Transient and Recurrent Random Walks on Integer Lattices

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May 2017

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1 Introduction

In Jonathan Novak's paper, he seeks to prove a result that is at the core of the theory of random walks, Polya's Theorem, through use of modernized methods. Novak aims to prove this theorem through use of concepts developed by Laplace and de Moivre. Novak's audience is those who have an understanding of basic probability theory, analysis, differential equations and who are studying random walks and are seeking insight on different methods of proof for a core concept in that field. The most interesting results that can be extrapolated from this paper are not only the results of the theorem itself, in their elegance of statement and implication, but are also the intersections created between differing fields of mathematics such as combinatorics, ODEs, and analysis.

2 Overview

In Probability Theory, an **event** is a set of outcomes from an experiment being performed. We say that an event occurs with probability ρ (where $\rho \in [0,1]$) if it occurs ρ% of the time the experiment is performed. When we are looking at finite **sample spaces** (the set of all outcomes of a specific experiment), not much care needs to be taken when dealing with finite sample spaces as events having probability zero and one respectively imply that they will **never** and **surely** happen. However, when we address infinite sample spaces ($\mathbb{R}, \mathbb{R}^2$, etc.) more care must be taken as events that have probability zero can occur and events with probability one do not always occur. This is because there can be non-empty subsets of this infinite sample space with probability zero. To denote these two ideas, we say that events with probability zero **almost never** happen and events with probability one **almost always** happen.

Suppose that we have an integer lattice of some dimension d ($\mathbb{Z} \times \overbrace{\cdots}^{d \text{ times}} \times \mathbb{Z}$). Take a particle at a point in this lattice. Suppose that at each unit of time, the particle will jump to a random neighboring lattice point with equal probability of jumping to any neighboring lattice point. This is known as a **simple random walk** on $\mathbb{Z}^d$.
We can categorize these walks as either being **recurrent** or **transient** with the former meaning the particle returns to its starting position with probability one and the latter categorizing random walks that are not recurrent.

Pólya's Theorem states that: **The simple random walk on $\mathbb{Z}^d$ is recurrent in dimensions $d = 1, 2$ and transient in dimension $d \geq 3$.[1]** and we will now go through the derivation of this result.

3 Notation

This paper uses these following notations.
-$prob(E)$, where E is an event, denotes the probability that E will occur.
-$\bigsqcup_I A_i$, with A_i denoting sets indexed by elements $i \in I$, denotes the disjoint union of these sets

$$\bigsqcup_I A_i = \bigcup_{i \in I} \{(x, i) : x \in A_i\}$$

-$\mathcal{B}(f(z))$ denotes the Borel Transform of a function which is defined by:

$$\mathcal{B}(f(z)) = \int_0^\infty f(tz)e^{-t}dt$$

4 Decomposition of Loops

Definitions: *A* ***generating function*** *is an infinite sequence of numbers that are treated as the coefficients of a series expansion. For a more rigorous treatment, the reader is referred to chapter 2, specifically sections 2.2 and 2.3 in* ***[3]***
Ordinary generating functions *are those which act as coefficients for a conventional power series* $\sum a_n x^n$.

In this section, we seek to represent the probability, $\mathbf{p}$*, that a random walk on* $\mathbb{Z}^d$ *returns to its original position* x_0 *as a relationship between two generating functions so that we can apply limiting operations and further mathematical methods to them to derive a result about transience/recurrence of random walks.*

Utilizing the mutual exclusivity of the events denoted by E_n and the fact the probability of an event that is the disjoint union of a sequence of events E_n is simply the sum of the probabilities p_n where $p_n = prob(E_n)$ **[Paper, pg. 1]**, we get that

$$E = \bigsqcup_{n\geq 0} E_n$$

and, as a result

$$p = \sum_{n\geq 0} p_n$$

We call a random walk on $\mathbb{Z}^d$ that starts and ends at the same point a **loop**. We consider walks of zero length to be loops and call these loops **trivial**. A nontrivial loop that is not the concatenation (concatenation meaning the "stitching" of said loops end point to the start of the next loop and the end point of the final loop to the start of the first) of two nontrivial loops is defined as **nondecomposable**.

