basic Probability Theory

Next, we'll need some basic concepts in probability theory so as to provide more background for the derivation and meaning behind the B.S.Eq. We'll use $P(X)$ to denote the probability of some event X occuring. $P(X \cup Y)$ denotes the probability that X and Y occur, and $P(X \mid Y)$ denotes the probability of X occuring given that Y already occured. The relationship between joint and conditional probability is like so:

$$P(X \cup Y) = P(X \mid Y)P(Y)$$

and from this we get Bayes' Theorem, a very flexible result:

$$P(X \mid Y) = \frac{P(Y \mid X)P(X)}{P(Y)}$$

Furthermore, this gives us a way to define independence. Two events X and Y are independent if

$$P(X \cup Y) = P(X)P(Y)$$

Notice that, using our original definition of joint probability, this means that $P(X \mid Y)$ is equal to $P(X)$ (and similarly $P(Y \mid X) = P(Y)$). Now, onto more definitions:

a. *Random Variable*: A random variable is a variable whose possible values are outcomes of a random phenomenon.

   More formally, a random variable is defined as a function that maps the outcomes of unpredictable processes to numerical quantities. To study random variables, we can characterize them with probability distributions.

   A simple example: Let X represent the result of a coin flip with equal probability heads and tails. Note that this is a discrete random variable (it can only be either 0 or 1). An example of a continuous random variable is the temperature of a city or something comparable.

   Furthermore, two random variables are *independent* of each other if the occurence of one does not affect the probability distribution of the other.

b. *Probability Mass/Density Function (PDF)*: A probability mass function is a function which corresponds to a certain random variable. This function gives the probability that an outcome will be within a certain interval of the random variable's possible outcomes. Thus, this function can be used to study the properties of the random variable and to do more math. For a continuous random variable, this is often simply a function defined on a certain range. For a discrete random variable, X, it is often a simple piecewise function like, for example,

$$f_X(x) = P(X = k) = \begin{cases} \frac{1}{4} & \text{k} = 0 \\ \frac{3}{4} & \text{k} = 1 \end{cases}$$

   mapping values in the random variable's codomain to probabilities.
   Note that, for discrete random variables, we use Probability Mass Function and for continuous we use Probability Density Function.

c. *Cumulative Distribution Function (CDF)*: This is simply the probability that a given random variable will take on a value less than or equal to a given value. Notice that this is effectively the integral of the Probability Mass Function from $-\infty$ to a certain a. This function in particular has many uses and allows easy computation of probabilities as it is common to want to know probability within an interval with continuously distributed random variables. It is calculated by:

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} P_X(t)\ dt$$

where the integrand is simply the corresponding random variable's probability mass/density function.

d. *Expected Value* or *Mean*: This is simply the mean value of a random variable or, in other words, the variable's expected value. Given a random variable X with probability density function $P_X(x)$, its expected value can be calculated like so:

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x P_X(x)\ dx$$

where the integrand is a value x multiplied by its probability in the probability mass function.

An important property of expected value is its linearity. Given two independent random variables X and Y:

$$\mathrm{E}[\alpha X + \beta Y] = \alpha \mathrm{E}[X] + \beta \mathrm{E}[Y]$$

e. *Variance/Standard Deviation*: The variance of a random variable corresponds to the variable's spread relative to its expected value. Given a random variable X with expected value μ, variance is calculated like so:

$$\mathrm{Var}(X) = \sigma^2 = \mathrm{E}[(X - \mu)^2]$$

There are a few very simple but important properties of Variance which we will use. Firstly,

$$\mathrm{Var}(\alpha X) = \alpha^2 \mathrm{Var}(X)$$

and, assuming X and Y are two independent variables,

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

Standard deviation, is often used in conjunction with variance, σ, and it is simply the square root of variance, denoted simply by σ.

f. *Probability Distribution*: A probability distribution is a specific and well observed type of probability. More specifically, its a well-studied 'distribution' in that it has a known probability mass function as well as other common properties. If a random variable X is characterized by a certain distribution Y, we can express this as $X \sim Y$. Let us list a few common ones and their properties. Note, that each distribution has certain parameters that it takes in its definition (often specifying a relevant probability or a frequency):

   i) Bernoulli Distribution: This distribution takes just one parameter, p, and is one of the most fundamental. It simply models the probability of an event occuring with probability p. It can be thought to model a simple coin flip with probability of heads equal to p.
   The expected value of a random variable $X \sim \text{Bernoulli}(p)$ is $\mu = p$ and the variance is $\sigma^2 = p(1-p)$.

   ii) Binomial Distribution: This distribution is simply a repeated coin flip, counting the number of heads. It takes two parameters p, probability of heads on a single flip, and n, number of coin flips.

