Haim Grebnev

Abstract: This is a letter that I wrote (I haven't sent it yet, I still have to review it) directed to Paul J. Nahin about the two book "An Imaginary Tale: The Story of $\sqrt{-1}$" and "Dr. Euler's Fabulous Formula" that he wrote. I read both books and I wrote this letter to this author in order to explain to him what I liked about these books. This letter mainly acts as a review of what he does in his two books, but almost in every case when I discuss certain results that he derives, I go ahead and generalize, explain or complete some of the proofs and derivation that he presents. Sometimes the author will leave a certain fact unproved and in this letter I will come in and show how one does prove these results. The first part of the letter is meant as a set up for the more difficult mathematics further in the letter. The letter starts off not very mathematically intensive, and as one moves farther along, the mathematics involved gets a lot more serious. The rigor of mathematics in this letter ranges from relatively loose to absolute straight and pure rigor. This letter is intended to be read by people who have a sophisticated knowledge of mathematics (Math 336), but the audience isn't expected to be an experienced Professor of Mathematics (Paul J. Nahin is a professor of electrical engineering, however his mathematical knowledge is outstanding and far-reaching). As one could infer from the title of the books, the topic of this letter can be summarized by one letter: i. However to be specific, this letter largely focuses on Fourier series and Complex analysis (two fields which are in fact deeply connected). This letter additionally portrays how I, as a young person, grew up and came to understand all of this material since everything here is a representation of a small part of my mathematical thoughts over the last couple years. I hope that anyone who has read both of these wonderful books by Paul J. Nahin will be able to learn something and find pleasure from reading this letter should they choose to read it. Enjoy!

Acknowledgment: I want to thank Professor Marshall for a wonderful talk we had on the Riemann-zeta function. I used some of the things that he showed me that day in this essay. Professor Marshall has the amazing talent of being able to listen, a trait that not all possess. I appreciate all of the time and patience he devoted to teach me calculus last year and to listen to some of my mathematical discoveries with an encouraging voice.

Dear Dr. Paul J. Nahin,

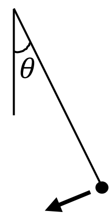
Hi Dr. Nahin (I didn't know how to address you so I went with "Dr. Nahin"), I am a student at the University of Washington and I have read both of your books "An Imaginary Tale: The Story of $\sqrt{-1}$" and "Dr. Euler's Fabulous Formula." I would like to tell you that I loved both of your books very much. In this letter I would like to explain to you what I enjoyed about these books.

Part I

Finding the Book

I read the first book "An Imaginary Tale: The Story of $\sqrt{-1}$" when I was in high school. One day when our class was visiting our school library (for some research project probably) I decided to amuse myself by looking at what physics and math books they had there. I first picked up a calculus physics book (which was weird because our school seemed to frown upon the use of calculus in science) in which I turned to the page which had the following statement (reworded):

"Some smart person proved that no elementary function satisfies the following differential equation for the pendulum ($g \approx 9.8\ m/s^2$):"



$$L \cdot \frac{d^2\theta}{dt^2} = g \cdot \sin(\theta)$$

All I said was "Wow! I don't know how to prove that" and put the book back on its shelf. Then I picked up another book titled "An Imaginary Tale: The Story of $\sqrt{-1}$." I looked inside and saw lots of these kinds of symbols: $\prod$ and so I immediately told myself that I've got to read this book! I checked it out of the library and decided to take it home to read. I thought if I won't understand something I can always ask my dad for help (my dad knows a lot of math).

For what follows, page numbers in "An Imaginary Tale: The Story of $\sqrt{-1}$" refer to the edition where your preface to the paperback edition was written in 2006.

The Cubic Equation

I started to read your book and I quickly got hooked. First I was really amazed at the fact that in just the first 3 pages you derive the cubic formula. I always considered it really difficult. In fact, when I was much younger I printed it out and glued a copy of it into my mathematics journal. The version I printed out looked very big and very complicated. But you gave a very easy and beautiful derivation of it that del Ferro thought of a long time ago. I had tried to derive the cubic formula for a long time before I read your book, and was not successful. In your book you started off with the cubic equation:

$$x^3 + px = q$$

Immediately somebody might have said "But wait! That isn't the general cubic equation!" But I knew better. You can always get from the general cubic equation:

$$ax^3 + bx^2 + cx + d = 0$$

To the previous cubic equation by scaling and shifting (I'll elaborate a little bit later when I'll present to you the full cubic equation). Returning to solve the first cubic equation above you proceed by doing a trick (on page 9) of saying $x = u + v$ and then you set (I skip steps):

$$3uv + p = 0$$

$$u^3 + v^3 = q$$

And pretty quickly you arrive at the solution:

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Obviously when you get one root of a cubic equation, you can get the other two by dividing the original cubic equation by x minus the first root and then use the quadratic formula in order to obtain the other two roots.

Later the cubic equation got involved in a rather amusing story. When I became a student at the University of Washington, I immediately signed up for Honors Calculus 1 (It was a really good class). One time our teacher gave us a homework problem which involved solving a cubic equation. He gave us the hint of just completing the cube (he gave us a really nice cubic equation) in order to solve it. But I of course did not do that. I decided to show off my skills by deriving the full cubic formula in the homework and plugged in the coefficients of the cubic equation into it. The formula I presented in the homework was:

"One solution of the general cubic equation $ax^3 + bx^2 + cx + d = 0$ is:

$$x = \sqrt[3]{-\frac{\frac{d}{a} + \frac{2\left(\frac{b}{a}\right)^3}{27} - \frac{bc}{3a^2}}{2} + \sqrt{\frac{\left(\frac{d}{a} + \frac{2\left(\frac{b}{a}\right)^3}{27} - \frac{bc}{3a^2}\right)^2}{4} + \frac{\left(\frac{c}{a} - \frac{\left(\frac{b}{a}\right)^2}{3}\right)^3}{27}}} + \sqrt[3]{-\frac{\frac{d}{a} + \frac{2\left(\frac{b}{a}\right)^3}{27} - \frac{bc}{3a^2}}{2} - \sqrt{\frac{\left(\frac{d}{a} + \frac{2\left(\frac{b}{a}\right)^3}{27} - \frac{bc}{3a^2}\right)^2}{4} + \frac{\left(\frac{c}{a} - \frac{\left(\frac{b}{a}\right)^2}{3}\right)^3}{27}}} - \frac{b}{3a}$$

You can actually recognize certain elements of the formula you gave in your book on page 10 in this formula. The $\frac{b}{3a}$ term is the shift needed to delete the quadratic term in the general cubic equation. Meaning that after you divide through $ax^3 + bx^2 + cx + d = 0$ by a and then plug in $x = y - \frac{b}{3a}$ you will get a cubic equation of the form $y^3 + py = q$, and then you can proceed as you do in your book.

When he returned our graded homework the next Friday, I came up to my TA and asked him what he thought of my derivation of the cubic formula. He looked at me with wide eyes (obviously lack of sleep) and said… that I need to take harder classes. He said that he was not able to follow all of the steps (he probably graded our homework late at night) and just looked at whether my answer was correct. I was laughing on the inside (victory!).

Later I told a graduate student how to solve the cubic equation, and the graduate student didn't know del Ferro's method of solving the cubic equation. He like everybody else kept showing me in return a much more complicated method of solving the cubic equation. I'm still surprised that no one seems to know the approach that you show in your book in solving the cubic equation.

Formulas I can't get over

You know, there are only a few formulas in mathematics that I just can't ever get over. Those are formulas that, when I learn them, come at me so unexpectedly and inspire much awe. Two such formulas that I have yet not been able to get over are:

$$\frac{\sin(\theta)}{\theta} = \prod_{k=1}^{\infty} \cos(\frac{1}{2^k}\theta)$$

(page 64) and its corollary

$$\prod_{k=1}^{\infty} \cos(\frac{1}{2^k}\theta) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \ldots = \frac{2}{\pi}$$

and the result (page 72):

$$\prod_{k=1}^{n-1} \sin(\frac{k\pi}{2n}) = \prod_{k=1}^{n-1} \cos(\frac{k\pi}{2n}) = \frac{\sqrt{n}}{2^{n-1}}$$

The last result is really amazing to me because I know how to sum cosines (in more ways than one). I can even take the limit of the sum of cosines on a finite interval (which is called the "integral"). But to be able to take such a product! That was completely unexpected.

Complex Geometry (Pun Intended)

The history you provide in your book is fascinating. It was especially interesting to learn about how people for a long time have tried to grapple with the geometric meaning of the square root of negative one (which I will henceforth denote by i). It was really funny to read about how (I unfortunately can't find the page) one person (who studied ballistics) was disapproving of Wessel's idea of representing complex numbers as vectors in a plane because he didn't like mixing geometry and algebra. To me that sounds silly because I always get excited when fields of mathematics merge together. I also found it really cool that you can use complex numbers to prove geometric facts such as the medians of any triangle meet at one point (page 89) and you can even say how far that point is from each of the vertices of the triangle. Complex numbers became a powerful tool of mine in solving all kinds of geometric problems.

Where did that come from?

In Chapter 4 you discuss the Leonardo Reoccurrence, which at the time blew my mind! It still does. My favorite phrase in that section is "I will start by <u>guessing</u> $u_n = k \cdot z^n$." There you defined $\{u_n\}$ is a sequence defined recursively by the formula $u_{n+2} = p \cdot u_{n+1} + q \cdot u_n$ with initial conditions u_0 and u_1. Of course the instant thought that comes to the mind of the reader is "how did somebody come up with that guess!" After reading further the reader is then amazed at how you with this guess in hand are able to derive a formula for u_n. That was exactly what was going through my mind. Though, this mystery of where the idea came from to assume $u_n = k \cdot z^n$ didn't last forever with me.

One time my dad tried to explain to me how matrices can be used to arrive at the guess just mentioned about the Leonardo Reoccurrence. Although I did not understand what he was talking about, his words stuck with me. When I became a freshman at the University of Washington, at one point (after I have gained some experience with matrices) I realized that the Leonardo Reoccurrence can be written down in the form of matrix-vector multiplications (in fact the solution to systems of linear differential equations borrow this idea as well!). We could in fact write down the Leonard Reoccurrence in the following way:

$$\text{If } u_{n+2} = p \cdot u_{n+1} + q \cdot u_n$$

Then let us write down the following sequence of vectors:

$$\left\{\begin{bmatrix} u_{2k} \\ u_{2k+1} \end{bmatrix}\right\} = \left\{\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} u_4 \\ u_5 \end{bmatrix}, \dots\right\}$$

Then we can notice that the reoccurrence can be rewritten in the following manner in matric form (you can check that this is true by multiplying it out):

$$\begin{bmatrix} q & p \\ pq & p^2+q \end{bmatrix}\begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} u_{n+2} \\ u_{n+3} \end{bmatrix}$$

I will admit that this is a guess too, but … Anyways, after one wrote down the previous recursive formula, it is almost an immediate result that:

$$for\ n\ even, \qquad \begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} q & p \\ pq & p^2+q \end{bmatrix}^{\frac{n}{2}} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

Thus can you see were the guess you made in the book $u_n = k \cdot z^n$ comes from? It comes from the above. Of course you still need to be able to take matrices to high powers. And I knew how to do that. In fact it was at this time that I was reading the section 1.2 in your book "Dr. Euler's Fabulous Formula" where you talked about how to take matrices to high powers. I haven't even started studying linear algebra in at the University of Washington! In section 1.2 in "Dr. Euler's Fabulous Formula," you demonstrate the fact that for a 2 by 2 matrix A (I is the identity matrix):

$$A^n = (k_1\lambda_1^n + k_2\lambda_2^n)A + (k_3\lambda_1^n + k_4\lambda_2^n)I$$

For some constants, possibly complex, (which you show how to calculate in "Dr. Euler's Fabulous Formula") $k_1, k_2, k_3, k_4, \lambda_1, \lambda_4$. Now if we say that $A = \begin{bmatrix} q & p \\ pq & p^2+q \end{bmatrix}$, then from the two formulas above we get that

$$\begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \left(\left(k_1\lambda_1^{n/2} + k_2\lambda_2^{n/2}\right)\begin{bmatrix} q & p \\ pq & p^2+q \end{bmatrix} + \left(k_3\lambda_1^{n/2} + k_4\lambda_2^{n/2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

For some constants $k_1, k_2, k_3, k_4, \lambda_1, \lambda_4$. Now if we multiply the right side in such a way as to only get the first component of the vector on the left side of the equation we get:

$$u_n = u_0\left(\left(k_1\lambda_1^{n/2} + k_2\lambda_2^{n/2}\right)q + \left(k_3\lambda_1^{n/2} + k_4\lambda_2^{n/2}\right)\right) + u_1\left(\left(k_1\lambda_1^{n/2} + k_2\lambda_2^{n/2}\right)p\right)$$

And collecting terms we get that u_n can be expressed as:

$$u_n = k_5\sqrt{\lambda_1}^n + k_6\sqrt{\lambda_2}^n$$

For some constants k_5, k_6. This is exactly the form that you present in your solution to the Leonardo Reoccurrence, and is an origin of the idea that u_n grows as an exponential function. With that guess in hand, one can now solve the Leonardo Reoccurrence in exactly the same manner as you did in your book.

Something for Later

I got to the point in the book when you started to talk about how imaginary numbers play a role in the theory of relativity. I did not understand the section very much at the time, and so I skipped it and said to myself that I would return to it after I've read about the theory of relativity in the Feynman Lectures (it was approximately at that time that I started to read the Feynman Lectures on Physics). The same went for uses of complex numbers in electrical circuits.

Celestial Mechanics

The section 5.3 is probably my favorite section of the whole book. Since I was little, I've always known about the 3 laws of Kepler, but I never knew how to prove them from the gravitation law that Newton came up with: $F_G = G\frac{mM}{r^2}$. I remember how when I was reading the beginning of this section I was telling myself, "No, you're not going to derive the 3 laws of Kepler. That just couldn't be. I heard that they're really hard to prove." And just like with your derivation of the cubic formula, you amazed me by showing how to derive a concept I thought was really difficult to prove. You showed once again how complex numbers can give beautiful derivations of laws that would otherwise be hard to prove.

About a year later I wondered what would happen if I would have used another gravitational law. For example what would have happened if I used the gravitational law:

$$F_G = G\frac{mM}{r^3}$$

What would be changed if there was a cube in the denominator? Well what I did was just carry out the calculation just the way you did in your book. I noticed that when I got to, just like you did on page 117, the formula

$$\frac{dA}{dt} = \frac{1}{2}c_1$$

Where A represents the function of how much area the planet sweeps out versus time. This was your proof of Kepler's second gravitational law. I realized that this equation was in fact invariant of the gravitational law that I picked in the first place. So Kepler's second law was true for any gravitational force. When I told this to my dad, he almost immediately said: "Well yeah, that is just the conservation of angular momentum." And then I right away understood that Kepler's second law just came out of the fact that since the gravitational force was always pointing towards the sun (or the planet that it is orbiting), there is no torque in the system of its rotation. This means that the angular momentum of the planet around the sun is always constant. Now angular momentum (usually denoted L) is given by the formula:

$$L = I\omega$$

Where I is the moment of inertia of the system and ω is the angular speed of the planet. Since in this system $I = mr^2$ and by definition: $\omega = \theta'(t)$, we have that:

$$L = mr^2 \frac{d\theta}{dt}$$

And since we said that L is constant (say $L = c_2$) and from freshman calculus we have $A'(t) = \frac{1}{2}r^2 \frac{d\theta}{dt}$, we have that:

$$\frac{dA}{dt} = \frac{c_2}{m}$$

So $A'(t)$ is constant, and this is Kepler's laws that hold in any gravitational law as long as the gravitation force is always pointed along the line that connects the sun and the planet. I was not able to arrive at a good answer at what the orbit would look like if the gravitational law was anything else other than the one where the gravitational law was proportional to the inverse of the square of the distance between two planets (or in our example the distance between the sun and the planet).

