The Game of Hex and Brouwer's Fixed-Point Theorem

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Contents

1. Abstract

This paper is a summary by a math student of **The Game of Hex and The Brouwer Fixed-Point Theorem** by David Gale. Section 2 introduces the game of Hex. Section 3 dives into a theorem about the game that in section 4 is used to show the equivalence of the Hex Theorem and Brouwer Fixed-Point Theorem. In the final section, the Hex Theorem is generalized from 2 to n dimensions, where we develop an algorithm for finding approximate fixed points of continuous mappings.

2. The Game of Hex

2.1 Introduction

Hex is traditionally a 2-player game invented by the Danish engineer and poet Piet Hein in 1942 and rediscovered at Princeton by John Nash in 1948. It was produced commercially for many years but has been out of print for many years in the US. In other countries, however, it is still very popular.

2.2 Setup and Rules

Figure 2.1 below shows a typical 11×11 Hex board. The two players are denoted by either the letter x or o, and the players move alternatively. Play is very simple, and the players take turns marking any previously unmarked hexagon with an x or o respectively. The game is won by the x- (resp. o-) player if he/she has succeeded in marking a connected set of tiles which joins the two boundary regions X and X' (resp. O and O'). A set S of tiles is determined to be "connected" if any two members h and h' of S can be joined by a path $P = (h = h^1, h^2, \cdots, h^m = h')$ where h^i and h^{i+1} are adjacent.

Figure 2.1 denotes a game in which neither player has won and it the o-player's move, but the x-player has a guaranteed win in 3 moves if he/she plays on the shaded tiles. The potentially winning connected set is indicated by the tiles with underlined x's.

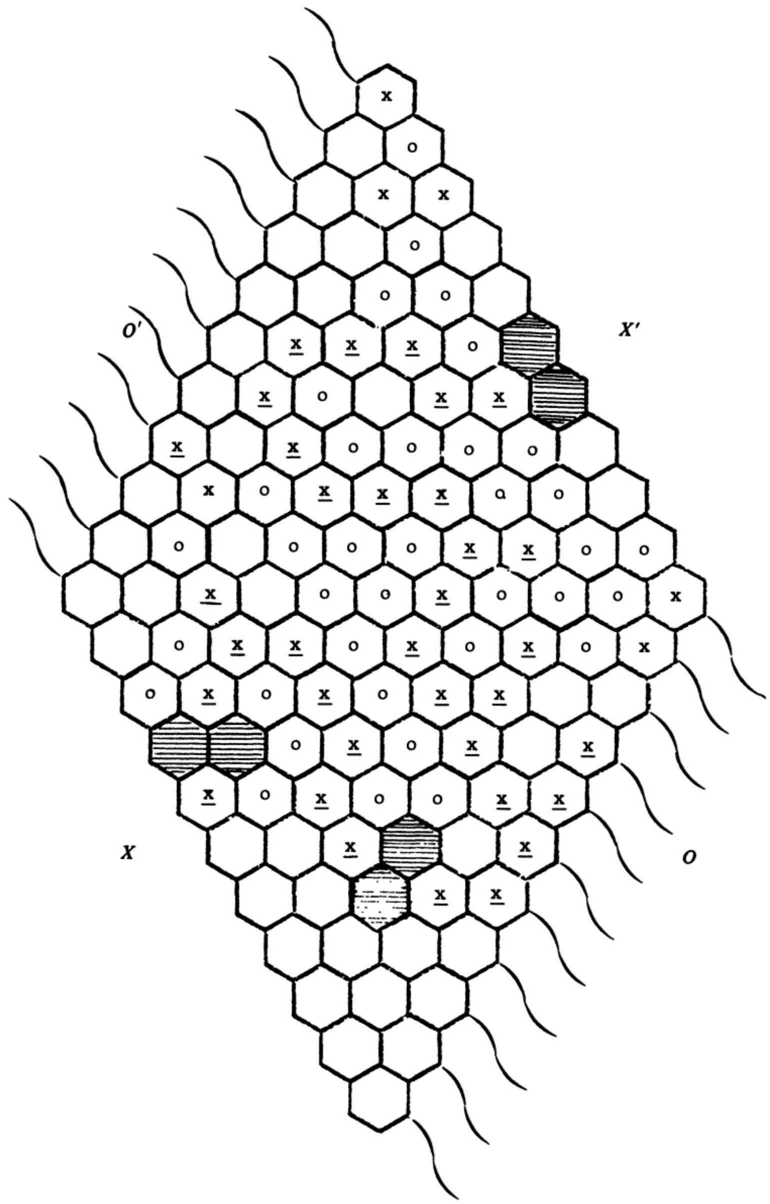

Figure 2.1: A garden-variety Hex game

3. Hex Theorem

One appealing feature of Hex is that it can never end in a draw. This is because one player can block the other player only by completing his/her own chain. More precisely,

