Outside ZF - Set Cardinality, the Axiom of Choice, and the Continuum Hypothesis

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June 6, 2017

Synopsis

In June 2002, "Two Classical Surprises Concerning the Axiom of Choice and the Continuum Hypothesis" by Leonard Gillman was published in the American Mathematical Monthly Vol. 109, No. 6. It offers an accessible slice of foundational set theoretic concepts, and ties together several large theorems in an intriguing way. Specifically, Gillman shows how both the Continuum Hypothesis and Trichotomy for cardinal numbers independently imply the Axiom of Choice. Gillman's clear style makes the proofs and discussion of these surprising results enjoyable and approachable for undergraduate readers.

This article underlines how mathematicians' view of set theory has evolved over time, as the objects and concepts that it consists of have been defined, extended, and analyzed. Our discussion will be held assuming the Zermelo-Frankel (ZF) axioms of set theory. And the theorems under consideration - the Continuum Hypothesis, Trichotomy, and the Axiom of Choice - are all major concepts which are independent of ZF. They cannot be derived from the ZF axioms, and so are explicitly taken or not taken as an assumption in every relevant situation. Making such significant assumptions does not cause a mathematical existential crisis - it is a common practice. It is equivalent to deciding the rules of a game. Every round could have different rules - and then you would have to be aware and not expect to get comparable results. In this way, the set of rules of Euclidean Geometry and the set of rules of Non-Euclidean Geometry differ, and they result in different implications for the concepts "line", "space", etc. Thus, it is all right that our discussion will deal with implications between statements that are unprovable in ZF. We are simply modifying the set of assumptions and therefore aggregating new results.

Austim Stromme mentioned that in practice, mathematicians only sometimes mention whether they are assuming the Axiom of Choice. In fact, he says that it is widely believed that it would be difficult to separate out the pieces of modern (i.e graduate level and above) math that rely on Choice, since so many intuitive claims about sets rely on it.

Set theory was born in December 1873 when Georg Cantor (1845-1918) established that the continuum is not countable [1], and hence entered the world of exploring infinite sets of different sizes. No matter what subset of $\mathbb{N}$ or $\mathbb{R}$ Cantor examined, he consistently identified only two sizes of infinities [3]. Thus, early on, he stated this dichotomy as a hypothesis.

Continuum Hypothesis (1878) - Every infinite set of real numbers either is countable or has the power of the continuum. Equivalently, any infinite subset of the set $\mathbb{R}$ of all real numbers can be put in one-to-one correspondence with $\mathbb{R}$ or with the set $\mathbb{N}$ of all natural numbers.

Privately, Cantor referred to this as the "Two-Class Theorem" [4]. He never published neither this name nor the term "Continuum Hypothesis". The latter did not even appear in any unpublished writing. The term was expressly coined by Felix Bernstein in 1901.

This famed hypothesis (a strongly backed up but unproved claim) is in fact unprovable in the Zermelo-Frankel axioms. Initially, Cantor viewed it in terms of number theory. I will refer to the statement of this Hypothesis throughout the paper, highlighting how its wording has evolved as other set theoretic concepts developed.

Axiom of Choice - For any collection of nonempty sets there exists a set containing an element from each set of the collection.

Trichotomy - Any two (infinite) cardinals $\mathfrak{a}$ and $\mathfrak{b}$ are comparable i.e. either $\mathfrak{a} > \mathfrak{b}$, $\mathfrak{a} = \mathfrak{b}$, or $\mathfrak{a} < \mathfrak{b}$.

When first creating cardinal numbers, Cantor recognized that trichotomy may not be taken for granted. From his construction of the cardinals, their comparability is neither evident nor provable. Initially, he stated that Trichotomy will be capable of proof "only after we have gained a survey over the ascending sequence of the transfinite numbers and an insight into their connection" [3]. Thirty years later, in 1915, Friedrich Hartogs showed that Trichotomy implies the Axiom of Choice and vice versa, a surprise considering how distinct the propositions seem to be. Therefore today, although Trichotomy still cannot be proved solely from the construction of the cardinals, Trichotomy is included by assumption within ZFC, the Zermelo-Frankel Axioms with the Axiom of Choice. Hartogs' result concerning Trichotomy contributed to the Continuum Problem debacle, since some mathematicians wondered "in desperation whether $\mathfrak{c}$ might actually be greater than all the alephs" [2], and Hartogs in fact showed that no cardinal can be greater than all the alephs.