Wow!

Wow! Sorry, I couldn't help myself. I was amazed with Euler's derivation of $\zeta(2)$ when I read your book. In fact, if the equation $e^{\pi i} = -1$ is Feynman's "remarkable formula," my "remarkable formula" would be non-other than:

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

To me I have never seen such an outstanding, remarkable, and beautiful formula in all of my life! In fact this formula has taken up so much of my focus that I have thought of 6 or 7 additional separate proofs of this formula. On pages 148-149, you show how Euler originally derived this fact. I then realized that what you said was true: that the very same method can be used to derive the value of $\zeta(4)$ and the rest of the values of the zeta function at even integers. Let me show how I was able to extend Euler's method of calculating $\zeta(2)$ to compute $\zeta(4)$:

Just like you do in your book, I started off with the Taylor Series for the sine function:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Then by substituting $\sqrt{x}$ into both sides and dividing both sides by $\sqrt{x}$ one easily gets:

$$\frac{\sin(\sqrt{x})}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(2k+1)!}$$

Now just like in your book we notice that the above function has the roots $\pi^2, (2\pi)^2, (3\pi)^2, \ldots$. Now, using the trick that a polynomial of the form:

$$P(x) = 1 + \sum_{k=1}^{\infty} c_k x^k$$

(Here the c_k's are constants) can be written as:

$$P(x) = (-1)^n \prod_{k=1}^{n} (1 - \frac{x}{r_k})$$

(Here the r_k's are the roots of $P(x)$) We can than rewrite the above series for $\frac{\sin(\sqrt{x})}{\sqrt{x}}$ as:

$$\frac{\sin(\sqrt{x})}{\sqrt{x}} = \left(1 - \frac{x}{(\pi)^2}\right)\left(1 - \frac{x}{(2\pi)^2}\right)\left(1 - \frac{x}{(3\pi)^2}\right) \ldots = \prod_{k=1}^{\infty} \left(1 - \frac{x}{(k\pi)^2}\right)$$

Now if you start multiplying this expression out using the distributive property in such a way as to get the x^2 term in the Taylor Series of this function, you will get:

$$\frac{1}{\pi^4} \sum_{k>j} \frac{1}{k^2 j^2} = \frac{1}{5!}$$

Multiply both sides by π^4 to get:

$$\sum_{k>j} \frac{1}{k^2 j^2} = \frac{\pi^4}{5!}$$

Now that we have this preliminary result in hand we go onto the main steps. Let us take the Taylor Polynomial of $\frac{\sin(\sqrt{x})}{\sqrt{x}}$ that we derived above and square both sides in order to get:

$$\left(\frac{\sin(\sqrt{x})}{\sqrt{x}}\right)^2 = \left(1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots\right)^2$$

Now it shouldn't be too hard to show that the right hand side can be written out as:

$$\left(1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+\cdots\right)^2 = 1-\frac{2^3}{4!}x+\frac{2^5}{6!}x^2-\frac{2^7}{8!}x^3+\cdots = \sum_{k=0}^{\infty}\frac{(-1)^k x^k 2^{2k+1}}{(2(k+1))!}$$

What is actually important to us is the fact that the coefficient of x^2 is $\frac{2^5}{6!}$. The fact that the coefficient of x^2 is $\frac{2^5}{6!}$ is not hard to verify as a standalone fact by just directly multiplying out $\left(1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+\cdots\right)^2$ by the distributive property. One can show that the first equality above holds by using the fact that (which can be proved by the binomial theorem):

$$\sum_{k=0}^{\lceil\frac{n+1}{2}\rceil-1}\left(\frac{1}{(2k+1)!\,(2(n+1-k)-1)!}\right)=\frac{2^{2n-1}}{(2n)!}$$

(Where $\lceil m\rceil$ denotes: take the next bigger than m integer if m is not an integer. If m is an integer $\lceil m\rceil = m$. $\lceil m\rceil$ is called the ceiling function.) But again, all we need is the fact that $\frac{2^5}{6!}$ is the coefficient of x^2.

Now, we also know that:

$$\left(\frac{\sin(\sqrt{x})}{\sqrt{x}}\right)^2 = \left(\left(1-\frac{x}{(\pi)^2}\right)\left(1-\frac{x}{(2\pi)^2}\right)\left(1-\frac{x}{(3\pi)^2}\right)\ldots\right)^2$$
$$= \left(1-\frac{x}{(\pi)^2}\right)\left(1-\frac{x}{(\pi)^2}\right)\left(1-\frac{x}{(2\pi)^2}\right)\left(1-\frac{x}{(2\pi)^2}\right)\ldots = \prod_{k=1}^{\infty}\left(1-\frac{x}{(k\pi)^2}\right)^2$$

And now if you multiply the expression $\prod_{k=1}^{\infty}\left(1-\frac{x}{(k\pi)^2}\right)^2$ out as to get the x^2 term of its Taylor Polynomial you will get:

$$\frac{1}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{k^4}+4\frac{1}{\pi^4}\sum_{k>j}\frac{1}{k^2j^2} =$$
$$\frac{1}{\pi^4}\zeta(4)+4\frac{1}{\pi^4}\sum_{k>j}\frac{1}{k^2j^2}$$

There is a 4 in front of the $\frac{1}{\pi^4}\sum_{k>j}\frac{1}{k^2j^2}$ term because if you look closely at the expression $\left(1-\frac{x}{(\pi)^2}\right)\left(1-\frac{x}{(\pi)^2}\right)\left(1-\frac{x}{(2\pi)^2}\right)\left(1-\frac{x}{(2\pi)^2}\right)$ ..., when you multiply it out by the distributive property there are 4 ways to get the term $\frac{1}{(k\pi)^2(j\pi)^2}$ where $k>j$.

Now since we know that $\frac{2^5}{6!}$ is the coefficient of the x^2 term in the Taylor Polynomial of the function $\left(\frac{\sin(\sqrt{x})}{\sqrt{x}}\right)^2$ we get that:

$$\frac{1}{\pi^4}\zeta(4) + 4\frac{1}{\pi^4}\sum_{k>j}\frac{1}{k^2j^2} = \frac{2^5}{6!}$$

Multiply both sides by π^4 to get:

$$\zeta(4) + 4\sum_{k>j}\frac{1}{k^2j^2} = \frac{2^5\pi^4}{6!}$$

But what do we do now? Now we use the preliminary result that we derived earlier:

$$\sum_{k>j}\frac{1}{k^2j^2} = \frac{\pi^4}{5!}$$

Plugging this into the previous expression gives us that

$$\zeta(4) + 4\frac{\pi^4}{5} = \frac{2^5\pi^4}{6!}$$

Rearranging gives:

$$\zeta(4) = \frac{2^5\pi^4}{6!} - 4\frac{\pi^4}{5!}$$

And simplifying gives us the next sequel in "remarkable formulas:"

$$\zeta(4) = \frac{\pi^4}{90}$$

Wow again! In your book "Dr. Euler's Fabulous Formula" you show yet another method of calculating these sums (specifically the zeta function at even integers) using Fourier series. But I will get to that in this letter in due course. Sorry, I have to say it one more time: wow!

I have something to say about the formula that Euler derived:

$$\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \cdots = \frac{\pi^2}{4}\ln(2) + 2\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx$$

But I will delay talking about it until I get to a section in your book "Dr. Euler's Fabulous Formula," where you talk about a problem that Ramanujan solved.

Outstanding Results

Although I will not comment in these in depth I was fascinated to learn about Euler's constant γ, Euler's proof that $\sum_{p\ prime}\frac{1}{p}$ diverges and the fact that $\lim_{n\to\infty}\frac{\pi(n)}{li(n)} = 1$ where $li(n) = \int_2^n \frac{dx}{\ln(x)}$. You talk about all of these subjects in your book. The last one, called the Prime Number Theorem, is really interesting because no matter how much I've tried, I have not yet been able to think of a proof it. I have not even been able to prove any weaker forms of this theorem. So far it's been a very frustrating problem for me.

Complex Complexities of Complicated Mathematics

I will always remember your section on Roger Cotes and a Lost Opportunity. It was so maddening to know that a person came so close to being known as the creator of the formula $e^{ix} = \cos(x) + i\,\sin(x)$, and never did because of not clear exposition in his paper. I've definitely learned the lesson of clear exposition! Never will make this mistake.

This section started to make me think about how someone could arrive at Euler's identity using integrals, just like Cotes did. In the case of Cotes, he had a geometric interpretation for his integrals. One day when I was sitting on the bus on my way to school, I thought of a new derivation of Euler's formula involving integrals. That day before my math class started, I derived Euler's formula on the black board (I was at the University of Washington already) in front of my peers.

I often before math class would derive formulas on the black board in front of my classmates (the teacher usually arrived just right when the bell rung, so I had time to start my derivations before he came in). My classmates almost never listened to me, and the reason I did this was because I like writing with chalk. I don't get to write and draw with chalk a lot!) Anyways here is the derivation of Euler's formula that I thought of:

I wondered what would happen if I integrated:

$$\int \frac{1}{1+x^2}dx$$

Two different ways and then equated my results (since they're supposed to give the same thing). The first way to integrate the above integral is to use the result taught in all freshman calculus classes, that:

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$$

$\tan^{-1}(x)$ is the arctangent function. So obvioudiously (Word doesn't seem to think that that's a word):

$$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$$

Where C is a constant. Now let us take the integral another way (this was in fact the period when we were studying about how to integrate rational functions, not that I didn't know how to do that already):

$$\int \frac{1}{1+x^2}dx = \int \frac{1}{(1+ix)(1-ix)}dx = \frac{1}{2}\int \left(\frac{1}{1+ix} + \frac{1}{1-ix}\right)dx$$

Now, we know that $\int \frac{1}{y}dy = \ln(y)$. So let us use this result in the above expression:

$$\frac{1}{2}\int\left(\frac{1}{1+ix}+\frac{1}{1-ix}\right)dx=\frac{1}{2i}(\ln(1+ix)-\ln(1-ix))+C_2=\frac{1}{2i}\ln\left(\frac{1+ix}{1-ix}\right)+C_2$$

Where C_2 is a constant. Let us just agree from henceforth in this section that C_k denotes some constant. Now since these two methods are supposed to give the same thing, namely $\int\frac{1}{1+x^2}dx$, we then get that:

$$\frac{1}{2i}\ln\left(\frac{1+ix}{1-ix}\right)=\tan^{-1}(x)+C_3$$

Getting the idea?! Ok let us plug in $x=\tan(s)$ into the above expression. We get that:

$$\frac{1}{2i}\ln\left(\frac{1+i\tan(s)}{1-i\tan(s)}\right)=\tan^{-1}(\tan(s))+C_3$$

So:

$$\ln\left(\frac{1+i\tan(s)}{1-i\tan(s)}\right)=2is+C_4$$

Let us first get rid of the annoying C_4 term. If you plug in $s=0$ in both sides you will get that $C_4=0$. So the expression becomes:

$$\ln\left(\frac{1+i\tan(s)}{1-i\tan(s)}\right)=2is$$

Now let us simplify the expression that is sitting inside of the logarithm. Write:

$$\frac{1+i\tan(s)}{1-i\tan(s)}=\frac{\frac{\cos(s)+i\sin(s)}{\cos(s)}}{\frac{\cos(s)-i\sin(s)}{\cos(s)}}=\frac{\cos(s)+i\sin(s)}{\cos(s)-i\sin(s)}=$$

$$\frac{[\cos(s)+i\sin(s)]*[\cos(s)+i\sin(s)]}{[\cos(s)-i\sin(s)]*[\cos(s)+i\sin(s)]}=\frac{\cos^2(s)-\sin^2(s)+2\cos(s)\sin(s)}{\cos^2(s)+\sin^2(s)}=$$

$$\cos(2s)+i\sin(2s)$$

So with the previous formula we have that:

$$\ln(\cos(2s)+i\sin(2s))=2is$$

Are we warm or what! Take the exponential function of both sides to get:

$$e^{2is}=\cos(2s)+i\sin(2s)$$

And plugging in $s=\frac{\theta}{2}$ will give is Euler's amazing identity:

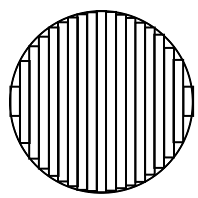
$$e^{i\theta}=\cos(\theta)+i\sin(\theta)$$

I don't know what was going through my professor's head when he saw this derivation of Euler's identity, but the first thing he said was: you need to define what you mean by $\ln(1+ix)$ and $\ln(1-ix)$ (he just loved taking out the fun out of things). Later I learned that he specialized in complex analysis and so he had full mastery of these things in his domain.

*After I wrote this section I looked back into the book and realized that you did a similar thing in your section "The Count Computes i^i. Here I used it in order to derive the general case, which is Euler's formula.

π

In middle school people they loved π day. I was amazed at how many digits of π my classmates could memorize. I never gave it much effort as I never found it very interesting to memorize the digits of π. One year I decided to compete in this competition of who could memorize more digits of π. I believe I placed around second place in the class, although there was no prize for that. I did though get a pencil as a participation award. But ever since 7^{th} grade I was wondering how did people even calculate the digits of π. I mean I could find websites with thousands upon thousands of digits of π. But how did people calculate them. My 7^{th} grade teacher told me that his favorite method was to approximate the area of a circle by summing the following rectangles:



It was the section "Calculating π From $\sqrt{-1}$" that finaly gave me the answer. I was astounded the simple expression $\frac{\pi i}{2} = \ln(i)$ could be used to create very quickly converging sequences to $\frac{\pi}{4}$. For this all you need is to think of an expression of the form:

$$i = \frac{\prod_{k=1}^{n}(a_k + ib_k)}{\prod_{k=1}^{n}(a_k - ib_k)}$$

Where the a_k's and the b_k's are real numbers. Up to that point the sequence:

$$\frac{\pi}{4} = 4\left(\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)5^{2k+1}}\right) - \left(\sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)239^{2k+1}}\right)$$

That you provide on page 174 was the fastest converging sequence to a rational multiple of π that I ever saw at that time. Amazing!

*In this letter it may seem like I am skipping over certain material in your book, but I will come back to some of them later.

How to Drive People Crazy!