Theorem 1 (Hex Theorem). *If every tile of the Hex board is marked either x or o, then there is either an x-path connecting X and X' or an o-path connecting O and O'.*

First, let's think about this theorem intuitively. We can imagine that the X-regions are portions opposite the banks of the river "O" (as depicted in Figure 2.1), and that player x is building a dam by putting down "stones". It is clear that if player x will be successful in damming the river if and only if he has placed his stones in a way that allows him to walk from one side (X) to the other (X'). In addition, one can strengthen the Hex Theorem by appending at the end of the statement "but not both". However, although intuitive (if player x succeeds in constructing a path from X to X', he will have dammed the river and prevented any flow from O to O'), it is much harder to prove. The analysis to follow and the relation to the Brouwer Theorem depends only on the no-draw property, so that is what we will prove below.

Proof. Suppose the board has been covered by x's and o's as in Figure 3.1. X-face will denote either a tile marked x or one of the regions X or X'. O-face is defined analogously.

We consider the edge graph Γ of the Hex Board, which includes additional edges ending in the vertices marked u, u', v, and v' to separate the four boundary regions, also shown in Figure 3.1. We now present an algorithm for finding a winning set on the completely marked board: We make a *tour* along Γ, starting from the vertex u and following the simple rule of always proceeding along an edge which is the boundary between an X-face and an O-face. The edge from u has this property since it separates X and O. Note further that this *tour* determines a unique path within Γ; suppose one has proceeded along some edge e and arrived at vertex w. Then two of the three faces incident to w are those of which e is the common boundary, so one is an X-face and one is an O-face. The third face is arbitrary, but in either case there is exactly one edge e' which satisfies the touring rule. Figure 3.1 shows this fact (in the next sections, the situation will be generalized so that no picture will be necessary).

Before we continue, we show an important result from graph theory:

Lemma 2 (Graph Lemma). *A (finite) graph whose vertices have degree at most two is the union of disjoint subgraphs, each of which is either (i) an isolated vertex, (ii) a simple cycle, (iii) a simple path.*

Proof. By structural induction on the number of edges, it is easy to see that the three options of subgraphs above are the only ones that are possible. □

If we consider only the subgraph Γ' of Γ consisting of edges which separate an X-face from an O-face, then the hypothesis of the Graph Lemma is satisfied and the conclusion shows that our *tour* of Γ will not cycle. In other words, our touring rule guarantees that *we will never revisit any vertex.* This, combined with the fact that there are a finite number of vertices, gives us the result that the tour must terminate; but the only possible terminals are the vertices u', v, and v'. Now note that each of these three vertices is incident to one of the regions X' or O'. This guarantees the existence of a connected path of either x's or o's across the terminal points, which in turn guarantees a winner of the game, which is what we wanted to show. □

Figure 3.1 shows the graph Γ' for the completely marked board where player o has won.