In the following paper, I will review definitions, present Gillman's proof of Hartogs' results, and then return to a discussion of how the Continuum Hypothesis evolved over time. I will attempt to make clear the difference between ordinal and cardinal numbers, and to show several of the claims that Gillman skips over as 'trivial'.

Ordering

A total order on a set S is a relation $<$ that satisfies the following three conditions for all x, y, and z in S:

i. either $x = y$ or $x < y$ or $x > y$;

ii. $x \nless x$;

iii. if $x < y$ and $y < z$, then $x < z$.

So a total ordering provides a way for the elements of S to be comparable, and is also irreflexive (ii) and transitive (iii). $\mathbb{R}$, with its natural ordering, is a familiar example of an ordered set. Conditions (ii) and (iii) by themselves define a partial ordering on S. It may occur that a set can be partially ordered but not totally ordered. Consider for example, the following set of four people: a couple with two children, where $x < y$ means that person x is a direct ancestor of person y. There is no way to compare the two parents to each other, nor the two children to each other, and therefore (i) cannot hold.

Ordering is significant because it is our means of comparing sets and set sizes. In particular, two ordered sets are *similar* if there is an order-preserving, one-to-one correspondence between them. The sets M and N have an order-preserving correspondence if the relative rank of arbitrary elements m_1, m_2 is the same as that of the corresponding elements n_1, n_2. So there is a map $f : M \to N$ which satisfies $n_1 < n_2 \implies f(n_1) < f(n_2)$. Similarity is an equivalence relation that sorts ordered sets into classes by order type. Order type will be defined below. Another important equivalence relation that sorts sets into classes by size is equipotency. It is a distinct concept from similarity. Two sets (not necessarily ordered) are *equipotent* if there is a one-to-one correspondence between them. For example, the interval [0,1] on the real line is equipotent to the unit square [0,1] x [0,1] if you let every real number T from the line map to the set of coordinates (x,y) where x consists of the even digits of T and y consists of the odd digits. Colloquially, "there are the same number of elements in both sets", however this mapping does not preserve relative ranking, and so does not show similarity.

Ordering by set inclusion means the following: Let S be a collection of sets. Then for sets $S_1, S_2 \in S$, we can define $S_1 < S_2$ to mean $S_1 \subset S_2$.

So for example, considering sets of integers, we can say {1,2,3}<{1,2,3,4}. This is a partial order because not all sets are comparable. For instance {1,2,3} and {1,3,5} are not comparable because neither is a subset of the other.

Cardinal Numbers

The cardinal number of a set is the generalization of the concept of "number of elements" to all sets, transfinite (non-finite) as well as finite. Gillman informally asks the reader to accept the principle that every set can be associated with an object called its cardinal number (or cardinal or cardinality or power). The cardinal of a finite set of n elements is denoted by n. The smallest infinite cardinal is $\aleph_0$, pronounced "aleph not." This symbol denotes the size of the set $\mathbb{N}$ of natural numbers. Since many sets will be associated with the same cardinal, a set is referred to as a *representative* of its cardinal.

Two sets have the same cardinal number if they are equipotent; if a one-to-one correspondence exists between them. So for instance, both the cardinal of the set of even natural numbers and the cardinal of the set of rational numbers is $\aleph_0$. This is so because a) the set of even natural numbers can be mapped using $2k \to k$ to the set of natural numbers, and then b) the natural numbers can be mapped to the set of rational numbers using the following algorithm: place the rational numbers into a grid with first diagonal 1, second diagonal 1/2, 2/1, third diagonal 1/3, 2/2, 3/1, fourth diagonal 1/4, 2/3, 3/2, 4/1, etc. The sum of the numerator and denominator of every rational number contained in a given 'diagonal' is constant, therefore all rational numbers will be contained in some diagonal. The elements of every 'diagonal' are ordered by increasing numerator. If we map the natural numbers to this enumeration of the rational numbers, skipping over repeat values, we create the 1-1 correspondence required.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinals with representative sets A and B, respectively. The relation '$<$' on cardinals, where $\mathfrak{a} < \mathfrak{b}$ means that A is equipotent with some subset of B, but B is not equipotent with any subset of A. Again, $|A| < |B|$ means there is an injection $f\colon A \hookrightarrow B$ but no such surjection. This is an incredibly interesting way to deal with these (potentially infinite) concepts. This relation is irreflexive and transitive, and so defines a partial ordering on the class of all cardinals. The relation '$\leq$' on cardinals signifies that A is equipotent with some subset of B, and no information is included on equipotency in the other direction.