   iii) Geometric Distribution: This distribution is similar to the binomial distribution but, instead, counts how many flips are expected until a heads is flipped (i.e. tails, tails, . . . , tails, heads).

   iv) Poisson Distribution: This distribution is different in that it takes a frequency (i.e. 5 occurrences per minute), and tells you the expected number of these events to happen in a specific time interval. It takes only a frequency, λ.

   v) Uniform Distribution: This is the simplest *continuous* distribution. It takes a lower bound and an upper bound and gives every value in between an equal probability.

   vi) **Normal Distribution**: This is one of the most useful distributions and ties into the Central Limit Theorem. It takes two parameters, mean value μ and standard deviation σ. As we will be using this distribution (and its Log) variant, we will go into its Probability Density Function and Cumulative Distribution Function. Given a normally distributed variable X with with parameters μ and σ (also expressible as $X \sim \mathcal{N}(\mu, \sigma)$), its PDF is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

   As the Normal Distributions CDF is impossible to precisely calculate, a CDF standardized version of the distribution is used. Given a 'standard normal distribution' with $\mu = 0$ and $\sigma = 1$, the CDF is defined to be $\phi(x)$. Values for $\phi(x)$ have been estimated and, by standardizing a normally distributed variable, ϕ can be used as a makeshift CDF. Standardizing a normally distributed random variable simply means to subtract μ and dividing by standard deviation σ to bring it in-line with the standard normal distribution:

$$F_X(a) = P(X \leq a) = P\left(\frac{X-\mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right) = \phi\left(\frac{a-\mu}{\sigma}\right)$$

The expected value of a normally distributed variable is simply equal to the mean value it takes as a parameter, μ, and similarly the variance is simply σ^2.

vii) **Log-Normal Distribution**: The Log-Normal Distribution is simply the probability distribution of a random variable whose logarithm is normally distributed. So, if X is log-normally distributed, $\log(X)$ is normally distributed. As such, it similarly takes parameters μ and σ^2 (defining the corresponding ordinary normal distribution). The PDF is similar to that of the ordinary Normal distribution and is given by:

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}$$

The mean of a log-normal distribution is equal to

$$e^{\mu+\frac{\sigma^2}{2}}$$

it's simple enough to move between the two distributions with substitutions and, so, they share most of their properties.

viii) Of course, there are other distributions but these are some of the most fundamental and pertinent ones.

Notice that each of these distributions can be easily applied to characterize a real-life scenario. Beyond this, you can often model unpredictable variables with a distribution which might not even seem immediately applicable, as we will be doing in the derivation of the Black-Scholes Equation.

g. *Central Limit Theorem* Given independent and identically distributed random variables $X_1, X_2, \ldots, X_n$, with $\mathrm{E}[X_i] = \mu < \infty$ and $\mathrm{Var}(X_i) = \sigma^2 < \infty$. Then, as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

In simple terms, this means that the normalized sum of many independent and identically distributed variables tends towards a normal distribution, *even*, if the original random variables are not normally distributed themselves.

This lets us approximate randomly distributed samples with the normal distribution and makes it easier to analyze the samples.

4 The Black-Scholes Equation

Now, let us examine the equation itself:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV = 0$$

where s is the stock price, V is the price of the call option as a function of s and time t, r is the risk-free rate of return, and σ is the volatility of the stock.
Next, we will derive the B.S. Equation using the basic principles we've introduced.

5 Derivation

To start[1], let us consider the price of a European call option. If we know that the price of the underlying asset will be S at time T and we chose a strike price E, then we can define the option price to be $P = \max(0, S - E)$. If the market price is greater than our strike price then their difference will be the profit we could make by exercising the option at strike price E and then selling the assets back at market price S.

Of course, we don't actually know what the price of the underlying asset will be at time T. The next logical step, then, is to find a probability density function $q(S)$ corresponding to the price of the asset at T. This would let us calculate its expected value, $V(S) = \int_{-\infty}^{\infty} P(S)q(S)\ dS$, and use this to find P. An important consideration to make, though, is that this would give us how much the option will be worth at time T but we want its worth right now. Assuming the interest rate to simply be the risk-free interest rate, r, we know that over the time $T - t$ the price increased by a factor of $e^{r(T-t)}$. So, to adjust for this, we have to multiply the calculated expected value by $e^{-r(T-t)}$ giving us:

$$V(s,t) = e^{-r(T-t)} \int_{-\infty}^{\infty} P(S)q(S)\ dS \tag{1}$$

Where s is the orinal price of the underlying asset.
Note that, if we want to work with a put option instead of a call option, we can simply redefine $P(S) = \max(0, E - S)$ and the rest of (1) remains the same.