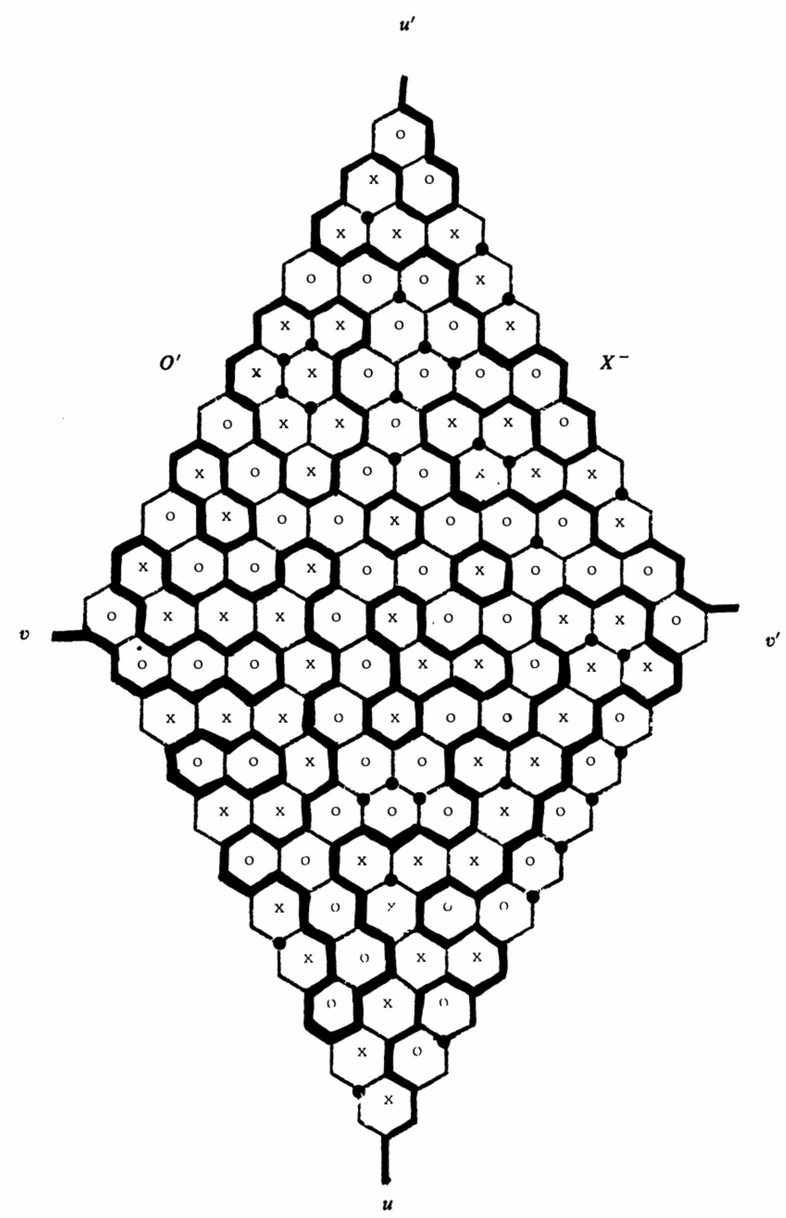

Figure 3.1: A garden-variety completed Hex game

4. Equivalence of the Hex and Brouwer Theorems

4.1 Brouwer Fixed-Point Theorem

In this section, we state the theorem at hand. Then, we will show that the Hex theorem described earlier is equivalent to the stated theorem.

Theorem 3 (Brouwer Fixed-Point Theorem)**.** *Let f be a continuous mapping from the unit square I^2 into itself. Then there exists $x \in I^2$ such that $f(x) = x$.*

4.2 New Representation and Some Notation

For analytic purposes it is convenient to use a different but equivalent model for the Hex board. When John Nash rediscovered the game in 1948, he thought of it as being played on a checkerboard where two squares were considered adjacent if they were next to each other horizontally, vertically, or along a positively sloping diagonal. It can easily be seen that this is equivalent to the original hexagon arrangement. This representation can also be easily "arithmetized" in a way which can generalize to n dimensions. We present some notation for this alternative representation of the board.

- Z^n denotes the lattice points of R^n
- For $x \in R^n$, $|x| = \max_i x_i$
- For $x \neq y \in R^n$, $x < y$ if $x_i \leq y_i$ for all i
- The points x and y are called *comparable* if $x < y$ or $y < x$
- The (two-dimensional) Hex board B_k of size k is a graph whose vertices consist of all z in Z^2 with $(1,1) \leq z \leq (k,k)$.
- Two vertices z and z' are *adjacent* (span an edge) in B_k if
  - (i) $|z - z'| = 1$
  - (ii) z and z' are comparable

Figure 4.1 gives the graphical representation of a Hex board of size 5. The boundary edges are labelled with the points of the compass (N, S, E, W) and consist of the vertices $z = (z_1, z_2)$ of B_k with $z_2 = k$, $z_2 = 0$, $z_1 = k$, and $z_1 = 0$ respectively. The horizontal player tries to make a path connecting E and W, while the vertical player tries to connect N and S. We can now restate the Hex Theorem:

Theorem 4 (Hex Theorem)**.** *Let B_k be covered by two sets H and V. Then either H contains a connected set meeting* E *and* W *or V contains a connected set meeting* N *and* S*.*

4.3 Equivalence of Hex and Brouwer

4.3.1 "Hex" implies "Brouwer"

First, we show that "Hex" implies "Brouwer". In other words, that the result of the Hex Theorem can be used to prove the Brouwer Fixed-Point Theorem.