Cantor referred to the essential difference between finite and transfinite sets as the following [3].

1. Every finite set is such that it is equivalent to none of its subsets.

2. Every transfinite set has subsets which are equivalent to it.

Here 'equivalence' exactly means equipotency, and the result is a neat and concise way to delineate infinity.

Bernstein's Equivalence Theorem - Two sets each equipotent with a subset of the other are equipotent. In other words, if $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a} \geq \mathfrak{b}$ then $\mathfrak{a} = \mathfrak{b}$.

This result is quite incredible, since it defines '$=$' equality of (infinite) cardinals. See [2] for a proof.

If we were to assume that for every pair of cardinals $\mathfrak{a}$ and $\mathfrak{b}$, either $\mathfrak{a} \leq \mathfrak{b}$ or $\mathfrak{a} \geq \mathfrak{b}$ holds, then it would be clear that either $\mathfrak{a} < \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} > \mathfrak{b}$. This is Trichotomy, and so the partial ordering relation '$<$' would be a total ordering on the class of all cardinals. However, this assumption is unprovable in ZF since we have no way to show that there does not exist a pair of cardinals for which $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a} \geq \mathfrak{b}$ both fail.

Cardinal Addition $\mathfrak{a} + \mathfrak{b} = |A \cup B|$. For addition to be associative and commutative, we require A and B to be disjoint. By [2], from arbitrary representative A and B we can always create disjoint representatives, for example the set of all pairs (a,0) for $a \in A$ and the set of all pairs (b,1) for $b \in B$.

Cardinal Multiplication $\mathfrak{a}\mathfrak{b} = |A \times B|$ where $A \times B$ is the normal Cartesian product of A and B. This multiplication is associate and commutative, is distributive over addition, and satisfies $\mathfrak{a} \times 0 = 0$ and $\mathfrak{a} \times 1 = mathfraka$.

Cardinal Exponentiation $\mathfrak{a}^{\mathfrak{b}} = |A^B|$ where A^B denotes the set of all mappings from B into A.

As an extension, note that one of Georg Cantor's most famous theorem states that the set of all subsets of L has a greater power (cardinal) than L itself. Employing cardinal exponentiation, the statement is:

Cantor's Theorem (1892) - Every cardinal $\mathfrak{m}$ satisfies $\mathfrak{m} < 2^{\mathfrak{m}}$.

Proof. Here 2 represents the set $\{0, 1\}$. By [1], consider an arbitrary set L and the corresponding set of functions M, where $f : L \to \{0, 1\}$. Such an M is essentially equivalent to the set of all subsets of L, since each f can be regarded as the characteristic function of a subset of L (f takes the value 1 for elements of L that belong to the subset). Now, it is clear that the power of M is not less than that of L, for the functions which take the value 1 for just one argument $l \in L$ each, form a subset of M which is obviously equipotent to L. Assuming Trichotomy, the power of M will be strictly greater than the power of L if we can show that $|M| \neq |L|$. Were both sets equipotent, we could index the functions in M with elements of L, so that each function would appear under the form f_l. But such an assumption is contradictory; it is possible to define a new function $g : L \to \{0, 1\}$ that differs from every f_l by the l'th element. Specifically, take $g(l) = 1$ if $f_l(l) = 0$ and take $g(l) = 0$ if $f_l(l) = 1$. By construction, g differs from every f_l by at least one element, and therefore cannot be any of the f_l's. So there is no one-to-one mapping of M into L and therefore $|M| \neq |L|$. So $|M| > |L|$. This argument style is dubbed the *diagonalization argument*. $\square$

Well Ordering

An ordered set is said to be *well ordered* if every nonempty subset has a least element. Equivalently, a 'first' element per the '$<$' relation defined on this set. Every ordering of a finite set is a well ordering, because it is clear which element is 'first' in the list. The fact that $\mathbb{N}$ is well ordered is a fact equivalent to the principle of mathematical induction [2].