Next, we want to construct a good probability density function $q(S)$ for the underlying asset price S at T given a starting price of s at t. We can understand the changes as a coin flip occuring every dt years. Meaning, every dt years we flip a coin and, if the coin is heads the price of the asset increases by a factor of r and if the

[1]Derivation from Granville Sewell's "Derivation of the Black–Scholes Equation from Basic Principles" in the College Mathematics Journal Volume 49 2018, Issue 3

coin is tails the asset price decreases by a factor of r. Put differently, the logarithm of the price increases/decreases by $\log(r)$[2].

Between t and T we'll have thrown $N = (T - t)/dt$ coins. Thus, we have N Bernoulli distributed samples, each denoted by X_i, with probability of increase $p = \frac{1}{2}$ and variance $\sigma^2 = p(1-p) = \frac{1}{4}$.

By the central limit theorem, the we can approximate the number of incremental increases in price with a Normal distribution like so. By linearity of Expectation and properties of Variance we can move the N over into the right side to produce:

$$\frac{1}{N}\sum_{k=1}^{N} X_i \to \mathcal{N}\left(\frac{1}{2}, \frac{1}{4N}\right)$$
$$\sum_{k=1}^{N} X_i \to \mathcal{N}\left(\frac{N}{2}, \frac{N}{4}\right)$$

Thus we get a normal distribution with mean $\mu = N/2$ and standard deviation $\sigma = \sqrt{N}/2$ to represent the number of increases in price ($N = (T - t)/dt$).

Then, due to our earlier use use of the logarithm, the price S will have a log-normal distribution $q(S)$ (centered at S) such that the probability of our price being within a range ds of S at time T is given by the probability density function of the log-normal distribution:

$$q(S)\ dS = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(\log S - \alpha)^2}{2\sigma^2}}$$

Letting $z = \log S - \alpha$, we can express this as a normal distribution in the logarithm of price like so:

$$q(S)\ dS = p(t, z)\ dz = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-z^2}{2\sigma^2}}$$

Where σ and α will be defined below (both dependent on t, z). Naturally, the most likely outcome is that half of the incremental changes will be increases and half decreases, leaving the final price unchanged from the original price. Intuitively, we might want to set $\alpha = \log(s)$ so that our distribution has a peak at the most likely outcome but, as the peak of a log-normal distribution is lower than its mean, it makes more sense to set the mean equal to the original price s. Furthermore, taking inflation into account we must scale up our expected final amount by a factor of $e^{r(T-t)}$ (See equation (1)). This gives us a mean $\mu = se^{r(T-t)}$.

[2] Note that using Bernoulli distributed incremental changes is just one way to express the change in price with time. There are certainly other approaches

Thus, we want α such that $\mathrm{E}[q(S)] = se^{r(T-t)}$. Referring to the mean of a log-normal distrubuted variable, we get:

$$\mathrm{E}[q(S)] = e^{\alpha + \frac{\sigma^2}{2}}$$

Setting

$$e^{\alpha + \frac{\sigma^2}{2}} = se^{r(T-t)}$$

We get $\alpha = \log(s) + r(T-t) - \frac{\sigma^2}{2}$.

Now, to choose a standard deviation σ for our normal distribution of $z = \ln(S) - \alpha$. We found earlier that the standard deviation for the number of increases in the price is $\sqrt{N}/2$ and each increase in $\log(S) - \alpha$ is

$$(\#\text{increases} - \#\text{decreases})dx = 2(\#\text{increases} - N)dx$$

Thus, a reasonable choice for the standard deviation is $\sigma = \frac{\sqrt{N}}{2} \cdot 2dx = \sqrt{N}dx$. So variance is equal to $\sigma^2 = N\ dx^2 = (T-t)\frac{dx^2}{dt}$.
Let $\sigma_1 = \frac{dx}{\sqrt{dt}} = \sigma/\sqrt{T-t}$. We call this the price's volatility. Notice that, as dt decreases (time between price variations) and dx increases (variation size), the price's variance increases so volatility is a measure of the rate of price fluctuation (just as we saw in our initial definitions). So we have

$$\alpha = \log(s) + \log(r(T-t)) - \frac{\sigma^2}{2}$$

and

$$\sigma = \frac{\sqrt{N}}{2} \cdot 2dx = \sqrt{N}dx, \qquad \sigma^2 = (T-t)\frac{dx^2}{dt} = (T-t)\sigma_1^2$$

Thus, our normally distributed $z = \log(S) - \alpha$ is

$$p(t,z) = \frac{1}{\sqrt{2\pi(T-t)\sigma_1^2}} \mathrm{Exp}\left[\frac{-z^2}{2\sigma_1^2(T-t)}\right] \tag{2}$$

and

$$z = \log(S) - \alpha = \log(S) - \log(s) + (\sigma_1^2/2 - r)(T-t)$$

6 Intermediary stage

Now, finally we can return to the formula for the European option price we saw earlier in equation (1), as $dz = \frac{dS}{S}$:

$$V(s,t) = e^{-r(T-t)} \int_{-\infty}^{\infty} P(S)p(t,z)\ dz = e^{-r(T-t)} \int_{0}^{\infty} \frac{P(S)}{S} p(t,z)\ dS \tag{3}$$

where $p(t,z)$ is as defined above and $P(S)$ is the option price. Notice that, without even having completely derived the Black-Scholes equation itself, we've gotten to the point that we can find the value of the option, V, simply by calculating this integral. We will go on, though, to complete the derivation as it yields further information.