Theorem 5. *"Hex" ⇒ "Brouwer"*

Proof. Let $f : I^2 \to I^2$ be given by $f(x) = (f_1(x), f_2(x))$. From compactness of I^2 it suffices to show that for any $\epsilon > 0$ there exists $x \in I^2$ such that $|f(x) - x| < \epsilon$. By uniform continuity of f we know that, given $\epsilon > 0$, there is a $\delta > 0$ such that $\delta < \epsilon$ and $|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \epsilon$.

Now consider the Hex board B_k where $1/k < \delta$. We will define four subsets H^+, H^-, V^+, and V^- of B_k as follows:

$$\begin{aligned}
H^+ &= \{z | f_1(z/k) - z_1/k > \epsilon\} \\
H^- &= \{z | z_1/k - f_1(z/k) > \epsilon\} \\
V^+ &= \{z | f_2(z/k) - z_2/k > \epsilon\} \\
V^- &= \{z | z_2/k - f_2(z/k) > \epsilon\}
\end{aligned}$$

where $z = (z_1, z_2)$. Intuitively, a vertex z belongs to H^+, H^-, V^+, and V^- according as z/k is moved by f at least ϵ units to the right, left, up, or down, respectively.

The theorem will be proved if we can show that these four sets do not cover B_k; if vertex z lies in none of them, then $|f(z/k) - z/k| < \epsilon$, which by the compactness of I^2 is enough to obtain $f(z/k) = z/k$. The key observation is now that the disjoint sets H^+ and H^- (V^+ and V^-) are not contiguous (in other words, the two sets in question do not have any pairwise adjacent elements). Explicitly, that means that if $z \in H^+$ and $z' \in H^-$, then

$$f_1(z/k) - z_1/k > \epsilon$$

and

$$z_1'/k - f_1(z'/k) > \epsilon$$

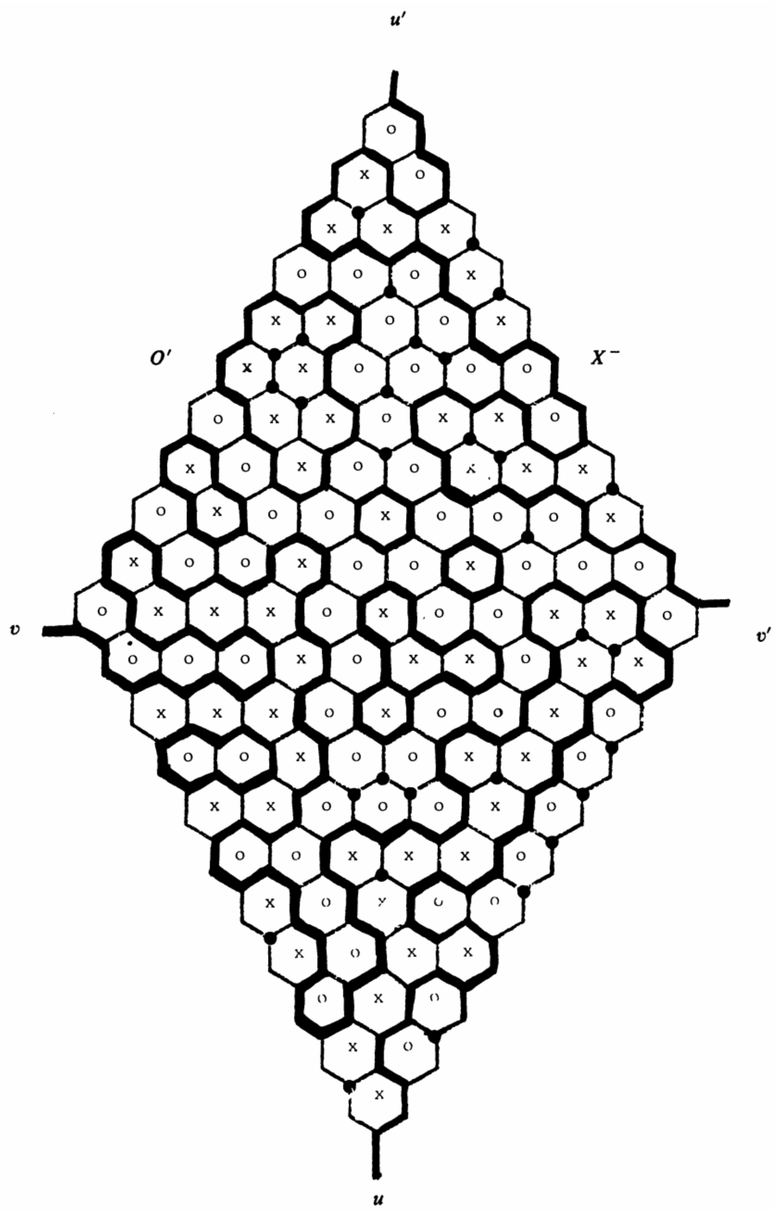

Figure 4.1: Graphical representation of size 5 Hex board

Adding these two gives

$$f_1(z/k) - f_1(z'/k) + z_1'/k - z_1/k > 2\epsilon$$

but by the choice of δ and k, we have that $z_1'/k - z_1/k < \delta < \epsilon$, so