Zermelo's Well-Ordering Theorem - Assuming the Axiom of Choice, every (infinite) set can be well-ordered.

Proved in 1904, this theorem was the first explicit application of the Axiom of Choice. Ernst Zermelo furthermore showed that assuming that every (infinite) set can be well-ordered, the Axiom of Choice flows out as a consequence. Therefore the Axiom of Choice $\leftrightarrow$ Well-Ordering. Today, we apply the Axiom of Choice when discussing, for example, the Wierstrass Nested Interval Theorem, accumulation points, and coverings. These are all cases in which we state "pick some element out of an infinite set, such that..." without giving a specific construction of exactly which element that is.

By [3], the following theorems hold:

1. Every subset of a well-ordered set is also a well-ordered set.

2. Every set which is similar to a well-ordered set is also a well-ordered set.

Ordinal Numbers

While the cardinal of a set measures its quantity, the ordinal of a (well-ordered) set measures its "length" [2]. We will see how ordinals correlate to cardinals in the upcoming section on alephs. An ordinal is recursively defines as the set of all preceding ordinals.

$$0 = \emptyset \quad 1 = \{0\} \quad 2 = \{0, 1\} \quad n + 1 = \{0, 1, ..., n\}$$
$$\omega = \{0, 1, ...n, ...\} \quad \omega + 1 = \{0, 1, ...n, ..., \omega\}$$

Ordinals are well-ordered sets - indeed all other well-ordered sets are similar to some unique corresponding ordinal. This ordinal is called the set's *order type*, and is the principle reason why ordinals are so significant as a foundation for defining set sizes. The class of all ordinals is well ordered by inclusion, $\alpha \subset \beta$ implying $\alpha \leq \beta$. The ordinal of $\mathbb{N}$ is ω. The subset $\mathbb{N}_0$ of even numbers and the subset $\mathbb{N}_1$ of odd numbers are also of type ω. By [2], ordinals beyond $\omega + 1$ are not commonly encountered in everyday life. However note that the ordered set that lists $\mathbb{N}_0$ in its natural order followed by $\mathbb{N}_1$ in its natural order has ordinal $\omega + \omega$.

Curiously, none of the familiar sets $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Z}$ are well ordered by their natural ordering. In $\mathbb{R}$, consider a set of points that tend to a lower limit not included in the set, for example $1/n \to 0$. That open set will not have a least element. For $\mathbb{Z}$, consider the set $0, 1, -1, 2, -2, 3, -3, ...$ Again, there is no least element. The Well-Ordering Theorem says that some relation $<$ exists that would make these infinite sets well ordered. However, we can not actually construct it, just as we cannot construct the object "the set of all sets."

Segments - For each element z of a well-ordered set S, the set $\{x \in S : x < z\}$ in which $<$ is the ordering relation on S, is knowns as a *segment* of S. These facts from [2] will be used later.

1. Every segment of an ordinal is an ordinal.

2. Every well-ordered set is similar to its set of segments ordered by set inclusion under the mapping that takes each element to the segment it determines.

The Alephs

An aleph is the cardinal number of an infinite well-ordered set (a set that is similar to some infinite ordinal). Since subsets of well-ordered sets are well ordered, we can state

Any infinite cardinal less than aleph is an aleph.

Note that while the finite ordinals correspond one-to-one to the finite cardinals, an infinite number of ordinals correspond to every infinite cardinal. Alephs are exactly the subset of the cardinals which have ordinals corresponding to them. In the presence of the Axiom of Choice,

All sets are well orderable and all infinite cardinals are alephs.