This section is actually a hyperbole of the hyperbole. Unfortunately I was never able to evoke the "crazy" emotions I was looking for in people when I showed them the following derivation. People either ignored or just nodded their heads in silence when I showed them this mathematics (that's actually the worst reaction a presenter could see: the slow silent nod. It usually means that the person isn't following what the presenter is saying). The only person that showed me some emotional reaction was my professor when he saw on the blackboard the formula (which I wrote up there… naturally):

$$\sum_{k=1}^{\infty}(-1)^k = \frac{1}{2}$$

Before I continue any further in my story, let me explain why I'm writing this section in the first place. In your book on page 185 you write that the crown jewel of mathematics is the formula (I will not disagree):

$$\zeta(s) = \zeta(1-s)\Gamma(1-s)2^s\pi^{s-1}\sin(\frac{1}{2}\pi s)$$

Then you made the comment that if you take the limit of both sides as $s \to 0$, the $\zeta(1-s)$ term goes to infinity while the $\sin(\frac{1}{2}\pi s)$ term goes to zero in such a way that the whole expression goes to $-\frac{1}{2}$. This would then say that $\zeta(0) = -\frac{1}{2}$.

Let me now return to what I wrote on the blackboard that morning before math class started. You may already know this "derivation," but I'll present it anyways. On the blackboard that morning I said: "take the Laplace transform of $\sin(x)$:"

$$I(\alpha) = \int_0^{\infty} e^{-\alpha x}\sin(x)\,dx \qquad for\ \alpha > 0$$

(We were in fact studying differential equations at the time) The fact that this was called a Laplace transform didn't bother me. I just wanted to construct this function $I(\alpha)$. Now we can evaluate the above integral by doing integration by parts two times (you can also do it easily by writing $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$):

$$\int_0^{\infty} e^{-\alpha x}\sin(x)\,dx = -\frac{e^{-\alpha x}}{\alpha}\sin(x)\Big|_0^{\infty} + \frac{1}{\alpha}\int_0^{\infty} e^{-\alpha x}\cos(x)\,dx =$$

$$\frac{1}{\alpha}\left(-\frac{e^{-\alpha x}}{\alpha}\cos(x)\Big|_0^{\infty} - \frac{1}{\alpha}\int_0^{\infty} e^{-\alpha x}\sin(x)\,dx\right) = \frac{1}{\alpha^2} - \frac{1}{\alpha^2}\int_0^{\infty} e^{-\alpha x}\sin(x)\,dx$$

So:

$$\left(1+\frac{1}{\alpha^2}\right)\int_0^\infty e^{-\alpha x}\sin(x)\,dx = \frac{1}{\alpha^2}$$

$$\int_0^\infty e^{-\alpha x}\sin(x)\,dx = \frac{\frac{1}{\alpha^2}}{\left(1+\frac{1}{\alpha^2}\right)} = \frac{1}{1+\alpha^2}$$

And so we get:

$$\int_0^\infty e^{-\alpha x}\sin(x)\,dx = \frac{1}{1+\alpha^2}$$

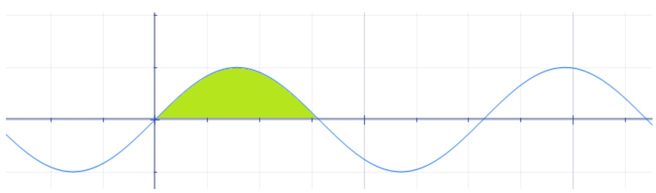
Now, take the limit of both sides as $\alpha \to 0^+$. We get that:

$$\lim_{\alpha\to 0^+}(I(\alpha)) = \lim_{\alpha\to 0^+}\int_0^\infty e^{-\alpha x}\sin(x)\,dx = \int_0^\infty \sin(x)\,dx = \lim_{\alpha\to 0^+}\frac{1}{1+\alpha^2} = 1$$

Hurray, so we get the result:

$$\int_0^\infty \sin(x)\,dx = 1$$

By this point the professor must have been thinking "What is this student thinking?! Bending and twisting the mathematics that I teach in this class in this manner." Now $\sin(x)$ looks like (all of the graphs in this letter were made in Microsoft Mathematics):



And all freshman calculus students who have been doing their homework know that the absolute area between each of the arcs and the x-axis (shown by the shaded region above) is 2. So we can write the integral in the last equation as:

$$\sum_{k=1}^{\infty}(-1)^k * 2 = 1$$

Dividing both sides by 2 will give the equation I wrote on the blackboard:

$$\sum_{k=1}^{\infty}(-1)^k = \frac{1}{2}$$

But we won't stop here. On page 153 you mention the subject of analytic continuation of the zeta function. On that page you say that one of the formulas that mathematicians use to analytically extend the zeta function beyond where it is classically not defined, is the formula:

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^z} = (1-2^{1-z})\zeta(z)$$

This is a formula that is very easy to derive for z such that (remember z could be complex too) $Re(z) > 1$ (The region where $Re(z) > 1$ is in fact the region in the complex plain where the series $\sum_{k=1}^{\infty}\frac{1}{k^z}$ converges). The left hand side of the equation is in fact called Dirichlet's eta function, and is denoted by:

$$\eta(z) = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^z}$$

So we have:

$$\eta(z) = (1-2^{1-z})\zeta(z)$$

Now looking back to the formula that I wrote on the board, we see that:

$$\eta(0) = \sum_{k=1}^{\infty}(-1)^k = \frac{1}{2}$$

So we have that:

$$\frac{1}{2} = (1-2^{1-0})\zeta(0)$$

Rearrangement finally gives us:

$$\zeta(0) = -\frac{1}{2}$$

Yeah, we're both thinking the same thing. Bold! And of course my use of the limit was absolutely wrong. To be specific, the step when the correctness of the derivation falls apart was when I did

$$\lim_{\alpha\to 0^+}\int_0^{\infty} e^{-\alpha x}\sin(x)\,dx = \int_0^{\infty}\sin(x)\,dx$$

But hey, it gave us the correct answer. Formal analytic continuation of the zeta function gives the same answer. I mean if my analysis teacher had the right to be bold when using the Dirac delta function when solving differential equations (although I will admit that he wasn't being as bold as I am in this case), I have the right to be bold as well.

I didn't stop here. Let us try to compute the zeta function at negative values. You show in your book that $\zeta(-2n) = 0$ for positive integers n. We could in fact try to compute the zeta function at all negative integer values. Our calculation will again require us to be bold. You know, I don't know whether boldness was good or bad trait that I picked up from you and Euler. Anyways, take the famous series expansion:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$$

This formula is in fact yet another source of the formula $\sum_{k=1}^{\infty}(-1)^k = \frac{1}{2}$. Just plug in -1 into x above. That should comfort us that there is some truth in all of this madness as two different methods yield the same answer. Anyways continuing, let us differentiate both sides:

$$\frac{-1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{k=0}^{\infty}(k+1)x^k$$

Now take the limit of both sides as $x \to -1^+$.The left hand side converges to $-\frac{1}{2}$, and the right hand side converges to the series:

$$1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots = \sum_{k=0}^{\infty}(-1)^k(k+1)$$

So we have:

$$-\frac{1}{4} = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots = \sum_{k=0}^{\infty}(-1)^k(k+1)$$

Now notice that the right hand side is the Dirichlet eta function, evaluated at -1. So we get:

$$\eta(-1) = -\frac{1}{4}$$

Using the formula $\eta(z) = (1 - 2^{1-z})\zeta(z)$, we get that:

$$\left(1 - 2^{1-(-1)}\right)\zeta(-1) = -\frac{1}{4}$$

And rearranging gives:

$$\zeta(-1) = -\frac{1}{12}$$

This also agrees with the values of the zeta function when one does analytic continuation to -1. Pretty cool, huh! To get the next value -2, one just has to multiply the above Taylor polynomial by x and then differentiate again to get:

$$\frac{d}{dx}\left(x * \frac{-1}{(1-x)^2}\right) = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots = \sum_{k=0}^{\infty}(k+1)^2x^k$$

And then take the limit as $x \to -1^+$ again. To get further negative values of the zeta function just repeat the same process over and over again. In fact one should get that:

$$\zeta(-n) = \frac{1}{(1-2^{1+n})} \cdot \lim_{x\to -1^+}\left(\frac{d}{dx}\left(x \cdot \frac{d}{dx}\left(x \cdot \frac{d}{dx}\left(x \cdot \ldots \cdot \frac{d}{dx}\left(\frac{1}{(1-x)}\right)\ldots\right)\right)\right)\right)$$

Where n is a nonnegative integer. Wikipedia has the following formula for $\zeta(-n)$:

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

Where B_n denotes the n$^{\text{th}}$ Bernoulli number. I naturally wanted to and have yet not been able to prove that:

$$-\frac{B_{n+1}}{n+1} = \frac{1}{(1-2^{1+n})} \cdot \lim_{x\to -1^+}\left(\frac{d}{dx}\left(x \cdot \frac{d}{dx}\left(x \cdot \frac{d}{dx}\left(x \cdot \ldots \cdot \frac{d}{dx}\left(\frac{1}{(1-x)}\right)\ldots\right)\right)\right)\right)$$

Related to the zeta function, Euler was able to prove that:

$$\zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!}$$

This is yet another formula that I have tried and have not been able to prove so far. But I will keep trying.

The Singularity of the Zeta Function at 1

In the last section in chapter 6 you derive the zeta functional relationship formula. At one point you say that in order to show that $\zeta(0) = -\frac{1}{2}$, one has to show that the singularity of the Riemann-Zeta function is powerful enough to overcome the zero of $\sin(x)$ at $x = 0$ in order to get a finite non-zero limit. More rigorously one has to show that:

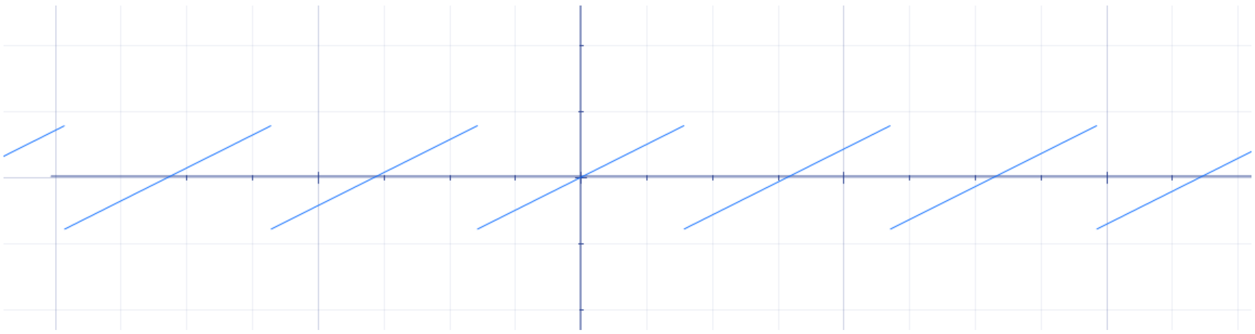
$$\lim_{x\to 1}\big(\zeta(x)(x-1)\big) \text{ exists and is non-zero.}$$

Let me discuss what I did in order to explore what kind of singularity the zeta function has at one.

One day I was sitting and I decided to explore something that one could call "hyper-Fourier series." By this I meant Fourier series that had t^3 in the arguments of the trigonometric function instead of just plain old t. Any undergraduate who has studied Fourier series knows the most famous of Fourier series: the saw tooth (here $\lfloor\quad\rfloor$ means the floor function):

$$\frac{x}{2} - \left\lfloor \frac{x+\pi}{2\pi} \right\rfloor \pi = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx)$$

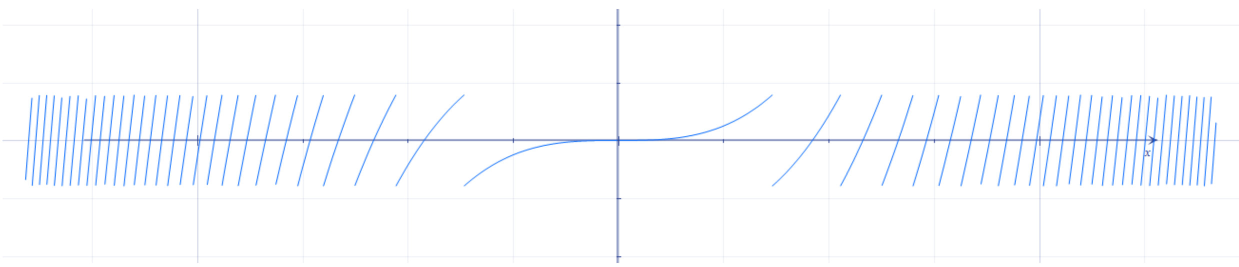
Which looks like:



Then I went ahead and plugged in t^3 into x on both sides of the equation, which will give us:

$$\frac{t^3}{2} - \left\lfloor \frac{t^3+\pi}{2\pi} \right\rfloor \pi = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kt^3)$$

If we plot the graph of this we get function we will get:



Then I integrated both sides of the equation from 0 to ∞. Integrating the left side is a little bit tricky, but correctly parametrizing and integrating each piece (separated by jumps) and then adding up all of the results will give:

$$\int_0^{\infty} \left(\frac{t^3}{2} - \left\lfloor \frac{t^3+\pi}{2\pi} \right\rfloor \pi \right) dt = \frac{\pi^{\frac{4}{3}}}{8} + \frac{\pi^{\frac{4}{3}}}{8} \sum_{k=2}^{\infty} \left((6k-5)(2k-3)^{\frac{1}{3}} - (6k-7)(2k-1)^{\frac{1}{3}} \right)$$

Integrating the right side (and using the fact that $\int_0^{\infty} \sin(t^n)\, dt = \sin\left(\frac{\pi}{2n}\right) \Gamma\left(\frac{n+1}{n}\right)$, something that I will show later in this letter how to derive using complex analysis) gives (just like you say in your book, I can interchange sums and integrals without hesitation and faster than anyone can snap their fingers – as long as I get the correct answer):

$$\int_0^{\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kt^3) \right) dt = \sum_{k=1}^{\infty} \left(\frac{(-1)^{k+1}}{k} \int_0^{\infty} \sin(kt^3)\, dt \right) =$$

$$\sin\left(\frac{\pi}{6}\right)\Gamma\left(\frac{4}{3}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{\frac{4}{3}}}=\frac{2^{\frac{4}{3}}-2}{2^{\frac{4}{3}}}\sin\left(\frac{\pi}{6}\right)\Gamma\left(\frac{4}{3}\right)\zeta\left(\frac{4}{3}\right)$$

In the last equality I used the relationship:

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{z}}=\frac{2^{z}-2}{2^{z}}\zeta(z)$$

Which can be easily derived by just rearranging terms. The left hand side is in fact called Dirichlet's Eta function, something I will touch upon later. So combining the two results gives us the pretty cool identity:

$$\frac{\pi^{\frac{4}{3}}}{8}+\frac{\pi^{\frac{4}{3}}}{8}\sum_{k=2}^{\infty}\left((6k-5)(2k-3)^{\frac{1}{3}}-(6k-7)(2k-1)^{\frac{1}{3}}\right)=\frac{2^{\frac{4}{3}}-2}{2^{\frac{4}{3}}}\sin\left(\frac{\pi}{6}\right)\Gamma\left(\frac{4}{3}\right)\zeta\left(\frac{4}{3}\right)$$

Or multiplying through by $\frac{8}{\pi^{\frac{4}{3}}}$ gives us:

$$1+\sum_{k=2}^{\infty}\left((6k-5)(2k-3)^{\frac{1}{3}}-(6k-7)(2k-1)^{\frac{1}{3}}\right)=\frac{8}{\pi^{\frac{4}{3}}}\cdot\frac{2^{\frac{4}{3}}-2}{2^{\frac{4}{3}}}\sin\left(\frac{\pi}{6}\right)\Gamma\left(\frac{4}{3}\right)\zeta\left(\frac{4}{3}\right)$$

An amazing looking formula! In fact one can check numerically that the left side converges to the quantity on the right, as it should.