$$z_1/k - z_1'/k > -\epsilon$$

Adding the two above inequalities gives

$$f_1(z/k) - f_z(z'/k) > \epsilon$$

The above shows that z and z' are not adjacent; if they were we would have $|z/k - z'/k| = 1/k < \delta$, which contradicts the choice of δ.

Similarly, V^+ and V^- are not contiguous. Now, we let $H = H^+ \cup H^-$, $V = V^+ \cup V^-$, and we suppose that Q is a connected set lying on H. From the previous paragraph Q must lie entirely in H^+ or H^-, because they are disjoint. But note that H^+ cannot meet E since f maps I^2 to itself, so no point on the right boundary can be mapped to the right. Similarly, H^- does not meet W, so Q cannot connect E and W. Similarly, V contains no connected set meeting both N and S. So by the Hex Theorem, the sets H and V do not cover B_k, proving the existence of the Brouwer Fixed-Point. □

4.3.2 "Brouwer" implies "Hex"

The reverse of the above. This proof makes use of the fact that the Hex board B_k gives a triangulation of the $k \times k$ square I_k^2 in R^2. In other words, each point of I^2 is uniquely expressible as a convex combination of some set of (at most three) vertices, all of which are pairwise adjacent. These vertices are the edges and triangles in Figure 4.1.

In addition, we use the fact that any mapping f from B_k into R^2 extends to a continuous piecewise linear map $\hat{f}$ on I_k^2. Specifically, if $x = \lambda_1 z^1 + \lambda_2 z^2 + \lambda_3 z^3$ where the λ_i are non-negative numbers summing to 1 and the z^i are the uniquely determined points used in the triangulation, then by definition, $\hat{f}(x) = \lambda_1 f(z^1) + \lambda_2 f(z^2) + \lambda_3 f(z^3)$.

Theorem 6. *"Brouwer" ⇒ "Hex"*

Proof. First, we assume that B_k is partitioned by two sets H and V. We define four sets as follows: let $\hat{W}$ be all vertices connected to W by an H-path, and let $\hat{E} = H - \hat{W}$. Let $\hat{S}$ be similarly defined as all vertices connected to S by a V-path, and let $\hat{N} = V - \hat{S}$. By definition, $\hat{W}$ and $\hat{E}$, and $\hat{N}$ and $\hat{S}$, are not contiguous.

For contradiction, assume there is no H-path from E to W and no V-path from N to S. Now let e^1 and e^2 be the unit vectors of R^2 and define $f : B_k \to B_k$ by

$$f(z) = \begin{cases} z + e^1, & z \in \hat{W} \\ z - e^1, & z \in \hat{E} \\ z + e^2, & z \in \hat{S} \\ z - e^2, & z \in \hat{N} \end{cases}$$

Each of the four cases of $f(z)$ can be shown to satisfy $f(z) \in I_k^2$. For the case of $z + e^1$, we have

$$z + e^1 \notin B_k \Leftrightarrow z \in \mathrm{E}$$

but by our assumption that there is no H-path from W to E, we can see that $\hat{W}$ does not meet E. Similarly for the other 3 cases, we have that $\hat{E}$ does not meet W, $\hat{N}$ does not meet S, and $\hat{S}$ does not meet N.

We now extend f with our piecewise linear map (which we know exists) to all of I_k^2 to obtain the contradiction by showing that f has no fixed point. We are able to do this through the following lemma:

Lemma 7. *Let z^1, z^2, z^3 be vertices of any triangle Δ in R^2 and let $\hat{\rho}$ be the simplical (piecewise linear) extension of the mapping ρ defined by $\rho(z^i) = z^i + v^i$ where v^1, v^2, v^3 are given vectors. Then f has a fixed point if and only if 0 lies in the convex hull of v^1, v^2, v^3.*

Proof. Let $x = \lambda_1 z^1 + \lambda_2 z^2 + \lambda_3 z^3$. Then $\hat{\rho}(x) = \lambda_1(z^1 + v^1) + \lambda_2(z^2 + v^2) + \lambda_3(z^3 + v^3)$ and x is fixed if and only if $\lambda_1 v^1 + \lambda_2 v^2 + \lambda_3 v^3 = 0$. □