Now consider a point $x_0 \in \mathbb{Z}^d$ and take ℓ_n to denote the number of loops of length n that are based at the x_0 and r_n to denote the number of said loops that are nondecomposable. We can see that $\ell_0 = 1$ (as a trivial loop occurs for $n = 0$) and $r_0 = 0$ as this trivial loop that occurs is not decomposable.

We can see that

$$\ell_n = \sum_{k=0}^{n} r_k \ell_{n-k} \quad (4.1)$$

for all $n \geq 1$ through a simple recursion argument. Consider any loop that has length n and take the first nondecomposable component of it, call the length of this component k, we can then see that there are r_k of these loops. We then have that the remaining $n-k$ steps along this walk also form a loop which may be decomposable, giving us that there are ℓ_{n-k} of these loops. Now, consider the quantity $(2d)^n$, which is the total number of n-length walks emanating from a point in $\mathbb{Z}^d$. This can be seen in any dimension d, there are $2d$ possible directions to move in and this "choice" can be made n times giving us this quantity. We can divide both sides of equality in 4.1 by this quantity to get a representation of q_n(the probability that the particle is at its original position after n steps) stated as.

$$q_n = \sum_{k=0}^{n} p_k q_{n-k}$$

We now consider the the ordinary generating functions $P(z)$ and $Q(z)$, which in formal notation, are represented by

$$P(z) = \sum_{k=1}^{\infty} p_k z^k \qquad Q(z) = \sum_{k=1}^{\infty} q_k z^k$$

Utilizing this power series notation, we can then see that the relationship between p_n and q_n is alternatively given by the expression $P(z)Q(z) = Q(z) - 1$

Noting that $p_n \leq q_n \leq 1$, we have that the radius of convergence of both of these power series is at least 1. Also, note that $Q(z)$ is not equal to 0 for z in the interval [0,1) which gives us that

$$P(z) = 1 - \frac{1}{Q(z)} \quad (z \in [0, 1)).$$

By noting that

$$P(1) = \sum_{k=1}^{\infty} p_k = p$$

we can apply Corollary 7.28 of Abel's Theorem **Folland,pg. 331** to see that

$$p = \lim_{\substack{z \to 1^- \\ z \in [0,1)}} \sum_{k=0}^{\infty} p_k z^k = \lim_{\substack{z \to 1^- \\ z \in [0,1)}} P(z) = 1 - \frac{1}{\lim_{\substack{z \to 1^- \\ z \in [0,1)}} Q(z)}$$

By considering the conditions placed on the coefficients p_n and q_n, we can see that the limiting operation performed on Q(z) will either converge to a finite real number (implying transience of said random walk) or will diverge to $+\infty$ (implying recurrence of said random walk).

$$p = \lim_{\substack{z \to 1^- \\ z \in [0,1)}} P(z) = 1 - \frac{1}{\lim_{\substack{z \to 1^- \\ z \in [0,1)}} Q(z)} \qquad (4.1)$$

5 Constructing an Exponential Loop Generating Function

We now seek to analyze the limit in question by creating a more manageable representation of $Q(z)$ through consideration of the loop generating function as an exponential generating function as this generating function admits a much simpler form. We can then take this simpler form and transform it back into a a simple form for the ordinary loop generating function through use of a Borel Transform so that we may perform further analysis on it.