Now remember how in an early step I plugged in t^3 into x? I could have plugged in t^n into x where n is an odd positive integer (I have a bad feeling about even n, so I didn't look at them) and one can derive the more general result:

$$1+\sum_{k=2}^{\infty}\left(\big(2nk-(2n-1)\big)(2k-3)^{\frac{1}{n}}-\big(2nk-(2n+1)\big)(2k-1)^{\frac{1}{n}}\right)=$$

$$\frac{2(n+1)}{\pi^{\frac{n+1}{n}}}\cdot\frac{2^{\frac{n+1}{n}}-2}{2^{\frac{n+1}{n}}}\sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)\zeta\left(\frac{n+1}{n}\right)$$

This also looks like a remarkable formula! But the cool thing is, is that I can take this formula and take the limit of both sides as $n\to\infty$ in order to analyze the point of singularity of $\zeta(z)$ on the real axis as you approach 1 from the right side. Many of the terms on the right side will go away. For example $\Gamma\left(\frac{n+1}{n}\right)$ will tend to 1 as $n\to\infty$. $\sin\left(\frac{\pi}{2n}\right)$ will tend to zero approximately like $\frac{\pi}{2n}$ will. So taking the limit of both sides as $n\to\infty$, rearranging terms, using L'Hopital's rule in the mix, and relabeling variables will give the result that:

$$\lim_{\varepsilon\to 0^{+}}\big(\varepsilon\cdot\zeta(1+\varepsilon)\big)=$$

$$\frac{1}{\ln(2)} \cdot \lim_{n\to\infty}\left(1+\sum_{k=2}^{\infty}\left(\left(2nk-(2n-1)\right)(2k-3)^{\frac{1}{n}}-\left(2nk-(2n+1)\right)(2k-1)^{\frac{1}{n}}\right)\right)$$

I think that this is an incredible looking formula. But it was especially amazing to me because using inequalities and approximating integrals, I was able to prove that the right hand side of the equation lies somewhere in the interval $[0.5, \frac{1}{\ln(2)}]$ (I will not burden you with those details as I will soon show you how to calculate the limit exactly). Thus was I was able to prove that as the Riemann Zeta function approaches 1 from the right on the real axis, it will have a singularity of order 1 there:

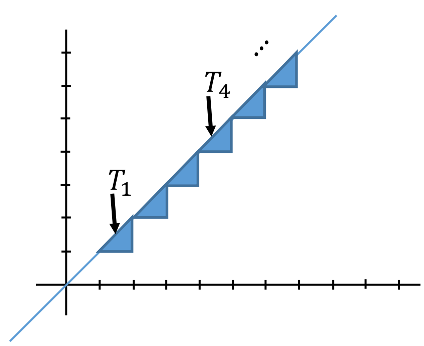
$$\lim_{\varepsilon\to 0^+}\left(\varepsilon\cdot\zeta(1+\varepsilon)\right)\in[0.5,\frac{1}{\ln(2)}]$$

One day I came into Professor Marshall's office (he was my teacher in first year calculus) and I showed him this result. He said that he found it most interesting and then he proceeded to show me how to calculate the above limit exactly. He was the one who showed me the following calculation.

First Professor Marshall started to talk about how one can extend the Riemann Zeta function onto the set $\{z\in\mathbb{C} : Re(z)>0\}$ in the complex plane. Professor Marshall wrote that for $\forall z\in\mathbb{C} : Re(z)>1$,

$$\zeta(z)+\frac{1}{1-z}=\sum_{k=1}^{\infty}\frac{1}{k^z}-\int_1^{\infty}\frac{1}{x^z}dx=\sum_{k=1}^{\infty}\left(\frac{1}{k^z}-\int_k^{k+1}\frac{1}{x^z}dx\right)=\sum_{k=1}^{\infty}\left(\int_k^{k+1}\left(\frac{1}{k^z}-\frac{1}{x^z}\right)dx\right)=$$
$$\sum_{k=1}^{\infty}\left(\int_k^{k+1}\left(\int_k^{x}zy^{-(z+1)}\,dy\right)dx\right)=z\sum_{k=1}^{\infty}\left(\iint_{T_k}y^{-(z+1)}\,dydx\right)$$

Where the $T_k{}'s$ are the triangles in the picture:



Let us call the set $S=\bigcup_{k=1}^{\infty}T_k$. Then we can rewrite the last integral as:

$$\zeta(z)+\frac{1}{1-z}=z\iint_S y^{-(z+1)}\,dA$$

Now noticing that this integral converges on $\{z \in \mathbb{C} : Re(z) > 0\}$, we then get that we can use this formula to analytically extend the Riemann Zeta function onto the region:

$$\{z \in \mathbb{C} : Re(z) > 0\}$$

By writing:

$$\zeta(z) = \frac{1}{1-z} + z \iint_S y^{-(z+1)}\, dA$$

Of course at first this might not look like a very convenient way to compute the value of the Riemann-zeta function in this analytically extended region, it does however make the calculation of the limit that we want trivial. Take the last equation and multiply through by $(z-1)$ to get:

$$(z-1)\zeta(z) = 1 + z(z-1) \iint_S y^{-(z+1)}\, dA$$

Now take the limit of both sides as z approaches the 1 from the right on the real axis. Since $\lim_{z \to 1^{+,\mathbb{R}}} \left(\iint_S y^{-(z+1)}\, dA \right)$ is finite we then get that the limit that we want is:

$$\lim_{x \to 1^+} (x-1)\zeta(x) = 1$$

Or rewritten in another form:

$$\lim_{\varepsilon \to 0^+} \big(\varepsilon \cdot \zeta(1+\varepsilon) \big) = 1$$

Once I understood this, I thought that this was really cool! But the most amazing part of all was that combining this with my result (and doing a little bit of rearranging) gives the identity:

$$\lim_{n \to \infty} \left(n \sum_{k=2}^{\infty} \left(\left(2k - 2 - \frac{1}{n} \right) (2k-1)^{\frac{1}{n}} - \left(2k - 2 + \frac{1}{n} \right) (2k-3)^{\frac{1}{n}} \right) \right) = 1 - \ln(2)$$

A result to truly be proud of. I later wrote this on the blackboard before class in Math 336.

Using this method of what I called "hyper-Fourier series" I was able to show that the singularity of the zeta function as you approach it from the right is of order 1. But with my method I was not however able to calculate precisely what the residue of the zeta function was at 1. Professor Marshall showed me how to do that precisely, and with that I was able combine his and my result to create a new identity (the above equation).

However I did think that Professor Marshall's method of extending the Riemann-zeta function was a little bit complicated and I wondered whether or not there was a simpler method. Indeed there is and it is in fact a method that you touch upon on page 153.

We first define the Dirichlet eta function:

$$\eta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^z}$$

As mentioned before, it is easy to prove by rearranging terms that for $\forall z \in \mathbb{C} : Re(z) > 1$:

$$\eta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^z} = \frac{2^z - 2}{2^z} \zeta(z)$$

So:

$$\zeta(z) = \left(\frac{2^z - 2}{2^z}\right)^{-1} \eta(z)$$

In order to use this to extend the Riemann Zeta Function onto the region $\{z \in \mathbb{C} : Re(z) > 0\}$ (except of course at 1, which is a point of disastrous singularities for the Riemann Zeta function), all we have to do is prove that the Dirichlet's Eta function converges on $\{z \in \mathbb{C} : Re(z) > 0\}$. This was a point that I was stuck on for some time. But then after hearing Professor Marshall's extension, I quickly realized that the convergence of the Dirichlet eta function can be proven by using a similar idea of rewriting the series as an integral. We write:

$$\eta(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^z} = \left(\frac{1}{1^z} - \frac{1}{2^z}\right) + \left(\frac{1}{3^z} - \frac{1}{4^z}\right) + \cdots = z \int_1^2 t^{-z-1} dt + z \int_3^4 t^{-z-1} dt + \cdots =$$

$$\eta(z) = z \cdot \int_{\cup_{k=0}^{\infty} [2k+1, 2k+2] = [1,2] \cup [3,4] \cup \ldots} t^{-z-1} dt$$

And we can see that the integral on the right hand side converges on $\{z \in \mathbb{C} : Re(z) > 0\}$ and so the Dirichlet Eta Function converges in this region. So we can indeed use the formula:

$$\zeta(z) = \left(\frac{2^z - 2}{2^z}\right)^{-1} \eta(z)$$

To extend the Riemann Zeta function onto the region $\{z \in \mathbb{C} : Re(z) > 0\}$ as well.

This extension through the Dirichlet Eta function does seem to be much simpler than the extension that Professor Marshall showed me. However, it doesn't seem to be very useful in figuring out the limit:

$$\lim_{z \to 1} \big((z-1) \cdot \zeta(z)\big) = 1$$

Complex Analysis

Subsection 1: The Hyper-Fresnel Integrals:

There will be 3 "subsections" in the section "Complex Analysis" since this section is so large.

Now we get to my currently favorite subject of mathematics: complex analysis. Although I will point out that I don't bias against any field of mathematics when I do personal research. So what I mean is that when say "my favorite math subject", well… that doesn't mean much. This chapter was the most astounding of them all! Chapter 6 was amazing, but this one was even greater in my opinion. In fact nearly everything I know about complex integration comes from this book. From the period after I read this chapter, I've learned a lot more about complex integration from discovering facts for myself or little independent readings from various sources. But the basis of my knowledge comes from this book. As I'm writing this section, I don't even know what I should start this section off with.

In your section on complex analysis you derive the Fresnel integrals (I was not satisfied with your derivation of the Fresnel integrals in chapter 6 though, so I consider the true derivation to be the one in the complex analysis section). These integrals are fantastic! I remember that when I showed them to my math professor for the first time, the first thing he told me was "these integrals don't even converge." Later a little thought convinced him that these integrals did in fact converge. See how deceptive these integrals are?

One of the things that one might wonder is why the two integrals must be equivalent:

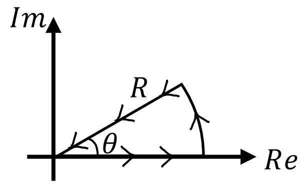
$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx$$

Another thing that someone might wonder is: can one generalize the Fresnel integrals any further. The answer to both questions are the equations:

$$\int_0^\infty \sin(x^n)\,dx = \sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$$

$$\int_0^\infty \cos(x^n)\,dx = \cos\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$$

Where $\Gamma(x)$ is the gamma function. The above formulas are important because they provide a good method for numerically calculating these integrals. Can you imagine how one would go about calculating numerically the above integrals? Direct numerical computation would converge extremely slowly for large n. But calculating $\sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$ and $\cos\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$ are easy. The way I derived the above integrals was to just generalize the calculation you did on pages 199 – 201. All I did was, instead of integrating the complex function e^{-z^2}, I integrated e^{-z^n}. The contour that integrated on was the following pie shaped slice in the complex plain:



Let the angle of the pie slice be $\theta = \frac{\pi}{2n}$. R is the radius of the pie slice. Let us call this contour upon which we integrate γ (Technically γ should be a function of R because we are later going to stretch the contour by letting $R \to \infty$. So we should be writing γ_R. But for brevity I will drop the subscript and just write γ). Now since e^{-z^n} is an analytic function and γ is a closed looped contour, we have that:

$$\oint_{\gamma} e^{-z^n} dz = 0$$

Now let us expand the integral:

$$\oint_{\gamma} e^{-z^n} dz = \int_0^R e^{-x^n} dx + i \int_0^{\frac{\pi}{2n}} e^{-R^3 e^{i3\theta}} Re^{i\theta} d\theta$$
$$- \int_0^R e^{-\left(\cos\left(\frac{\pi}{2n}\right) + i \sin\left(\frac{\pi}{2n}\right)\right)^n x^n} \left(\cos\left(\frac{\pi}{2n}\right) + i \sin\left(\frac{\pi}{2n}\right)\right) dx$$

By DeMoivre's Formula, we know that the quantity in the exponent in the last integral is:

$$\left(\cos\left(\frac{\pi}{2n}\right) + i \sin\left(\frac{\pi}{2n}\right)\right)^n = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) = i$$

So we have that:

$$\oint_{\gamma} e^{-z^n} dz = \int_0^R e^{-x^n} dx + i \int_0^{\frac{\pi}{2n}} e^{-R^n e^{in\theta}} Re^{i\theta} d\theta - \int_0^R e^{-ix^n} \left(\cos\left(\frac{\pi}{2n}\right) + i \sin\left(\frac{\pi}{2n}\right)\right) dx$$

Now take the limit of both sides as $R \to \infty$ (remember the left hand side is constantly zero and so its limit is zero) and get that:

$$\int_0^{\infty} e^{-x^n} dx + \lim_{R\to\infty} \left(i \int_0^{\frac{\pi}{2n}} e^{-R^n e^{in\theta}} Re^{i\theta} d\theta \right) - \int_0^{\infty} e^{-ix^n} \left(\cos\left(\frac{\pi}{2n}\right) + i \sin\left(\frac{\pi}{2n}\right)\right) dx = 0$$

It isn't hard to show (following the same logic in your book) that:

$$\lim_{R\to\infty}\left(i\int_0^{\frac{\pi}{2n}} e^{-R^n e^{in\theta}} Re^{i\theta}\,d\theta \right) = 0$$

So we get that:

$$\int_0^\infty e^{-x^n}\,dx = \int_0^\infty e^{-ix^n}\left(\cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right)\right)dx$$

Now let us handle the right hand side first:

$$\int_0^\infty e^{-ix^n}\left(\cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right)\right)dx = \int_0^\infty (\cos(x^n) - i\sin(x^n))\left(\cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right)\right)dx =$$

$$\left(\cos\left(\frac{\pi}{2n}\right)\int_0^\infty \cos(x^n)\,dx + \sin\left(\frac{\pi}{2n}\right)\int_0^\infty \sin(x^n)\,dx\right)$$
$$+\, i\left(\sin\left(\frac{\pi}{2n}\right)\int_0^\infty \cos(x^n)\,dx - \cos\left(\frac{\pi}{2n}\right)\int_0^\infty \sin(x^n)\,dx\right)$$

Notice that this last expression has to be real because the integral $\int_0^\infty e^{-x^n}dx$ is a real number. This means that the imaginary component in the previous expression is equal to zero:

$$\sin\left(\frac{\pi}{2n}\right)\int_0^\infty \cos(x^n)\,dx - \cos\left(\frac{\pi}{2n}\right)\int_0^\infty \sin(x^n)\,dx = 0$$

So:

$$\int_0^\infty \cos(x^n)\,dx = \cot\left(\frac{\pi}{2n}\right)\int_0^\infty \sin(x^n)\,dx$$

Plugging this result into the expression:

$$\int_0^\infty e^{-x^n}\,dx = \cos\left(\frac{\pi}{2n}\right)\int_0^\infty \cos(x^n)\,dx + \sin\left(\frac{\pi}{2n}\right)\int_0^\infty \sin(x^n)\,dx$$