The key fact is again the non-contiguousness of $\hat{W}$ and $\hat{E}$ and $\hat{S}$ and $\hat{N}$. Applying the above lemma here means that if one considers the three vertices of any triangle of mutually adjacent vertices, then it will never be the case that one of these vertices is translated by e^i and another by $-e^i$, so the three vertices are translated by vectors which all lie in the same quadrant of R^2. Hence, they do not have 0 in their convex hull. Because no points are mapped to themselves, we have obtained a fixed-point-free mapping, contradicting the Brouwer Fixed-Point Theorem. Because we assumed the negation of "Hex", we have shown that "Brouwer" ⇒ "Hex". □

5. The n-dimensional Hex Theorem

Now we generalize the game of Hex into n dimensions and prove the corresponding version of the Hex theorem. Then, we will show an application of the theorem to find the fixed points described in Brouwer's theorem.

5.1 n-dimensional Hex theorem

To begin our discussion of the n-dimensional Hex theorem, we must introduce a formal definition of an n-dimensional Hex board and associated terms. The following definitions are direct generalizations of the ones used previously for $n = 2$.

Definition 1 (n-dimensional Hex board). *The n-dimensional Hex board of size k, H_k^n, consists of all vertices (denoted as vectors) $z = (z_i, \ldots, z_n) \in Z^n$ such that $1 \leq z_i \leq k$, $i = 1, \ldots, n$. To avoid notational clutter, we will from now on denote H_k^n by H.*

Definition 2 (Adjacent vertices). *A pair of vertices z and z' is called adjacent if $|z - z'| = 1$ and z and z' are comparable (similarly to above meaning that $z_i \geq z_i'$ or $z_i' \geq z_i$ for all i).*

Definition 3. *For each i, we define*

$$H_i^- = \{z|z \in H, z_i = 1\}$$
$$H_i^+ = \{z|z \in H, z_i = k\}$$

Definition 4 (Labeling). *A labeling of H is a mapping L from H to $N = \{1, 2, \ldots, n\}$.*

Now we are properly equipped to state our theorem:

Theorem 8 (Hex Theorem). *For any labeling L there is at least one $i \in N$ such that $L^{-1}(i)$ contains a connected set which meets H_i^- and H_i^+. Such a set will be called a winning i-set.*

Before continuing with the proof of this theorem, we chip in that the proof that the n-dimensional Hex and Brouwer Theorems are equivalent is obtained by generalizing mechanically the two-dimensional proof from the last section. The proof is left to the reader as an exercise. We now give a proof of the n-dimensional Hex Theorem.

Proof. Let the *augmented Hex board* $\hat{H}$ to be all $z \in Z^n$ such that $0 \leq z_i \leq k+1$. Further, let

$$F_i^- = \{z | z \in \hat{H}, z_i = 0\}$$
$$F_i^+ = \{z | z \in \hat{H}, z_i = k+1\}$$

be the *faces* of $\hat{H}$. Let e_i be the ith unit vector in R^n, and let e be the n-vector, all of whose components are 1. In other words, $e = (1, 1, \ldots, 1)$.

We provide a definition of a *simplex* and associated terms to aid us in our proof:

Definition 5 (Simplex)**.** *A simplex of R^n is an $(n+1)$-tuple of vertices $\sigma = (z^0, \ldots, z^n)$, where $z^i \in Z^n$ and both*

(i) $z^{i+1} - z^i = e^r$ *for some* $r \in N$

(ii) $z^{i+1} - z^i \neq z^{j+1} - z^j$ *for* $i \neq j$

Note that for $\sigma \subset \hat{H}$, every pair of z^i and z^j are adjacent.

Definition 6 (i-facet)**.** *The i-facet of σ is the n-tuple*

$$\tau^i = (z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^n)$$

This can practically be thought of as σ with the i-th term removed.

Here we draw attention to an important simplex to follow: $\sigma^0 = (0, e^1, e^1 + e^2, \ldots, e)$. Note two important properties of σ^0. First, all vertices of σ^0 lie in H. Second, its n-facet $\tau^0 = (0, \ldots, e^1 + e^2 + \ldots + e^{n-1})$ satisfies $\tau^0 \in F_n^-$.