We can see that this requires that we find a representation for the loop generating function

$$L(z) = \sum_{n=0}^{\infty} \ell_n z^n$$

as $Q(z) = L(\frac{z}{2d})$. Although the ordinary loop generating function does not lend itself to analysis very easily, the exponential generating function

$$E(z) = \sum_{n=0}^{\infty} \ell_n \frac{z^n}{n!}$$

certainly does. This results from the fact that any loop on $\mathbb{Z}^d$ is a shuffle of loops on $\mathbb{Z}^d$ or in other words, that each pair of counteracting movements on $\mathbb{Z}^d$ can be thought of as acting on a copy of $\mathbb{Z}^1$ and that looking at a walk on a d-dimensional lattice is just like combining all of the movement commands into a single string of operations on the particle.

This is exactly what we need as the multiplication of exponential generating functions corresponds to shuffles which is a property that will be discussed here, for a more robust treatment, the reader is referred to [**Enum Comb, Chapter 5**].

We now take a moment to solidify some notational additions. We let $\ell_n^{(d)}$ denote the number of length n in $\mathbb{Z}^d$ and take $E_d(z)$ to denote the exponential generating function of $\ell_n^{(d)}$. Now, for the purpose of this proof, consider the case when $d = 2$. We know that a loop on the two-dimensional integer lattice is a walk that takes unit steps both horizontally and vertically and ends where it started. A loop of length n on $\mathbb{Z}^2$ is made of a number k of horizontal steps along with $n-k$ vertical steps. From our previous results, we can see that the unit steps in the horizontal direction create a loop of length k, likewise, that the steps in the vertical direction create a loop of length $n-k$. Thus, we have that the number of length n loops on $\mathbb{Z}^2$ that results from taking k horizontal steps and $n-k$ horizontal steps is

$$\binom{n}{k} \ell_k^{(1)} \ell_{n-k}^{(1)}$$

The reasoning for this being that by specifying the time when the k horizontal steps are performed determines when th $n-k$ vertical steps occur (all of which possibilities must be taken into account). As a result, we have that the total number of n-length loops on $\mathbb{Z}^2$ is

$$\ell_n^{(2)} = \sum_{k=0}^{n} \binom{n}{k} \ell_k^{(1)} \ell_{n-k}^{(1)}$$

as we sum up the number of loops for each corresponding length k for all k such that $0 \leq k \leq n$. Now, for the purpose of drawing a conclusion about the multiplication of exponential generating functions, we will count loops in one dimension giving us that

$$\ell_n^{(1)} = \begin{cases} \binom{2k}{k} & \text{if } n = 2k \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Since any loop on $\mathbb{Z}$ is a combination of k positive unit steps and k negative units for some $k \geq 0$ and similar to before, the times where the positive steps occur determine the times when the negative steps occur, we get that

$$E_1(z) = \sum_{k=0}^{\infty} \ell_k^{(1)} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!k!}$$

Now, note that

$$E_2(z) = \sum_{n=0}^{\infty} \ell_n^{(2)} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \ell_k^{(1)} \ell_{n-k}^{(1)} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k}$$

We can apply the same line of reasoning to the general dimension d as by moving up by one dimension, we just add another axis to step along allowing us to make similar combinatoric arguments giving us that

$$E_d(z) = E_1(z)^d$$

and

$$E_1(z) =$$

This fact is utilized through combinatorial arguments to notice that the exponential generating function for lattice walks in $\mathbb{Z}^1$ is a **modified Bessel function of the first kind** (generally denoted as $I_\alpha(z)$) which satisfies the following second-order differential equation:

$$\left(z^2 \frac{d^2}{dz^2} + z\frac{d}{dz} - (z^2 + \alpha^2)\right) F(z) = 0 \quad (\alpha \in \mathbb{C}).$$

We call this differential equation the **modified Bessel equation** and although we will not go indepth here, the reader is referred to [**Special functions, Chapter 4**] for a more exhaustive reference. Novak[Paper, pg.5] elicits both a series and integral representation of the modified function from previous resources which will both prove useful to us and as stated are

rip reference from Novak and explain bessel .