(Remember, the imaginary component is gone because it's equal to zero) we get that:

$$\int_0^\infty e^{-x^n}\,dx = \left(\cos\left(\frac{\pi}{2n}\right)\cot\left(\frac{\pi}{2n}\right) + \sin\left(\frac{\pi}{2n}\right)\right)\int_0^\infty \sin(x^n)\,dx$$

Simplification gives that:

$$\int_0^\infty \sin(x^n)\,dx = \sin\left(\frac{\pi}{2n}\right)\int_0^\infty e^{-x^n}\,dx$$

Now the only part that remains to show is that the integral $\int_0^\infty e^{-x^n}\,dx$ is connected with the Gamma Function. But that's easy! Just take:

$$\Gamma\left(\frac{1}{n}\right) = \int_0^\infty \frac{e^{-x}}{x^{\frac{n-1}{n}}}\,dx$$

Making the substitution $x = u^n$, one will get (remembering that $dx = n \cdot u^{n-1}du$)

$$\Gamma\left(\frac{1}{n}\right) = \int_0^\infty \frac{e^{-u^n}}{u^{n-1}} n * u^{n-1}du = n\int_0^\infty e^{-u^n}\,du$$

So we get that:

$$\int_0^\infty e^{-x^n}\,dx = \frac{1}{n}\Gamma\left(\frac{1}{n}\right) = \Gamma\left(\frac{n+1}{n}\right)$$

Substituting this into the integral $\int_0^\infty e^{-x^n}\,dx$ a couple of equations back will finally give us what we want:

$$\int_0^\infty \sin(x^n)\,dx = \sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{1}{n}\right)$$

A couple of equations back if would have solved for $\int_0^\infty \cos(x^n)\,dx$ instead of $\int_0^\infty \sin(x^n)\,dx$, we would have gotten the other result:

$$\int_0^\infty \cos(x^n)\,dx = \cos\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{1}{n}\right)$$

Putting them together we get the amazing pair of equations side by side:

$$\int_0^\infty \sin(x^n)\,dx = \sin\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$$

$$\int_0^\infty \cos(x^n)\,dx = \cos\left(\frac{\pi}{2n}\right)\Gamma\left(\frac{n+1}{n}\right)$$

Amazing right! I really don't know how I would have gone about proving these results without the use of complex analysis. If I were to give these integrals a name, I would call them Hyper-Fresnel integrals. You know, because they look like the Fresnel integrals but they are "hyper" since they instead involve the raising to the n^{th} power.

Subsection 2: Extension of a Really Important Integral:

In your derivation of the gamma reflection formula you use the very important integral:

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^\beta}dx = \frac{\pi}{\beta\sin(\frac{\alpha}{\beta}\pi)}$$

In chapter 6 you state that you will prove this integral in chapter 7, and indeed you do prove it there using complex analysis. However your derivation in chapter 7 only covers the case when α is an integer of the form $\alpha = 2m + 1$ where m is a positive integer and β is of the form $\beta = 2n$ for a non-negative integer n such that $n > m$. I later started to think about how one could extend this equation to positive real numbers α and β. Here I will show an approach on how to do that:

Let us take any $\alpha > 0$ and $\beta > 0$ such that $\beta > \alpha$. First notice that the integral:

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^\beta}dx$$

converges. The convergence of the integral as the upper limit goes to ∞ is given by the fact that $\beta > (\alpha - 1) + 1$. The convergence of the integral as the lower limit goes to zero from the right is given by the fact that $(\alpha - 1) > -1$ and $\left(1 + x^\beta\right) \approx 1$ when x is near zero from the right. So now that we've got the convergence question figured out, let us compute the integral.

The idea for the following computation came from the book "Complex Analysis" by Theodore W. Gamelin, page 207. I have a reference to this book at the end of this letter.

First let us do a change of variables. Substitute in $u = x^\alpha$:

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^\beta}dx = \int_0^\infty \frac{x^{\alpha-1}}{1+(x^\alpha)^{\frac{\beta}{\alpha}}}dx = \frac{1}{\alpha}\int_0^\infty \frac{1}{1+u^{\frac{\beta}{\alpha}}}du = \frac{1}{\alpha}\int_0^\infty \frac{1}{1+x^{\frac{\beta}{\alpha}}}dx$$

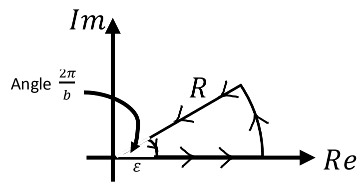
(in the last equality I just changed the variable u to x. I like the letter x more than the letter u). So now let us instead compute the integral:

$$\int_0^\infty \frac{1}{1+x^b}dx \quad where \quad b > 1$$

And then we will go back and substitute in $b = \frac{\beta}{\alpha}$. To do this we will integrate:

$$\oint_{\gamma_{\varepsilon,R}} \frac{1}{1+z^b}\,dz$$

Where $\gamma_{\varepsilon,R}$ is the following contour in the complex plain:



Let us agree that the cut for the function z^b goes along the set $\{z \in \mathbb{C} : Im(z) \in \mathbb{R}_+ \cup \{0\}\}$. Essentially this means that the cut runs down on the imaginary axis starting at zero. First we show that the integrand has a simple pole at $z = e^{\frac{i\pi}{b}}$ in the region bounded by $\gamma_{\varepsilon,R}$ (Let ε and R be small and large enough respectively in order for $\gamma_{\varepsilon,R}$ to contain the point $z = e^{\frac{i\pi}{b}}$). The fact that the integrand has simple pole at $z = e^{\frac{i\pi}{b}}$ can be seen by:

$$\lim_{z \to e^{\frac{i\pi}{b}}} \frac{\left(z - e^{\frac{i\pi}{b}}\right)}{1+z^b} = \lim_{z \to e^{\frac{i\pi}{b}}} \frac{\left(z - e^{\frac{i\pi}{b}}\right)}{z^b - (-1)} = \frac{1}{\left.\frac{d}{dz}(z^b)\right|_{z=e^{\frac{i\pi}{b}}}} = -\frac{e^{\frac{i\pi}{b}}}{b}$$

So $\frac{\left(z - e^{\frac{i\pi}{b}}\right)}{1+z^b}$ is bounded near $e^{\frac{i\pi}{b}}$. Using Riemann's Theorem on Removable Singularities, we see that the integrand indeed has a simple pole at $e^{\frac{i\pi}{b}}$ and:

$$Res\left[\frac{1}{1+z^b}, e^{\frac{i\pi}{b}}\right] = -\frac{2\pi i e^{\frac{i\pi}{b}}}{b}$$

With this we get that:

$$\oint_{\gamma_{\varepsilon,R}} \frac{1}{1+z^b}\,dz = \int_{\varepsilon}^{R} \frac{1}{1+x^b}\,dx + \int_{0}^{\frac{2\pi}{b}} \frac{1}{1+(Re^{i\theta})^b} Rie^{i\theta}\,d\theta - \int_{\varepsilon}^{R} \frac{1}{1+\left(xe^{i\frac{2\pi}{b}}\right)^b} e^{i\frac{2\pi}{b}}\,dx$$

$$-\int_{0}^{\frac{2\pi}{b}} \frac{1}{1+(\varepsilon e^{i\theta})^b} \varepsilon i e^{i\theta}\,d\theta = Res\left[\frac{1}{1+z^b}, e^{\frac{i\pi}{b}}\right] = -\frac{2\pi i e^{\frac{i\pi}{b}}}{b}$$

By the ML-Estimate we get that:

$$\lim_{\varepsilon\to 0^+}\left(\int_0^{\frac{2\pi}{b}} \frac{1}{1+(\varepsilon e^{i\theta})^b}\varepsilon i e^{i\theta} d\theta\right) \le \lim_{\varepsilon\to 0^+}\left(\frac{\varepsilon}{1-\varepsilon^b}\int_0^{\frac{2\pi}{b}} d\theta\right) = 0$$

And:

$$\lim_{R\to\infty}\left(\int_0^{\frac{2\pi}{b}} \frac{1}{1+(Re^{i\theta})^b} Rie^{i\theta} d\theta\right) \le \lim_{R\to\infty}\left(\frac{R}{R^b-1}\int_0^{\frac{2\pi}{b}} d\theta\right) = 0$$

This then gives us that:

$$\lim_{\varepsilon\to 0^+, R\to\infty}\left(\oint_{\gamma_{\varepsilon,R}} \frac{1}{1+z^b} dz\right) = \int_0^{\infty} \frac{1}{1+x^b} dx - e^{i\frac{2\pi}{b}}\int_0^{\infty} \frac{1}{1+x^b} dx =$$

$$\left(1-e^{i\frac{2\pi}{b}}\right)\int_0^{\infty} \frac{1}{1+x^b} dx = -\frac{2\pi i e^{\frac{i\pi}{b}}}{b}$$

So:

$$\int_0^{\infty} \frac{1}{1+x^b} dx = \frac{2\pi i e^{\frac{i\pi}{b}}}{b\left(e^{i\frac{2\pi}{b}}-1\right)} = \frac{\pi}{b\cdot\left(\frac{e^{\frac{i\pi}{b}}-e^{-\frac{i\pi}{b}}}{2i}\right)} = \frac{\pi}{b\cdot\sin\left(\frac{\pi}{b}\right)}$$

Going back and plugging this into the original integral gives us that:

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x^{\beta}} dx = \frac{1}{\alpha}\int_0^{\infty} \frac{1}{1+x^{\frac{\beta}{\alpha}}} dx = \frac{1}{\alpha}\cdot\frac{\pi}{\frac{\beta}{\alpha}\sin\left(\frac{\alpha\pi}{\beta}\right)} = \frac{\pi}{\beta\cdot\sin\left(\frac{\alpha\pi}{\beta}\right)}$$

And so we finally get the result that we wanted:

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x^{\beta}} dx = \frac{\pi}{\beta\cdot\sin\left(\frac{\alpha\pi}{\beta}\right)} \quad for \quad \alpha,\beta > 0 \quad such\ that \quad \beta > \alpha$$

This is the extension of the formula in your book.

Subsection 3: The Celestial Integral:

I did want to make one comment about the integral that you take in section 7.7 on page 218. In chapter 5 you show how this integral is connected with proving Kepler's 3rd law. In section 7.7 you compute this integral using complex analysis:

$$\int_0^{2\pi} \frac{1}{(1+E\cos(\theta))^2}\,d\theta = \frac{2\pi}{(1-E^2)^{\frac{3}{2}}} \quad for \quad 0 \leq E < 1$$

I remember when I first read your derivation I was amazed at how well complex analysis was able to handle this integral. I was also amazed by the generalization of Cauchy second integral theorem:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

I remember that later I sat down and took the integral by just substituting $E\cos(\theta)$ into x in the Taylor series expansion of $\frac{1}{(1+x)^2}$ (The following details are just supposed to be an illustration of the calculation, and are not in presentation mode. Meaning I skip a lot of steps and so these details are not meant to be followed, at least easily that is):

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + \cdots = \sum_{k=0}^{\infty} kx^{k-1}$$

$$\frac{1}{(1+E\cos(\theta))^2} = 1 - 2\cos(\theta) + 3\cos^2(\theta) - 4\cos^3(\theta) + 5\cos^4(\theta) - \cdots = \sum_{k=1}^{\infty} k\cos^{k-1}(\theta)$$

Now integrating both sides from 0 to 2π and using the fact that:

$$\int_0^{2\pi} \cos^n(x)\,dx = \begin{cases} 0 \quad if \quad n \equiv 1\ (mod\ 2) \\ \dfrac{\prod_{k=1}^{\frac{n}{2}} 2k-1}{\prod_{k=1}^{\frac{n}{2}} 2k} 2\pi \quad if \quad n \equiv 0\ (mod\ 2) \end{cases}$$

One will get that:

$$\int_0^{2\pi} \frac{1}{(1+E\cos(\theta))^2}\,d\theta = 2\pi\left(1 + \sum_{n=1}^{\infty}\left((2n+1)E^{2n}\frac{\prod_{k=1}^{n} 2k-1}{\prod_{k=1}^{n} 2k}\right)\right) =$$

$$2\pi\sum_{n=0}^{\infty}\left(E^{2n}\frac{(2n+1)!}{2^{2n}(n!)^2}\right)$$

Now using the binomial expansion of $(1+x)^\alpha$ for real value α (This is just its Taylor series expansion around $x=0$):

$$(1+x)^{\alpha} = 1 + \sum_{n=0}^{\infty}\left(\left(\prod_{k=1}^{n}\alpha-(k-1)\right)x^n\right)$$

One can notice that:

$$(1-x^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty}\left(x^{2n}\frac{(2n+1)!}{2^{2n}(n!)^2}\right)$$

And so plugging in E into x above we get that:

$$2\pi\sum_{n=0}^{\infty}\left(E^{2n}\frac{(2n+1)!}{2^{2n}(n!)^2}\right) = \frac{2\pi}{(1-E^2)^{-\frac{3}{2}}}$$

And so I finally got the result:

$$\int_0^{2\pi}\frac{1}{(1+E\cos(\theta))^2}d\theta = \frac{2\pi}{(1-E^2)^{\frac{3}{2}}}$$

The value of the integral $\int_0^{2\pi}\cos^n(x)\,dx$ that I wrote out above is in fact the connection between my derivation here and yours using complex analysis. In fact, the value of the integral is also deeply connected with Fourier series, thus hinting at some underlying connection between Fourier series and complex analysis, a connection that I will talk about later.

Part II:

When I finished the first book "An Imaginary Tale: The Story of $\sqrt{-1}$," I immediately wanted to get the second book "Dr. Euler's Fabulous Formula." Once I got my hands on it I wasted no time and started to read it (The page numbers below refer to the hard cover edition of "Dr. Euler's Fabulous Formula" that was published in 2006).