Definition 7 (i-neighbor)**.** *For $0 < i < n$, the i-neighbor of σ is the simplex $\tilde{\sigma}$ with the same vertices as σ, but z^i is replaced with $\tilde{z}^i = z^{i-1} + z^{i+1} - z^i$. $\tilde{z}^i$ is called the* mate *of z^i with respect to σ. We define the 0-neighbor of σ to be $\tilde{\sigma} = (z^1, \ldots, z^n, \tilde{z}_0)$, where $\tilde{z}^0 = z^1 + z^n - z^0$ and the n-neighbor of σ to be $\tilde{\sigma} = (\tilde{z}^n, z^0, \ldots, z^{n-1})$, where $\tilde{z}^n = z^{n-1} + z^0 - z^n$.*

Note that $\tilde{\sigma}$ satisfies the conditions of a simplex. Also note that $\tilde{\sigma}$ is the i-neighbor of σ if and only if σ is the i-neighbor of $\tilde{\sigma}$, with their intersection being their i-facets. Also note that if $\tilde{\sigma}$ is the 0-neighbor of σ, then σ is the n-neighbor of $\tilde{\sigma}$, and vice versa. The intersection of these 0-neighbor, n-neighbor pairs is the n-facet of σ and the 0-facet of $\tilde{\sigma}$.

We extend our labeling L to $\hat{H}$ by defining for z on the faces of $\hat{H}$:

$$\begin{aligned} L(z) &= \min\{i | z \in F_i^-\} && \text{if } z \in \cup_{i=1}^n F_i^- \\ &= \min\{i | z \in F_i^+\} && \text{otherwise} \end{aligned}$$

Below is our last definition before hopping into the proof.

Definition 8 (Completely labeled)**.** *A simplex σ or facet τ is completely labeled (c.l.) if L maps σ or τ onto all of N. Note that σ^0 and its corresponding τ^0 are c.l. since from the above extension, the vertices of τ^0 have labels 1 through n. This is easily verified for all vertices.*

Now that we have laid the groundwork, we can continue with our proof. Define a graph Γ with nodes being all the c.l. simplexes in $\hat{H}$. Simplexes are adjacent if they are neighbors an their intersection is a c.l. facet. We proceed in the n-dimensional extension of the original Hex Theorem proof with the following lemma.

Lemma 9. *Every node σ of Γ has degree $n \leq 2$.*

Proof. Let $\sigma = (z^0, \ldots, z^n)$ be an arbitrary c.l. simplex in $\hat{H}$. Then since there are $n+1$ vertices and n possible labels, there must be exactly two vertices z^i, z^j with the same label. Then $\tilde{\sigma}$ is a c.l. neighbor of σ if and only if it is the i- or j-neighbor of σ. That way the i- or j-facet will still contain a vertex with label for each of 1 through n, since there was initially a duplicate label. Any other case will not be c.l. □

Next, we can show that the simplex σ^0 has exactly one c.l. neighbor. Suppose $L(e) = i > 1$. Then the n-neighbor of σ^0 is $\tilde{\sigma}^0 = (-e^n, 0, \ldots, e^1 + \ldots + e^{n-1}) \notin \hat{H}$, since $-e^n < 0$. The other vertex of σ^0 with label i is $e^1 + \ldots + e^{i-1}$, with mate in $\hat{H}$, so σ^0 only has degree 1. If $L(e) = 1$, it can be seen that $L(0) = 1$ as well, and from the above argument for the n-neighbor, only the $0 - neighbor$ remains in $\hat{H}$.

Applying **Lemma 2** from above, we can show that σ^0 is the initial node of a simple path $P = (\sigma^0, \sigma^1, \ldots, \sigma^m)$. This is because it has only one adjacent node in the graph. We can now prove the following.

Lemma 10. *The c.l.-facet of σ^m lies on some face F_i^+ of $\hat{H}$.*

Proof. Let $\sigma^m = (z^0, \ldots, z^n)$, where σ^m has only one neighbor. We know such a σ^m must exist, otherwise σ^0 would have two neighbors. Then it must be that for some i, the mate $\tilde{z}^i$ of z^i is not in $\hat{H}$. For $0 < i < n$, $z^{i-1} < \tilde{z}^i < z^{i+1}$, so $\tilde{z}^i \in \hat{H}$.

Now suppose σ^m has no 0-neighbor. Then $\tilde{z}^0 = z^1 + z^n - z^0 \notin \hat{H}$. Let $z^1 - z^0 = e_r$. Then $\tilde{z}^0$ is not in $\hat{H}$ only if $z_r^n = k+1$. In other words, this is only possible if the r-th element of z^n is $k+1$. However, from condition (ii) of simplexes, we see that $z_r^i = k+1$ for all $i > 0$. So the 0-facet of σ^m is c.l. and lies on F_r^+.