The alephs are indexed ($\aleph_0$, $\aleph_1$, $\aleph_2$, etc.) and their sizes well ordered by the ordinals; hence any two alephs are comparable, and every nonempty set of alephs has a least member. To define the aleph's iteratively, we proceed as follows. The set Z_0 of all ordinals of cardinal $\aleph_0$ is uncountable. The number $\aleph_1$ is defined to be the cardinal of Z_0. Next, for each finite n, $\aleph_{n+1}$ is defined to be the cardinal of Z_n, the set of all ordinals of cardinality $\aleph_n$. The cardinal $\mathbb{N}_\omega$ is defined to be the sum of the $\aleph_n$ over all finite n, i.e., the cardinal of the union of the Z_n for all finite n.

For an arbitrary ordinal α, $\mathbb{N}_{\alpha+1}$ is the cardinal of Z_α, the set of all ordinals of cardinal $\aleph_\alpha$. Finally, the limit ordinal λ (an ordinal that, like ω, has no immediate predecessor), $\aleph_\lambda$, is the sum of all alephs of smaller index.

So for every transfinite cardinal number, there exists a next one greater, and also to every infinite ascending well-ordered set of transfinite cardinal numbers, there is a next one greater [3]. We will show a proof in the next section as to why there does not exist a greatest cardinal.

Trichotomy $\leftrightarrow$ Axiom of Choice

Gillman's article highlights the beautiful result that assuming Trichotomy and ZF, the Axiom of Choice falls out as a logical consequence. I will provide the details of this proof here. He begins with the idea that

Lemma. *The Axiom of Choice is equivalent to the proposition that every infinite cardinal is an aleph.*

Proof. It was discussed above that the Axiom of Choice is equivalent to the Well-Ordering Theorem. The Well-Ordering Theorem states that every infinite set can be well ordered. Hence, every infinite cardinal becomes a cardinal of a well-ordered set, and is by definition an aleph. So Well-Ordering is equivalent to the proposition that every infinite cardinal is an aleph, and we are done. $\square$

Gillman then proves the following theorem.

Theorem 1. *Hartogs-Sierpinski Theorem To each infinite cardinal* $\mathfrak{m}$ *is associated an aleph* $\aleph(\mathfrak{m})$ *satisfying the relations* $\aleph(\mathfrak{m}) \not\leq \mathfrak{m}$ *and* $\aleph(\mathfrak{m}) \leq 2^{2^{2^{\mathfrak{m}}}}$.

Proof. Let $\mathfrak{m}$ be an infinite cardinal, and let M be a set of cardinal $\mathfrak{m}$. Since 2^M is the set of all subset of M, a member of 2^M is a subset of M and a subset N of 2^M is a set of subset of M. Now, it may happen that the members of N are well ordered by set inclusion. Let W denote the set of all subsets N of s^M whose members are well ordered by set inclusion (as subsets of M). Since W is a set of subsets of 2^M, $W \subset 2^{2^M}$. Each member of W is a well-ordered collection of sets. We may therefore partition W into similarity classes. Let E denote the set of these classes. Now to each such class we can associate the ordinal common to all its members and interpret E as the set of these ordinals. The E is well ordered in the natural way, since all ordinals are comparable.

We now show that $|E| \not\leq |M|$. Suppose on the contrary that this is so. Then E is equipotent with some subset M_1 of M, The one-to-one correspondence between E and M_1 defines a well ordering of M_1 similar to that of E. Let S denote the set of segments of M_1. Then

$$\text{ord } S = \text{ ord } M_1 = \text{ ord } E$$

"ord X" denoting the ordinal of X. Since S is a set of subsets of M well ordered by set inclusion, it belongs to W; hence S belongs to some similarity class K in the collection E. The segment of E determined by K is then similar to S. But then by the double equality stated above, this segment of E is similar to E itself, which we know is not possible. By contradiction, it must be that $|E| \not\leq |M|$.

Observe next that, since M is infinite, E must be infinite as well; otherwise we would have $|E| < |M|$, which has just been ruled out. Thus E is an infinite well-ordered set, so $|E|$ is an aleph; we define it to be $\aleph(\mathfrak{m})$. Recalling that $|M| = \mathfrak{m}$, we see that $|E| \not\leq \mathfrak{m}$.