Ramanujan sum

When you look at the series on page 28 that Ramanujan summed we immediately see that this is a Fourier series:

$$P(x) = \sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{(n+1)(n+2)}$$

Once I read about how Ramanujan summed this series I wanted to see if I can use his method in order to discover new things. Indeed I did! First of all I set out to derive the above sum in a more generalized form. Specifically I calculated the sum:

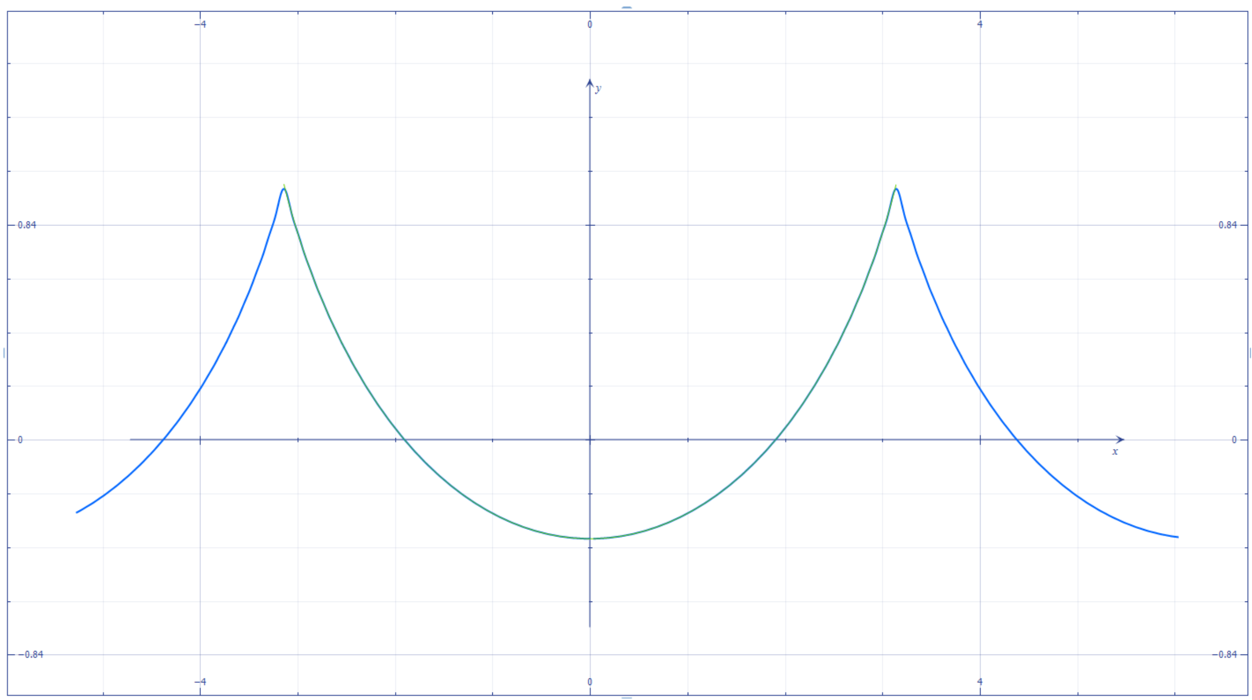
$$P_{m,n}(x) = \sum_{k=1}^{\infty}\frac{(-1)^k\cos(kx)}{(k+n)(k+m)}$$

Where m and n are two non-equal integers. I got the result that (by just following the same exact steps as Ramanujan did):

$$P_{m,n}(x) = \frac{1}{m-n}\left(\ln\left(2\cos\left(\frac{x}{2}\right)\right)[(-1)^m\cos(mx) - (-1)^n\cos(nx)] + \frac{x}{2}[(-1)^m\sin(mx) - (-1)^n\sin(nx)] + (-1)^n\sum_{k=1}^{n}\frac{(-1)^{k+1}\cos\big((n-k)x\big)}{k} - (-1)^m\sum_{k=1}^{m}\frac{(-1)^{k+1}\cos\big((m-k)x\big)}{k}\right)$$

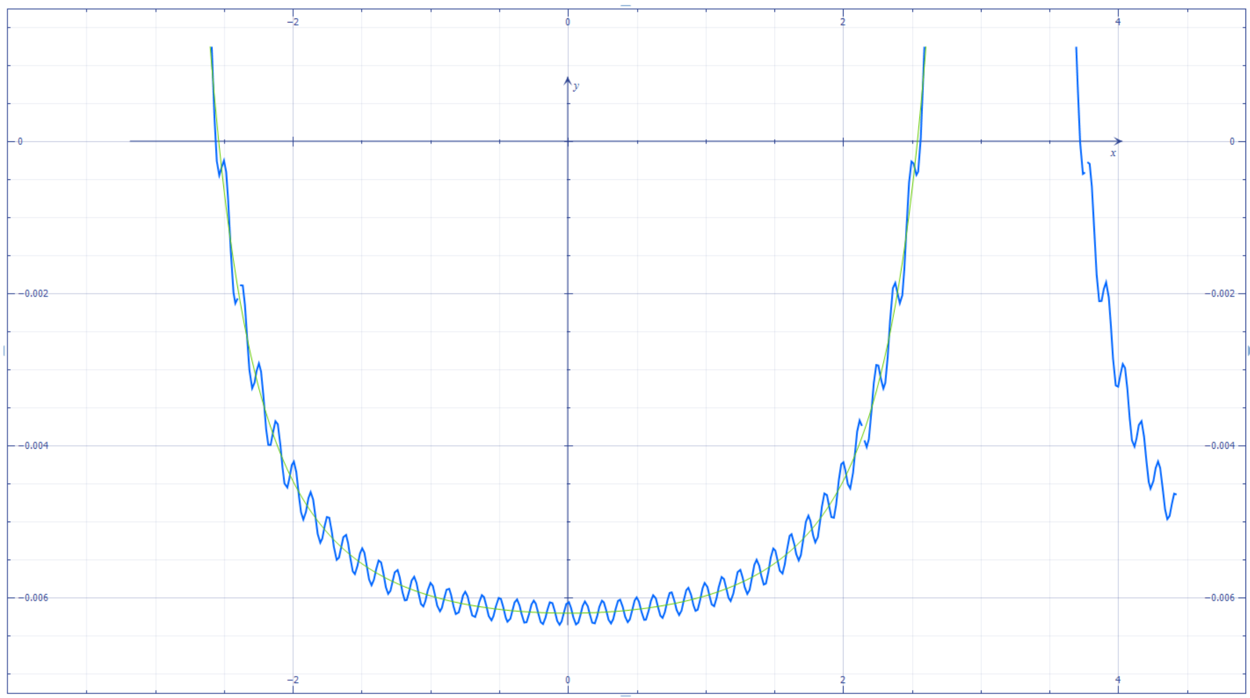
I personally think that it's really interesting to look at some graphs of these functions. So let us do exactly that.

Taking $n = 0$ and $m = 1$ we get the graph:

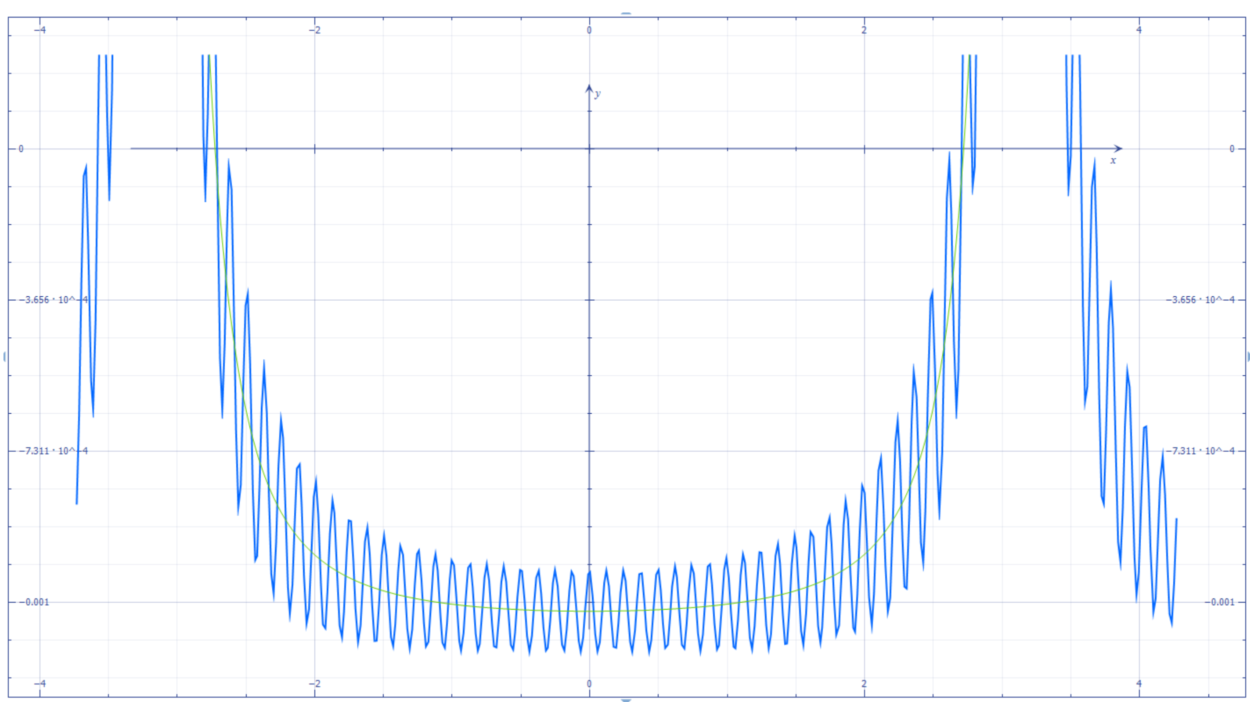


The blue curve is the series of $P_{1,0}(x)$ added up to 50 terms. The green curve is the exact evaluated answer for this series presented in the last equation. The fact that it is hard to distinguish the two graphs is due to the amazing agreement of the two results!

Now let us look at the case when $n = 7$ and $m = 11$:



Again here the series for $P_{7,11}(x)$ is added up to 50 terms. Finally let us look at the case when $n = 17$ and $m = 25$:



As before the series for $P_{17,25}(x)$ is added up to 50 terms. One of the things that I find fascinating and interesting in this sequence of graphs is that the amount of wiggling doesn't decrease at all as the integers m and n grow.

After this I was interested in whether I could derive some more interesting results by integrating both sides of the equation for $P_{m,n}(x)$. This is exactly what I did, and let us see what kind of things I discovered in the next section.

Integrating Both Sides of the Ramanujan Sum Equation

In this section I analyze a special case, namely $P_{0,2}(x)$. I started off with:

$$P_{0,2}(x) = \sum_{k=1}^{\infty} \frac{(-1)^k \cos(kx)}{k(k+2)} = \frac{1}{2}\ln\left(2\cos\left(\frac{x}{2}\right)\right)[\cos(2x) - 1] + \frac{x}{4}\sin(2x) - \frac{1}{2}\cos(x) + \frac{1}{4}$$

Now let us integrate both sides from 0 to $\pi/2$. Integrating the left side of this equation gives us:

$$\int_0^{\frac{\pi}{2}} \sum_{k=1}^{\infty} \frac{(-1)^k \cos(kx)}{k(k+2)} dx = \left(\sum_{k=1}^{\infty} \frac{(-1)^k \sin(kx)}{k^2(k+2)}\right)\Bigg|_0^{\frac{\pi}{2}} = \sum_{k=1}^{\infty} \frac{(-1)^k \sin\left(\frac{k\pi}{2}\right)}{k^2(k+2)} =$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2(2k+3)}$$

Integrating the right hand side of the equation gives us:

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(2\,cos\left(\frac{x}{2}\right)\right)\cdot\cos(2x)\,dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(2\cos\left(\frac{x}{2}\right)\right)dx + \int_0^{\frac{\pi}{2}} \left(\frac{x}{4}\sin(2x) - \frac{1}{2}\cos(x) + \frac{1}{4}\right)dx =$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(2\,cos\left(\frac{x}{2}\right)\right)\cdot\cos(2x)\,dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(2\cos\left(\frac{x}{2}\right)\right)dx + \frac{3\pi}{16} - \frac{1}{2}$$

Now the first two integrals can be computed separately in pretty much exactly the same manner as you compute the integral $\int_0^{\pi/2} \ln(2\cos(x))\,dx$ on pages 32 – 33. The results you get are (I won't bore you with the details of the actual calculations):

$$\int_0^{\frac{\pi}{2}} \ln\left(2\cos\left(\frac{x}{2}\right)\right)\cdot\cos(2x)\,dx = \frac{4-\pi}{8}$$

and

$$\int_0^{\frac{\pi}{2}} \ln\left(2\cos\left(\frac{x}{2}\right)\right)dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = K$$

The last constant K is called Catalan's constant and it is just like $\zeta(3)$ in the fact that no one knows whether there is a closed form expression that is equal to it. Plugging in the values of the two integrals gives us:

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)^2(2k+3)}=\frac{1}{2}\left(\frac{4-\pi}{8}\right)-\frac{1}{2}K+\frac{3\pi}{16}-\frac{1}{2}=\frac{\pi}{8}-\frac{1}{4}-\frac{1}{2}K$$

So we get the result:

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)^2(2k+3)}=\frac{\pi-2}{8}-\frac{1}{2}K$$

A pretty result that can also be derived by separation of fractions.

The jewel of the above calculations however that I was super excited about was the discovery of the fact that:

$$K=\int_0^{\frac{\pi}{2}}\ln\left(2\cos\left(\frac{x}{2}\right)\right)dx$$

I was excited about this because I have for a long time have been trying to find the closed form expression for Catalan's constant. This isn't progress, but it is still a pretty cool result. In fact if we do the substitution $u=\frac{x}{2}$ we get that:

$$K=\int_0^{\frac{\pi}{2}}\ln\left(2\cos\left(\frac{x}{2}\right)\right)dx=2\int_0^{\frac{\pi}{4}}\ln(2\cos(u))\,du=2\int_0^{\frac{\pi}{4}}\ln(2)\,du+2\int_0^{\frac{\pi}{4}}\ln(\cos(u))\,du=$$

$$K=\frac{\pi}{2}\ln(2)+2\int_0^{\frac{\pi}{4}}\ln(\cos(u))\,du$$

A little bit cleaner formula for K in my opinion. It's interesting to note that this integral has a striking resemblance to the last integral on page 33 in your book. Is there a deep connection somewhere?

It was actually at this point that I realized immediately how to prove the formula that you present on page 149 in "An Imaginary Tale: The Story of $\sqrt{-1}$:"

$$\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\cdots=\frac{\pi^2}{4}\ln(2)+2\int_0^{\frac{\pi}{2}}x\ln(\sin(x))\,dx$$

It is easy to show that $\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\cdots=\frac{7}{8}\zeta(3)$, so the above equation really is:

$$\zeta(3)=\frac{2\pi^2}{7}\ln(2)+\frac{16}{7}\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx$$

The proof of this result is really similar to the proof of the integral equation above that is equal to Catalan's constant. The result comes out pretty quickly and so let me show you how it's done in the next section.