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)}$$

$$I_\alpha(z) = \frac{\left(\frac{z}{2}\right)^{\alpha}}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^{\pi} e^{(cos(\theta))z} (sin(\theta))^{2\alpha} d\theta$$

Note that

$$I_0(2z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!\Gamma(k+1)} = E_1(z)$$

Which gives us that $E(z) = I_0(2z)^d$

6 Application of the Borel Transform

In this section, we would like to find a representation of the ordinary loop generating function, $L(z)$, in terms of a standard mathematical object by applying the Borel Transform to the expression for the exponential generating function, $E(z)$, which we already have a definition for in terms of standard *mathematical objects.*

A Borel Transform ($(\mathcal{B}f)(z) = \int_0^\infty f(tz)e^{-t}dt$) is then used to turn this exponential generating function into an ordinary generating function **[1]**(a generating function where the coefficients correspond to the coefficients, a_n, of a power series expansion $\sum a_n x^n$) giving us an integral form for $Q(z)$.

We can see how this occurs by writing out the Maclaurin series of $f(tz)$ and interchanging integration and summation (WHICH IS VALIDATED THROUGH UNIFORM CONVERGENCE Prove this) giving us that

$$(\mathcal{B}f)(z) = \int_0^\infty f(tz)e^{-t}dt = \sum_{n=1}^{\infty} f^{(n)}(0)\frac{z^n}{n!}\int_0^\infty e^{-t}t^n dt$$

and by noting that $\int_0^\infty e^{-t}t^n dt = n!$, we get that

$$(\mathcal{B}f)(z) = \sum_{n=1}^{\infty} f^{(n)}(0)z^n$$

We now apply the Borel Transform to the exponential loop generating function $E(z)$ to retrieve an integral representation for the ordinary loop function $L(z)$ giving us that

$$L(z) = \mathcal{B}(E(z)) = \mathcal{B}\left(I_0(2z)^d\right) = \int_0^\infty I_0(2zt)^d e^{-t}dt$$

7 Utilization of Laplace's Formula

In this section, we will now seek to show that the integral representation of the ordinary loop generating function diverging/converging for varying values of d is equivalent to the divergence or convergence of a much easier to evaluate elementary integral giving us our wanted result about transience/recurrence of a random walk on $\mathbb{Z}^d$.

We now have an integral representation for $Q(z)$ which is useful as these sorts of integrals lend themselves nicely to tests for convergence which will prove useful in evaluation of the original limit of the generating function. As there are no ill-behaved points in the finite portion of the integral, the convergence of the integral is dependent on the tail of the integral

$$\int_N^\infty I_0(2zt)^d e^{-t} \quad (N \gg 0).$$

and, furthermore, that the behavior of this integral is determined by the behavior of the integrand as $t \to \infty$

To assist us in evaluation of this integral, we note that

$$I_0\left(\frac{tz}{d}\right) = \frac{\left(\frac{tz}{2d}\right)^0}{\sqrt{\pi}\Gamma(0+1/2)}\int_0^\pi e^{cos(\theta)\frac{tz}{d}} sin(\theta)^{2\cdot 0}d\theta = \frac{1}{\pi}\int_0^\pi e^{tf(\theta)}d\theta \quad (7.1)$$

where $f(\theta) = cos(\theta)\frac{z}{d}$.
We will now use methods from Asymptotic Analysis to approximate this integral and to ensure that the error does not affect our end result. To put it succinctly, Asymptotic Analysis is a method of

describing the limiting behavior of a variety of equations and relations. We will be using an elementary result, so only a small portion of the field will be discussed here, but the reader is directed to [**GEN FUNC**] for a more robust treatment.