The other possibility is that σ^m has no n-neighbor. This would mean that $\tilde{z}^n = z^0 + z^{n-1} - z^n$ is not in $\hat{H}$. Now let $z^n - z^{n-1} = e_r$. This implies that $z_r^0 = 0$, which as above, implies that $z_r^i = 0$ for $i < n$, so the n-facet τ of σ^m satisfies $\tau \in F_r^-$. From our definition of the label extensions onto $\hat{H}$, τ can only have labels $i \leq r$, so because τ is c.l., it must be the case that $r = n$, and $\tau \in F_r^-$. Next, we show that if $\tau = (z^0, \ldots, z^{n-1})$ is c.l. and lies in F_n^-, then

$\tau = \tau^0$. First, consider z^0. If $z^0 > 0$, then $z^i_r > 0$ for all i, and r could not be a label of τ. Therefore, $z^0 = 0$. Likewise for $z^1 - 0 = e^1$, and so forth. Thus, if σ^m has no n-neighbor, then $\tau = \tau^0$. However, σ^m cannot have τ^0 as a facet because then the path to σ^0 would be complete, and we know that P is a path. $\square$

From these lemmas the Hex Theorem is now proved. From the way P was constructed, if we choose the vertices labeled i from each simplex in P, then we have our winning i-set. Since adjacent simplexes in the path P are neighbors, the vertices labeled i are adjacent by definition. Thus, they form a connected path in H. Next, the vertex $e^1 + \ldots + e^{i-1}$ of σ^0 lies on F_i^- and has label i, and σ^m has a c.l. facet on F_i^+, so the constructed set meets F_i^- and F_i^+. Because it meets the faces of $\hat{H}$, it also meets the faces of H. $\square$

5.2 One Consequence of Theorem

Below is an algorithm using the n-dimensional Hex Theorem to find points which are arbitrarily close to fixed points of the mappings described in Brouwer's Theorem.

Given the labeling L, look at the label of e. If $L(e) = i$, bring in the mate $\tilde{z}^i$ of the vertex z^i (with respect to σ^0). Call the new simplex σ^1. Now again look at the label of $\tilde{z}^i$. There is exactly one other vertex of σ^1 with this label. Replace it with its mate, and keep going. Each time a new "mate" vertex is brought into a simplex σ^k, drop its mate from σ^k. Our proof above shows that with this method, a winning i-path will be constructed (these steps are the exact same as finding the next neighbor simplex in P an traversing to it).

With the ideas discussed thus far, we are able to locate for any continuous function f of I^n into itself points which are moved by arbitrarily small amounts. To be more precise, we say the point x in the n-cube I^n is *moved in the direction* i if

$$|f(x) - x| = |f_i(x) - x_i|$$

To find a nearly fixed point, first choose an arbitrary Hex board H^n_k. k can be arbitrary, but larger values will give better approximations of the points in question. Let the labels $L(z)$ be defined to be the direction in which $\frac{z}{k}$ is moved under f. Then, starting with σ^0, run the algorithm discussed above to find the winning i-path. By the Hex Theorem $\Rightarrow$ Brouwer Theorem proof discussed previously, we are guaranteed that there will be two vertices, z and z', which are adjacent and such that both are moved in the direction i, but in different sign. Precisely, $f_i(z) - z_i \geq 0$, and $f_i(z') - z'_i \leq 0$, or vice versa. This follows from the fact that points on the two faces cannot be moved further into the faces (outside the cube). However, if k is large, then neither z nor z' could have been moved a large distance if they are moved in opposite directions. Thus, z and z' are points close to the fixed points stated to exist by Brouwer's Theorem.

6. Closing Remarks

There are many proofs of Brouwer's Fixed Point Theorem. Many are much too hard for me to understand. However, I thoroughly enjoyed reading Gale's discussion of this very well known theorem. Not only did he use something that I initially believed to be completely unrelated (the game of Hex) to prove an abstract theorem, but he did it in a way that someone with a basic understanding of analysis could understand. I hope that this short summary of Gale's article will pique your interest as his article did for me. His discussion touches more on game theory and the subject of "fixed-point chasing". I urge all who found this paper interesting to read Gale's article cited below, as he goes much more in depth and provides more material for the truly motivated learner.

7. References

1. Gale, David. "The Game of Hex and the Brouwer Fixed-Point Theorem." *The American Mathematical Monthly* 86, no. 10 (1979): 818-27.