7 Finishing the derivation

Note that, as we defined z above, $\frac{\partial}{\partial t}z = r - \sigma_1^2/2$ and $\frac{\partial}{\partial s}z = -1/s$. Thus, we can differentiate V as in (3) using the above partial derivatives of z and differentiation under the integral:

$$\frac{\partial}{\partial t}V(s,t) = V_t = rV + e^{-r(T-t)}\int_0^\infty \frac{P(S)}{S}\left[p_t + (r - \sigma_1^2/2)p_z\right] dS \tag{4}$$

and

$$sV_s = -e^{-r(T-t)}\int_0^\infty \frac{P(S)}{S}p_z\ dS \tag{5}$$

Differentiating again with respect to s:

$$s(V_s)_s = e^{-r(T-t)}\int_0^\infty \frac{P(S)}{S}p_{zz}\ DS \tag{6}$$

Combining (4) and (6) produces

$$V_t + \frac{1}{2}\sigma_1^2 s(sV_s)_s = rV + e^{-r(T-t)}\int_0^\infty \frac{P(S)}{S}\left[p_t + \frac{1}{2}\sigma_1^2 p_{zz} + (r - \sigma_1^2/2)p_z\right] dS$$

Now, let us show that $p(t,z)$ satisfies the equation $p_t + \frac{1}{2}\sigma_1^2 p_{zz} = 0$ (this equation is known as the backward diffusion equation):

$$\begin{aligned} p_t &= \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{2\pi(T-t)\sigma_1^2}}\text{Exp}\left[\frac{-z}{2\sigma_1^2(T-t)}\right]\right) \\ &= \frac{1}{2\sqrt{2\pi}(T-t)\sqrt{\sigma_1^2(T-t)}}e^{\frac{-z^2}{2\sigma_1^2(T-t)}} \\ &\quad - \frac{z^2}{2\sqrt{2\pi}\sigma_1^2(t-t)^2\sqrt{\sigma_1^2(T-t)})}e^{\frac{-z^2}{2\sigma_1^2(T-t)}} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\sigma_1^2 p_{zz} &= \frac{z^2}{2\sqrt{2\pi}\sigma_1^2(T-t)^2\sqrt{\sigma_1^2(T-t)}}e^{\frac{-z^2}{2\sigma_1^2(T-t)}} \\ &\quad - \frac{1}{2\sqrt{2\pi}(T-t)\sqrt{\sigma_1^2(T-t)}}e^{\frac{-z^2}{2\sigma_1^2(T-t)}} \end{aligned}$$

Thus, summing the two yields zero:

$$p_t + \frac{1}{2}\sigma_1^2 p_{zz} = 0$$

This simplies the integral in our equation to:

$$V_t + \frac{1}{2}\sigma_1^2 s(sV_s)_s = rV + (r - \sigma_1^2/2)e^{-r(T-t)} \int_0^\infty \frac{P(S)}{S} p_z \, dS$$

Substituting in (5):

$$V_t + \frac{1}{2}\sigma_1^2 s(sV_s)_s = rV - (r - \sigma_1^2/2)sV_s$$

And, as $(sV_s)_s = sV_{ss} + V_s$:

$$V_t + \frac{1}{2}\sigma_1^2 s^2 V_{ss} + \frac{1}{2}\sigma_1^2 s_1^2 s V_s = rV - rsV_s + \frac{1}{2}\sigma_1^2 s V_s$$
$$V_t + \frac{1}{2}\sigma_1^2 s^2 V_{ss} + rsV_s - rV = 0$$

With s as the stock price, V the price of the call option as a function of s and t, r the risk-free rate of return, and σ_1 the volatility of the asset as we defined above. Comparing this to what we had in section 4, this is exactly the Black-Scholes equation we wanted. Note that, as we know how much an option is worth at some time T and want the value at a previous time t, this equation is solved 'backwards in time' in that we start with a final condition on the option's worth.

Although this is a relatively simple result, it was part of a larger movement at the time. Moreover, there are many other ways of approaching derivatives trading such as tree-like structures and other probability based techniques. Really, the B.S. model is fairly unique in having a clean differential equation.

References

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