Finally, because E is a set of subsets of W, we have $E \subset 2^W$. Recalling that $W \subset 2^{2^M}$, it follows that $E \subset 2^{2^{2^M}}$. In terms of cardinals, $\aleph(\mathfrak{m}) \leq 2^{2^{2^\mathfrak{m}}}$. □

From $\aleph(\mathfrak{m}) \not\leq \mathfrak{m}$, we have that given any cardinal, there exists an aleph not smaller or equal to it. Since $\mathfrak{m}$ was arbitrary, we obtain the corollary

Corollary 1. *No cardinal is greater than all the alephs.*

Theorem 2. *Trichotomy is equivalent to the Axiom of Choice and to the Well-Ordering Theorem.*

Proof. First we derive Trichotomy from the other two propositions. If either one (and hence both) hold, then every infinite cardinal is an aleph. Since any two alephs are comparable, we conclude that any two cardinals are comparable - which is exactly Trichotomy. Conversely, assume Trichotomy, and consider any infinite cardinal $\mathfrak{m}$. From Hartogs-Sierpinski Theorem, $\aleph(\mathfrak{m}) \not\leq \mathfrak{m}$. Hence by Trichotomy, $\aleph(\mathfrak{m}) > \mathfrak{m}$. Thus $\mathfrak{m}$ is less than some aleph and therefore is itself an aleph. Since $\mathfrak{m}$ was arbitrary, all infinite cardinals are alephs. This proposition implies both the Axiom of Choice and the Well-Ordering Theorem. Incredibly, we have shown the following relations between statements which are unprovable within the Zermelo-Frankel axioms. □

$$\text{Trichotomy} \leftrightarrow \text{Axiom of Choice} \leftrightarrow \text{Well-Ordering Theorem}$$
$$\leftrightarrow \text{"every infinite cardinal is an aleph"}$$

Historical Evolution of the Continuum Hypothesis

Now to apply all the definitions we have covered. Note that all the following restatements of the Continuum Problem and Hypothesis feature cardinals and ordinals. These developed in step with the creation and development of these concepts.

Continuum Problem (1878) - How many distinct cardinals $\bar{\bar{A}}$ exist, where A is an infinite subset of $\mathbb{R}$ [4].

Continuum Problem (today) - Where does $\mathfrak{c}$, the cardinal of the continuum (the set $\mathbb{R}$ of real numbers) lie in the hierarchy of the alephs $\aleph_1, \aleph_2, ...\aleph_\alpha, ...$? Note that Cantor ruled out $\aleph_0 = |\mathbb{N}|$ when he showed that $|\mathbb{R}| > |\mathbb{N}|$.

Cantor's second form of the Continuum Hypothesis appeared in 1883, after he had developed his theory of transfinite ordinal numbers.

Continuum Hypothesis (1883) - The set $\mathbb{R}$ has the cardinality of the set of all countable ordinal numbers.

In the 1890s, Cantor delivered his third and final form of the Continuum Hypothesis. This occurred after he developed the notion of exponentiating cardinal numbers and established the symbolism for alephs (i.e. the cardinal numbers of infinite well-ordered sets). This version abbreviates to:

Continuum Hypothesis (1890s) - $2^{\aleph_0} = \aleph_1$.

The left hand side of the expression used his new concept of cardinal exponentiation to represent the real numbers as binary sequences. In particular, by [3], Cantor proved that $2_0^{\aleph} = \mathfrak{c}$. The idea is that a real number can be represented in binary by an uncountable string of 1's and 0's. Therefore $2_0^{\aleph}$, where $2 \equiv \{0, 1\}$ in the cardinal sense of exponentiation, represents all such possible strings, and so is equivalent to the set of the continuum.

In 1882, Cantor generalized CH slightly when he asserted that the set of all real functions has the cardinality of the third number-class, $\aleph_2$. However he never generalized CH further [4]. Today, the Generalized Continuum Hypothesis is what happens when the subscript in CH is replaced by an arbitrary ordinal α.

Generalized Continuum Hypothesis - $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. In other words, for every cardinal $\mathfrak{m}$, no cardinal lies strictly between $\mathfrak{m}$ and $2^{\mathfrak{m}}$.

This form was developed by Charles S. Peirce. The Continuum Problem is no longer an open question since it has been proven that GCH can neither be proved nor disproved within the Zermelo-Frankel axioms. An argument persists about whether or not GCH should be appended as a formal axiom in ZF. The current tradition is simply to state it as an assumption whenever one explicitly assumes it.

References

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