Proof of $\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\cdots=\frac{\pi^2}{4}\ln(2)+2\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx$

To derive this equation we start off by evaluating the following integral:

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx$$

We apply the same trick that you did on page 32 of your book "Dr. Euler's Fabulous Formula" to this integral. Namely we write:

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx=\int_0^{\frac{\pi}{2}} x\ln\left(\frac{e^{ix}-e^{-ix}}{2i}\right)dx=\int_0^{\frac{\pi}{2}} x\ln\left(\frac{e^{ix}-e^{-ix}}{2i}\right)dx=$$

$$\int_0^{\frac{\pi}{2}} x\big(\ln\left(e^{ix}-e^{-ix}\right)-\ln(2i)\big)dx=\int_0^{\frac{\pi}{2}} x\big(\ln\left(e^{ix}-e^{-ix}\right)-\ln(2i)\big)dx=$$

$$\int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx-\ln(2i)\int_0^{\frac{\pi}{2}} x\,dx=\int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx-(\ln(2)+\ln(i))\frac{\pi^2}{8}$$

Using the fact that $\ln(i)=\frac{\pi}{2}i$, we get that:

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx=\int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx-\left(\ln(2)+\frac{\pi}{2}i\right)\frac{\pi^2}{8}$$

Let us compute the integral on the right hand side separately:

$$\int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx=\int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}\left(1-e^{-2ix}\right)\right)dx=\int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}\right)+\ln\left(1-e^{-2ix}\right)dx=$$

$$\int_0^{\frac{\pi}{2}} ix^2\,dx + \int_0^{\frac{\pi}{2}} x\ln\left(1-e^{-2ix}\right)dx = i\frac{\pi^3}{24} + \int_0^{\frac{\pi}{2}} x\ln\left(1-e^{-2ix}\right)dx$$

Let us now compute the very last integral separately (Here I will use the Taylor series expansion of $\ln(1-z)$ and set $z = e^{-2ix}$):

$$\int_0^{\frac{\pi}{2}} x\ln\left(1-e^{-2ix}\right)dx = -\int_0^{\frac{\pi}{2}} x\sum_{k=1}^{\infty}\frac{e^{-i2kx}}{k}\,dx = -\sum_{k=1}^{\infty}\left(\frac{1}{k}\int_0^{\frac{\pi}{2}} xe^{-i2kx}dx\right)$$

Integration by parts easily yields that:

$$\int_0^{\frac{\pi}{2}} xe^{-i2kx}dx = i\frac{\pi e^{-ik\pi}}{4k} + \frac{e^{-ik\pi}-1}{4k^2} = i\frac{(-1)^k\pi}{4k} + \frac{(-1)^k-1}{4k^2}$$

So we have that:

$$\int_0^{\frac{\pi}{2}} x\ln\left(1-e^{-2ix}\right)dx = -\sum_{k=1}^{\infty}\left(\frac{1}{k}\left(i\frac{(-1)^k\pi}{4k} + \frac{(-1)^k-1}{4k^2}\right)\right) =$$

$$-i\frac{\pi}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} - \frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^3}$$

Using the easily proven facts that $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3} = -\frac{3}{4}\zeta(3)$ and $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} = -\frac{1}{2}\zeta(2)$ (actually these are just special cases of the formula $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^n} = -\frac{2^n-2}{2^n}\zeta(n)$) we get that the above expression is equal to:

$$-i\frac{\pi}{4}\left(-\frac{1}{2}\zeta(2)\right) - \frac{1}{4}\left(-\frac{3}{4}\zeta(3)\right) + \frac{1}{4}\zeta(3) = i\frac{\pi}{8}\zeta(2) + \frac{7}{16}\zeta(3) = i\frac{\pi^3}{48} + \frac{7}{16}\zeta(3)$$

So we have:

$$\int_0^{\frac{\pi}{2}} x\ln\left(1-e^{-2ix}\right)dx = \frac{7}{16}\zeta(3) + i\frac{\pi^3}{48}$$

Plugging all of this in into an earlier result gives us:

$$\int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx = i\frac{\pi^3}{24}+\int_0^{\frac{\pi}{2}} x\ln\left(1-e^{-2ix}\right)dx = i\frac{\pi^3}{24}+\left(\frac{7}{16}\zeta(3)+i\frac{\pi^3}{48}\right)=$$

$$\int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx = \frac{7}{16}\zeta(3)+i\frac{\pi^3}{16}$$

Plugging this into a yet earlier result gives us that:

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx = \int_0^{\frac{\pi}{2}} x\ln\left(e^{ix}-e^{-ix}\right)dx-\left(\ln(2)+\frac{\pi}{2}i\right)\frac{\pi^2}{8}=$$

$$\left(\frac{7}{16}\zeta(3)+i\frac{\pi^3}{16}\right)-\left(\ln(2)+\frac{\pi}{2}i\right)\frac{\pi^2}{8}=\frac{7}{16}\zeta(3)-\frac{\pi^2}{8}\ln(2)+i\frac{\pi^3}{16}-i\frac{\pi^3}{16}=$$

$$\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx = \frac{7}{16}\zeta(3)-\frac{\pi^2}{8}\ln(2)$$

And so we get the result we wanted:

$$\zeta(3)=\frac{2\pi^2}{7}\ln(2)+\frac{16}{7}\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx$$

As mentioned before this is equivalent to the statement:

$$\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\cdots=\frac{\pi^2}{4}\ln(2)+2\int_0^{\frac{\pi}{2}} x\ln(\sin(x))\,dx$$

I find all of this really cool. My professor once mentioned that all of this can be rigorously justified using complex analysis (notice I did interchange a sum and an integral at one point), and I kind of do see how. In any case, we have just re-derived a result due to Euler. He would have been proud.

That Was a Bit Weird

I was confused why on page 148 you say that "we will derive it [you were referring to $\zeta(2)=\frac{\pi^2}{6}$], too, later in this section," when you had already derived the result:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

On page 33 of your book. And it is only a couple of a quick steps away (which you actually do on page 153) to show that:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{3}{4}\zeta(2)$$

Or more generally:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^n} = \frac{2^n-1}{2^n}\zeta(2)$$

So after page 33 you should have immediately have had the result $\zeta(2) = \frac{\pi^2}{6}$. In any case, both of the derivations on pages 32 – 33 and pages 150 – 153.

Fourier Series

We have come to the last section, and indeed it is a climax of all of the mathematics done in this letter. The foremost thing that I liked about Fourier series was that they allowed one to calculate the values of the Riemann-Zeta function for even values. All you had to do was integrate and plug in correct the correct constant of integration. But Fourier series can do so much more. Let me show you what else I discovered that they can do. We will start with the bravest and the boldest of calculations.

I absolutely loved the calculations that you presented on pages 134 – 135 on how Euler was able to discover his very own Fourier series through geometric sums. In fact I believe that it was really brilliant to use this calculation as an introduction to the topic of Fourier series. Let me show you how one can use this same exact calculation to derive a pretty cool integral.

I discovered the following calculation while eating lunch at UW. Let us start with a variant of the problem that Euler considered on page 134 of your book. Specifically let us consider and calculate the sum:

$$S(t) = e^{it} + e^{3it} + e^{5it} + \cdots = \sum_{k=0}^{\infty} e^{(2k+1)it}$$

Let us use the standard trick of geometric sums to sum of the above series. We have:

$$e^{2it}S(t) = e^{3it} + e^{5it} + e^{6it} + \cdots = \sum_{k=1}^{\infty} e^{(2k+1)it}$$

So:

$$S(t) - e^{2it}S(t) = \left(1 - e^{2it}\right)S(t) = e^{it}$$

And so we get:

$$S(t) = \frac{e^{it}}{(1 - e^{2it})}$$

Let us turn this into a more convenient from. Multiplying the top and bottom of the fraction on the right by $\left(1 - e^{-2it}\right)$ gives us:

$$S(t) = \frac{e^{it}\left(1 - e^{-2it}\right)}{(1 - e^{2it})(1 - e^{-2it})} = \frac{e^{it} - e^{-it}}{2 - e^{2it} - e^{-2it}} = \frac{2i\,\frac{e^{it} - e^{-it}}{2i}}{2 - 2\,\frac{e^{2it} - e^{-2it}}{2}} = i\,\frac{\sin(t)}{1 - \cos(2t)} =$$

$$i\,\frac{\sin(t)}{1 - \cos^2(t) + \sin^2(t)} = i\,\frac{\sin(t)}{1 - \cos^2(t) + (1 - \cos^2(t))} = \frac{1}{2}i\,\frac{\sin(t)}{\sin^2(t)} = \frac{1}{2}i\csc(t)$$

And so we have:

$$S(t) = \sum_{k=0}^{\infty} e^{(2k+1)it} = \frac{1}{2}i\csc(t)$$

Let us integrate both sides of the last equality and get that (here I use the fact from high school calculus that $\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + C$):

$$-\frac{1}{2}i\ln|\csc(t) + \cot(t)| + C = \frac{1}{i}\sum_{k=0}^{\infty} \frac{e^{(2k+1)it}}{2k+1}$$

In order to determine the constant of integration C, just plug in $t = \pi/2$ into both sides and get that:

$$0 + C = \frac{1}{i}\sum_{k=0}^{\infty} \frac{(-1)^k i}{2k+1} = \frac{\pi}{4}$$

(Here I used the fact that $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$). So we get that $C = \frac{\pi}{4}$. Plugging this into our previous result finally gives us the Fourier series:

$$-\frac{1}{2}i\ln|\csc(t) + \cot(t)| + \frac{\pi}{4} = \frac{1}{i}\sum_{k=0}^{\infty} \frac{e^{(2k+1)it}}{2k+1}$$

Let us multiply through by i in order to make the above result look just a bit nicer:

$$\frac{1}{2}\ln|\csc(t) + \cot(t)| + \frac{\pi}{4}i = \sum_{k=0}^{\infty} \frac{e^{(2k+1)it}}{2k+1}$$

Now that we have this result, let us do something with it. Take the conjugate of both sides to get that:

$$\frac{1}{2}\ln|\csc(t)+\cot(t)|-\frac{\pi}{4}i=\sum_{k=0}^{\infty}\frac{e^{-(2k+1)it}}{2k+1}$$

Now let us multiply the last two equations by each other and integrate from 0 to 2π. Multiplying the left sides of the last two equations and then integrating from 0 to 2π gives us:

$$\int_0^{2\pi}\left(\frac{1}{2}\ln|\csc(t)+\cot(t)|+\frac{\pi}{4}i\right)\left(\frac{1}{2}\ln|\csc(t)+\cot(t)|-\frac{\pi}{4}i\right)dx=$$

$$\int_0^{2\pi}\left(\frac{1}{4}[\ln|\csc(t)+\cot(t)|]^2+\frac{\pi^2}{16}\right)dx=\frac{1}{4}\int_0^{2\pi}[\ln|\csc(t)+\cot(t)|]^2dx+\frac{\pi^3}{8}$$

On the other hand multiplying the right hand sides of the two equations and then integrating from 0 to 2π gives us:

$$\int_0^{2\pi}\left(\sum_{k=0}^{\infty}\frac{e^{(2k+1)it}}{2k+1}\right)\left(\sum_{k=0}^{\infty}\frac{e^{-(2k+1)it}}{2k+1}\right)dx=\sum_{k=0}^{\infty}\frac{2\pi}{(2k+1)^2}=2\pi\frac{3}{4}\zeta(2)=2\pi\frac{3}{4}\cdot\frac{\pi^2}{6}=\frac{\pi^3}{4}$$

So equating the two sides gives us that:

$$\frac{1}{4}\int_0^{2\pi}[\ln|\csc(t)+\cot(t)|]^2dx+\frac{\pi^3}{8}=\frac{\pi^3}{4}$$

Rearranging finally gives us the beautiful result:

$$\int_0^{2\pi}[\ln|\csc(t)+\cot(t)|]^2dx=\frac{\pi^3}{2}$$

We can simplify this result even further. Noticing that by the periodicity of the trigonometric functions $\csc(t)$ and $\cot(t)$, the area under the curve of the function $[\ln|\csc(t)+\cot(t)|]^2$ is the same on the intervals $[0,\pi/2]$, $[\pi/2,\pi]$, $[\pi,3\pi/2]$, $[3\pi/2,2\pi]$. Since there are four intervals of these kinds we get that our beautiful result can be rewritten as:

$$\int_0^{\frac{\pi}{2}}[\ln|\csc(t)+\cot(t)|]^2dx=\frac{\pi^3}{8}$$

I think that this is an absolutely fascinating and non-trivial integral equation. I remember I showed this equation to a group of graduate math students during lunch. The next day I asked whether anyone was able to prove the above statement and one student told me that he was able to prove it “by integration by parts, and then lots and lots of u-substitutions, lots and lots of u-substitution.” I didn’t believe him. To prove the above result without having known that it came from Fourier series would require some complex analysis at least.

Anyways, by similar methods one can also derive the pretty fact that:

$$K = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln|\csc(t) + \cot(t)|\, dt$$

Where K again is Catalan’s constant.

The Connection between Complex Analysis and Fourier Series

I have done a lot of complex analysis and Fourier series computations before and whenever I would do them I would always feel like there was an uncanny connection between the two. It took me yet another lunch break to help me find out why I kept on getting these uncanny chills. As it turns out, Dr. Euler magical formula turned out to be the cure.

One day eating my lunch (I took some delicious food that mom cooked) I wrote out the Taylor series of a function $f(z)$:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

This is the Taylor series of $f(z)$ (where z of course can be complex. Yes, I know more than you might think – and there are unsaid stuff here, but let’s not get hung up on boring details). Anyways let us shift the function z_0 in the direction of $-z_0$. In explanation, make the change of variables $(z - z_0) \to z$ and set $g(z) = f(z + z_0)$ to get:

$$g(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} z^k$$

Essential we shifted the Taylor series to the origin. Now, let us plug in $e^{i\theta}$ into z in the above equation. We will then get that:

$$g\left(e^{i\theta}\right) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} e^{ik\theta}$$

This is a Fourier series! Now let us multiply both sides by $e^{-in\theta}$ and integrate from 0 to 2π. We will get:

$$\int_0^{2\pi} g\left(e^{i\theta}\right)e^{-in\theta}\,d\theta = \int_0^{2\pi}\left(\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}e^{ik\theta}\right)e^{-in\theta}\,d\theta = \int_0^{2\pi}\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}e^{i(k-n)\theta}\,d\theta$$
$$= \sum_{k=0}^{\infty}\left(\frac{f^{(k)}(z_0)}{k!}\int_0^{2\pi} e^{i(k-n)\theta}\,d\theta\right) = 2\pi\frac{f^{(n)}(z_0)}{n!}$$

And so we have:

$$\int_0^{2\pi} g\left(e^{i\theta}\right)e^{-in\theta}\,d\theta = 2\pi\frac{f^{(n)}(z_0)}{n!}$$

Now we do the really important step. If we set $z = e^{i\theta}$, we can then rewrite the left integral as a contour integral over the unit circle in the complex plane. With this substitution we get that $dz = ie^{-i\theta}d\theta$ and so the left integral becomes (∂u means the unit circle):

$$\int_0^{2\pi}\frac{g\left(e^{i\theta}\right)}{e^{in\theta}}\,d\theta = \frac{1}{i}\int_0^{2\pi}\frac{g\left(e^{i\theta}\right)}{e^{i(n+1)\theta}}ie^{i\theta}\,d\theta = \frac{1}{i}\oint_{\partial u}\frac{g(z)}{z^{n+1}}\,dz = 2\pi\frac{f^{(n)}(z_0)}{n!}$$

Now shifting everything back z_0 (meaning making the change of variables $z \to (z - z_0)$) and using the fact that $g(z - z_0) = f\left((z - z_0) + z_0\right) = f(z)$ we get that (here $\partial u(z_0)$ denotes the unit circle centered at z_0):

$$i\oint_{\partial u(z_0)}\frac{f(z)}{(z - z_0)^{n+1}}\,dz = 2\pi\frac{f^{(n)}(z_0)}{n!}$$

And rearranging finally gives us the amazing result:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{\partial u(z_0)}\frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

Called Cauchy's Formula. Of course we don't need to only stick with integrating over $\partial u(z_0)$. We can integrate over any contour γ that includes z_0 and such that $f(z)$ is analytic and does not blow up in the region enclosed by γ in order for the above formula to hold true. So we can rewrite Cauchy's Formula in the more general form:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{\gamma}\frac{f(z)}{(z - z_0)^{n+1}}\,dz$$

How exciting! All of this shows how integration in the complex plane is closely related to Taylor series expansions and works on concepts that are very much related to Fourier series. For example, one of the biggest analogies between complex analysis and Fourier series that make all of the above possible are the facts that:

For integers m and n, $\int_0^{2\pi} e^{i(m-n)x}dx = \begin{cases} 0 & if \quad m \neq n \\ 2\pi & if \quad m = n \end{cases}$

and

For integers m and n, $\oint_{\partial u} \frac{1}{z^{m-n}}dz = \begin{cases} 0 & if \quad m \neq n-1 \\ 2\pi & if \quad m = n-1 \end{cases}$

One Last Integral

The Fourier transform has a beautiful application to the evaluation of certain integrals through Rayleigh's energy formula. One of the amazing examples of this that you present in your book is the formula:

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx = \frac{\pi}{2}$$

The other integrals that you derive using the energy formula can also be trivially computed using complex analysis. The above integral equation however doesn't seem to lend itself to complex integration as easily. I however did finally figure out how to use complex analysis to compute the above integral, and so let me show how I did it.