We start by noting that the function defined by $f(\theta)$ is strictly maximized over at $\theta = 0$ in the interval $[0, \pi]$. We can talk about maximizing a complex-valued function despite the lack of an ordering being present on C as, although $z \in \mathbb{C}$, recall from 4.1 that we are looking at the interval $[0, 1)$ on the real line, which is ordered. Thus, the function $e^{tf(\theta)}$ is maximized at $\theta = 0$ as $t > 0$. Although t can be equal to 0, we cannot say that there is a maximum at $\theta = 0$ because there are no extrema present. If we look at what happens as $t \to \infty$, we can see that this maximization effect is amplified LOCALIZES DEFINITION . This localization can be seen numerically by analyzing the second degree Taylor polynomial of f centered at $\theta = 0$

$$f(\theta) \approx P_{(2,0)}(f(\theta)) = \sum_{j=0}^{2} f^{(k)}(0)\frac{\theta^k}{k!} = f(0) + f'(0)\theta + f''(0)\frac{\theta^2}{2}$$

Notice that $f'(0) = 0$ and that $f''(0) = -1 < 0$ giving us that

$$f(\theta) \approx f(0) - |f''(0)|\frac{\theta^2}{2}$$

We now will substitute this approximation for f back into the integrand of the right-most integral in 7.1 giving us that

$$\frac{1}{\pi}\int_0^{\pi} e^{tf(\theta)}d\theta \approx \frac{1}{\pi}\int_0^{\pi} e^{t(f(0)-|f''(0)|\frac{\theta^2}{2})}d\theta = \frac{e^{tf(0)}}{\pi}\int_0^{\pi} e^{-t|f''(0)|\frac{\theta^2}{2}}d\theta \quad (7.2)$$

If we extend the right-most integral over the $[0, \infty)$ and ignore the rapidly decaying error incurred by the higher-order terms from the Taylor polynomial, we receive half of a Gaussian integral

$$\frac{e^{tf(0)}}{\pi}\int_0^{\infty} e^{-t|f''(0)|\frac{\theta^2}{2}}d\theta$$

which as a result of [**PAPER, Pg. 6**] can be calculated exactly to give us

$$\frac{e^{tf(0)}}{\pi}\int_0^{\infty} e^{-t|f''(0)|\frac{\theta^2}{2}}d\theta = \frac{e^{tf(0)}}{\pi}\sqrt{\frac{\pi}{2t|f''(0)|}}$$

From this, we can expect this function of t to be an approximation of our integral from 7.1, the accuracy of which, increases as t tends towards ∞. This result is actually ensured by Laplace's Formula [**6, Section 5.1**] giving us that

$$\frac{1}{\pi}\int_0^{\pi} e^{tf(\theta)}d\theta \sim \frac{e^{tf(0)}}{\pi}\sqrt{\frac{\pi}{2t|f''(0)|}} \qquad (t \to \infty).$$

With the notation $f(t) \sim g(t)$ denoting that $\lim_{t\to\infty}\frac{f(t)}{g(t)} = 1$. We can now bring everything together to evaluate our integral in an approximate fashion giving us that

$$I_0\left(\frac{tz}{d}\right)^d e^{-t} \sim C \cdot e^{t(z-1)} \cdot tz^{-\frac{d}{2}}$$

where C is a negligible constant. Through application of the monotone convergence theorem

is then analyzed using Laplace's method from asymptotic analysis. The resulting asymptotic formula is then shown to diverge for $d = 1, 2$ and to converge for $d \geq 3$ by the monotone convergence theorem giving us the result of Polya's theorem.

8 Notable Results

Having proved this result for $\mathbb{Z}^d$, the natural question to ask next would be, to what other vector spaces does this apply?
As it turns out, this result can also be applied to random motion in $\mathbb{R}^d$ as well, which is known as a form of **Brownian Motion**, giving us similar results of recurrence and transience of random walks in a continuous setting.
Although the proof behind this result is a bit more invovlved utilizing results from probability theory, the intuition behind this extension is rather simple in that we are basically looking at a lattice in dimension d with infinitessimal lengths separating each connected node which makes it easy to see why Polya's Theorem still applies.

To elaborate on the details of this, we can reference a web article written by Alex Chinco. Chinco interprets Brownian Motion in $\mathbb{R}^d$ as a vector-valued process as a function of time with d components

9 Citations

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