Again, I believe I discovered the following calculation while eating lunch.

I started off by pulling off a variant of a famous trick: let us construct the function $I(\alpha)$:

$$I(\alpha) = \int_0^\infty e^{-\alpha x^2}\frac{\sin^2(x)}{x^2}dx$$

To calculate the integral $\int_0^\infty \frac{\sin^2(x)}{x^2}dx$, all we will have to do is simply evaluate $I(\alpha)$ at $\alpha = 0$. Simple, right? But how do we do that? We first differentiate our function $I(\alpha)$ (In the second step here I pull the derivative in under the integral sign):

$$\frac{dI(\alpha)}{d\alpha} = \frac{d}{d\alpha}\left(\int_0^\infty e^{-\alpha x^2}\frac{\sin^2(x)}{x^2}dx\right) = \int_0^\infty \frac{\partial}{\partial\alpha}\left(e^{-\alpha x^2}\frac{\sin^2(x)}{x^2}\right)dx =$$

$$-\int_0^\infty e^{-\alpha x^2}(\sin(x))^2dx = -\int_0^\infty e^{-\alpha x^2}\left(\frac{e^{ix}-e^{-ix}}{2}\right)^2 dx = \frac{1}{4}\int_0^\infty e^{-\alpha x^2}\left(e^{2ix}+e^{-2ix}-2\right)dx =$$

Equation 1: $$\frac{dI(\alpha)}{d\alpha} = \frac{1}{4}\int_0^\infty e^{-\alpha x^2+2ix}dx + \frac{1}{4}\int_0^\infty e^{-\alpha x^2-2ix}dx - \frac{1}{2}\int_0^\infty e^{-\alpha x^2}dx$$

Now, let us analyze the sum of the first two integrals in the last equation. Let us complete the square in the exponents in each of these integrands:

$$\frac{1}{4}\int_0^\infty e^{-\alpha x^2+2ix}dx+\frac{1}{4}\int_0^\infty e^{-\alpha x^2-2ix}dx=$$

$$\frac{1}{4}e^{-\frac{1}{\alpha}}\int_0^\infty e^{-\alpha x^2+2ix+\frac{1}{\alpha}}dx+\frac{1}{4}e^{-\frac{1}{\alpha}}\int_0^\infty e^{-\alpha x^2-2ix+\frac{1}{\alpha}}dx=$$

$$\frac{1}{4e^{\frac{1}{\alpha}}}\left[\int_0^\infty e^{-\left(\sqrt{\alpha}x-\frac{i}{\sqrt{\alpha}}\right)^2}dx+\int_0^\infty e^{-\left(\sqrt{\alpha}x+\frac{i}{\sqrt{\alpha}}\right)^2}dx\right]$$

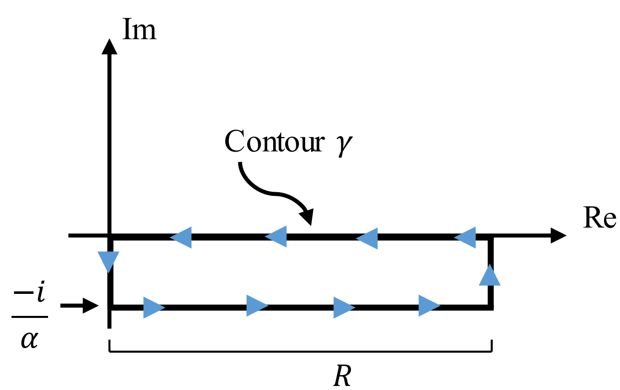
And so we get that:

Equation 2: $$\frac{1}{4}\int_0^\infty e^{-\alpha x^2+2ix}dx+\frac{1}{4}\int_0^\infty e^{-\alpha x^2-2ix}dx=\frac{1}{4e^{\frac{1}{\alpha}}}\left[\int_0^\infty e^{-\left(\sqrt{\alpha}x-\frac{i}{\sqrt{\alpha}}\right)^2}dx+\int_0^\infty e^{-\left(\sqrt{\alpha}x+\frac{i}{\sqrt{\alpha}}\right)^2}dx\right]$$

Let us first calculate the first integral in the above equation:

$$\int_0^\infty e^{-\left(\sqrt{\alpha}x-\frac{i}{\sqrt{\alpha}}\right)^2}dx$$

How should we approach calculating the above integral? We look to the complex plane for help. Let us take the contour integral of the function $e^{-(\sqrt{\alpha}z)^2}=e^{-\alpha z^2}$ on the following contour in the complex plane:



Let us call the width of the above contour R and say that we integrate the function $e^{-(\sqrt{\alpha}z)^2}$ counterclockwise on the above contour γ. By breaking up the contour integral into the four integrals on the sides of the rectangle above we get that

$$\oint_\gamma e^{-\alpha z^2}dz=i\int_0^{-\frac{1}{\alpha}} e^{(\sqrt{\alpha}y)^2}dy+\int_0^R e^{-\left(\sqrt{\alpha}\left(x-\frac{i}{\alpha}\right)\right)^2}dx+i\int_{-\frac{1}{\alpha}}^0 e^{-(\sqrt{\alpha}(R+iy))^2}dy+\int_R^0 e^{-(\sqrt{\alpha}x)^2}dx$$

Now, since e^{-z^2} is an analytic function and γ is a closed loop, we get that the contour integral $\oint_\gamma e^{-z^2}\,dz$ is equal to zero. So the left hand side of the last equation is zero and so we have:

$$i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2}\,dy + \int_0^R e^{-\left(\sqrt{\alpha}x-\frac{i}{\sqrt{\alpha}}\right)^2}\,dx + i\int_{-\frac{1}{\alpha}}^0 e^{-\alpha(R+iy)^2}\,dy - \int_0^R e^{-\alpha x^2}\,dx = 0$$

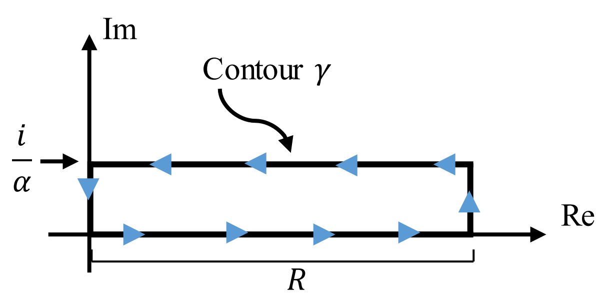
(I flipped the limits on the last integral) Now let us stretch the rectangular contour in the right direction to infinity. In other words, let $R\to\infty$. The third integral in the last expression goes to zero and so as we take the limit of both sides as $R\to\infty$ we get that:

$$i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2}\,dy + \int_0^\infty e^{-\left(\sqrt{\alpha}x-\frac{i}{\sqrt{\alpha}}\right)^2}\,dx - \int_0^\infty e^{-\alpha x^2}\,dx = 0$$

And so we get:

Equation 3: $$\int_0^\infty e^{-\left(\sqrt{\alpha}x-\frac{i}{\sqrt{\alpha}}\right)^2}\,dx = -i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2}\,dy + \int_0^\infty e^{-\alpha x^2}\,dx$$

If you do the same type of calculation of integrating $e^{-\alpha z^2}$ but on the contour:



You will get the similar result involving the other integral:

$$\int_0^\infty e^{-\left(\sqrt{\alpha}x+\frac{i}{\sqrt{\alpha}}\right)^2}\,dx = i\int_{\frac{1}{\alpha}}^0 e^{\alpha y^2}\,dy + \int_0^\infty e^{-\alpha x^2}\,dx = -i\int_0^{\frac{1}{\alpha}} e^{\alpha y^2}\,dy + \int_0^\infty e^{-\alpha x^2}\,dx$$

Since $\int_0^{\frac{1}{\alpha}} e^{\alpha y^2}\,dy = -\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2}\,dy$ by symmetry of $e^{\alpha z^2}$, we get that the above expression is equivalent to:

Equation 4: $$\int_0^\infty e^{-\left(\sqrt{\alpha}x+\frac{i}{\sqrt{\alpha}}\right)^2} dx = i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2} dy + \int_0^\infty e^{-\alpha x^2} dx$$

Now let us plug in the results Equation 3 and Equation 4 into Equation 2. We will then get that:

$$\frac{1}{4}\int_0^\infty e^{-\alpha x^2+2ix} dx + \frac{1}{4}\int_0^\infty e^{-\alpha x^2-2ix} dx =$$

$$\frac{1}{4e^{\frac{1}{\alpha}}}\left[-i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2} dy + \int_0^\infty e^{-\alpha x^2} dx + i\int_0^{-\frac{1}{\alpha}} e^{\alpha y^2} dy + \int_0^\infty e^{-\alpha x^2} dx\right] = \frac{1}{4e^{\frac{1}{\alpha}}}\left[2\int_0^\infty e^{-\alpha x^2} dx\right] =$$

$$\frac{1}{2e^{1/\alpha}}\int_0^\infty e^{-\alpha x^2} dx$$

And so we get the equation:

$$\frac{1}{4}\int_0^\infty e^{-\alpha x^2+2ix} dx + \frac{1}{4}\int_0^\infty e^{-\alpha x^2-2ix} dx = \frac{1}{2e^{1/\alpha}}\int_0^\infty e^{-\alpha x^2} dx$$

Plugging in this result into Equation 1 gives us:

$$\frac{dI(\alpha)}{d\alpha} = \frac{1}{2e^{\frac{1}{\alpha}}}\int_0^\infty e^{-\alpha x^2} dx - \frac{1}{2}\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\left(\frac{1}{e^{\frac{1}{\alpha}}} - 1\right)\int_0^\infty e^{-\alpha x^2} dx$$

Now we know that the integral $\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$. This then give us that:

$$\frac{dI(\alpha)}{d\alpha} = \frac{1}{2}\left(\frac{1}{e^{\frac{1}{\alpha}}} - 1\right)\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

And so:

$$\frac{dI(\alpha)}{d\alpha} = \frac{\sqrt{\pi}}{4\sqrt{\alpha}}\left(e^{-\frac{1}{\alpha}} - 1\right)$$

Now, what do we do from here? How do we use this expression to get the value of $\int_0^\infty \frac{\sin^2(x)}{x^2} dx$. Simple! We integrate both sides of the last equation from 0 to ∞:

$$\int_0^\infty \frac{dI(\alpha)}{d\alpha} d\alpha = I(\infty) - I(0) = \frac{\sqrt{\pi}}{4}\int_0^\infty \left(e^{-\frac{1}{\alpha}} - 1\right)\frac{1}{\sqrt{\alpha}} d\alpha$$

(The $d\alpha$'s cancel out in the leftmost integral ☺) One can easily notice (and prove) that:

$$I(\infty) = \lim_{\alpha \to \infty} \left(\int_0^\infty e^{-\alpha x^2} \frac{\sin^2(x)}{x^2} dx \right) = 0$$

And so we have that:

$$-I(0) = -\int_0^\infty \frac{\sin^2(x)}{x^2} dx = \frac{\sqrt{\pi}}{4} \int_0^\infty \left(e^{-\frac{1}{\alpha}} - 1 \right) \frac{1}{\sqrt{\alpha}} d\alpha$$

Or more elegantly written:

$$\int_0^\infty \frac{\sin^2(x)}{x^2} dx = \frac{\sqrt{\pi}}{4} \int_0^\infty \left(1 - e^{-\frac{1}{\alpha}} \right) \frac{1}{\sqrt{\alpha}} d\alpha$$

But this is just one integral written as another integral, how does this help us (what did you expect, that's what complex integration does essentially: write one integral as another)? But notice that $\left(1 - e^{-\frac{1}{\alpha}}\right)\frac{1}{\sqrt{\alpha}}$ is a pretty interesting looking integrand. Let us do some change of variables to turn this into something more familiar. Fist set $u = \sqrt{\alpha}$ and so the integral on the right will become:

$$\frac{\sqrt{\pi}}{2} \int_0^\infty \left(1 - e^{-\frac{1}{u^2}} \right) du$$

This is an even more interesting looking integrand! Remind you of anything? It should! Let us now do the substitution $y = \frac{1}{u}$ so that the above integral becomes:

$$\frac{\sqrt{\pi}}{2} \int_0^\infty \frac{\left(1 - e^{-y^2}\right)}{y^2} dy$$

Do integration by parts gives us:

$$\frac{\sqrt{\pi}}{2} \int_0^\infty \frac{\left(1 - e^{-y^2}\right)}{y^2} dy = \frac{\sqrt{\pi}}{2} \left[\left(-\frac{1 - e^{-x^2}}{x} \right) \Bigg|_0^\infty + 2 \int_0^\infty e^{-x^2} dx \right] =$$

$$\sqrt{\pi} \int_0^\infty e^{-x^2} dx = \sqrt{\pi} \left(\frac{1}{2} \sqrt{\pi} \right) = \frac{\pi}{2}$$

So we finally get that (drum rolls please):

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx = \frac{\pi}{2}$$

Amazing! Right?

Although I have shown you a derivation of the above integral using complex analysis, there is a much, much more simple derivation of the above integral equation. Simply integrate by parts to get that (in the last step I do the substitution $u = 2x$):

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx = \left(-\frac{\sin^2(x)}{x}\right)\Bigg|_0^\infty + \int_0^\infty \frac{2\sin(x)\cos(x)}{x}dx = \int_0^\infty \frac{\sin(2x)}{x}dx = \int_0^\infty \frac{\sin(u)}{u}du$$

And in pages 64 – 65 you give a very simple proof of the fact that: $\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$. Thus we do have a simple proof of the equation:

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx = \frac{\pi}{2}$$

I point this out because it proves that you were wrong on page 209 of your book when you say that this equation "is *not* easily derived by other means."

I showed you the complex analysis derivation before the integration by parts proof because had I showed the second much simpler proof, you probably would not have paid attention to the derivation involving complex analysis. ☺

Concluding Words

I really liked your two books "An Imaginary Tale: The Story of $\sqrt{-1}$" and "Dr. Euler's Fabulous Formula." I have learned so much from these books and they have given me starting points from which I can go and invent new mathematics. From these two books I have learned about topics that my math courses at the University of Washington wouldn't ever dream of covering. Part of the reason for this is that they have to prove everything rigorously (which is right because math is rigor) and are never brave enough to cover material that require Eulerian boldness in their derivations. Some of the topics are inaccessible because they require much more advanced mathematics in order to prove 100% rigorously. But with these books and a little bit of reliance on intuition, I have been able to learn how to arrive at so many surprising, and beautiful results.

References

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