BLACK SCHOLES OPTION PRICING MODEL

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Primer

Introduction. A European call option is a financial security that gives its owner the right, but not the obligation, to purchase a pre-determined asset at a pre-determined price and future time.

The owner of a European call option on stock A has the right, but not the obligation, to purchase A at a fixed exercise price K on the expiration date T. Let S_T be the value of the stock on the expiration date, V_T be the value of the option on expiration. Then $V_T = \max\{0, S_T - K\}$. Observe that the value of an option is always non-negative, since the owner of the option will not exercise the option if it will not be profitable to do so, but that since S_T is theoretically unbounded, so V_T is too.

Call options are a particularly interesting financial instrument because they offer unlimited potential for gain and zero potential for loss. Options generally are useful because they enable parties to transfer personally unacceptable risk to other parties who are willing to take on the exposure for a premium. As both parties want a fair price, a consistent way to price options would facilitate such transactions.

In fact, we will see that a European call option has a unique no-arbitrage price, given a number of simplifying assumptions about investors' preferences and market factors.

Definitions

Risk-neutral investors. Risk-neutral investors choose investments solely on the basis of net present value of expected value of cash flows. Thus, if investments A and B are mutually exclusive and satisfy $PV(E[A]) < PV(E[B])$, a risk-neutral investor would prefer investment B. (Note: $E[A]$ is a stream of cash flows. Each period lump sum represents expected value of CFs in that period. See Present Value.)

No Arbitrage. The net present value of all investments is 0.

It follows that if an asset's cash flows can be replicated exactly, the asset's price is equal to the present value of the replicating cash flows.

Present Value. A cash flow C today is worth more than the same cash flow in the future. Intuitively, this is because I can invest C today and receive C plus interest in the future (since interest rates are, with rare exceptions, positive in the US). For example, if the prevailing yearly interest rate were 5%, \$100 today would be equivalent to \$105 in a year: I could take the \$100 and invest it to receive \$105 in a year, or I could take the \$105 in a year and borrow \$100 now, paying 5% on the principal. A present value function maps a stream of cash flows at various times into a single dollar amount. Let CF be a stream of cash flows $\{C_t\}_{t=0}^{\infty}$, $\{r_t\}_{t=0}^{\infty}$. Then

$$PV(CF) = \sum_{t=0}^{\infty} \frac{C_t}{(1+r_t)^t}.$$

Net present value, NPV, is the present value of all positive and negative cash flows involved in an investment (i.e. includes negative cash flow to purchase the cash flow stream).

When computing continuous time, a cash flow C_t to be received t units of time later has a present value of

$$PV(C_t) = C_t e^{-rt}.$$

Note that the no arbitrage principle can be stated simply in terms of PV: For any cash flow stream resulting from investment CF_A,

$$NPV(CF_A) = 0.$$

Arbitrage Theorem. (Ross p.72). Given an experiment with outcomes {1,2,...,m} and n possible wagers, let x_i be the amount bet on wager i, $r_i(j)$ be the percentage return if the experiment's outcome is j, where $j \in \{1, 2, ..., m\}$. Then $x = (x_1, x_2, ..., x_n)$ is a betting strategy with return $\sum_{i=1}^{n} x_i r_i(j)$.

The arbitrage theorem states that either

a) there is a probability vector $p = (p_1, p_2, ..., p_m)$ so that $\sum_{j=1}^{m} p_j r_i(j) = 0$ for all $i \in \{1, 2, ..., n\}$, or

b) there is a betting strategy $x = (x_1, x_2, ..., x_n)$ so that $\sum_{i=1}^{n} x_i r_i(j) > 0$ for all $j \in \{1, ..., m\}$. (Ross proves this on p.78 via the Duality Theorem of Linear Programming.)

Central Limit Theorem (Ross p.27-28). Let $\{X_i\}_{i=1}^{n}$ be independent, identically distributed random variables with expected value μ, variance σ^2, $S_n = \sum_{i=1}^{n} X_i$. Then for large n,S_n is approximately normal with expected value $n\mu$, variance $n\sigma^2$.

Example. Suppose a stock is currently priced at \$100, and will be worth either \$50 or \$200 next period. Let r be the one period interest rate. By the arbitrage theorem, no sure win exists if there exists a probability vector $(p, 1-p)$ on the future stock prices so that the expected present value is zero, whether an investor purchases the stock or the option.

The expected return on the stock is given by $E[return] = p\frac{150}{1+r} + \frac{50}{1+r} - 100$. Set this equal to zero and solve for p to get $p = \frac{1+2r}{3}$. Thus, there is a unique probability vector so that no arbitrage exists for the stock.

Now consider the option. Let C be the fair price of a call option on this stock. Since the probability vector $(p, 1-p)$ with $p = \frac{1+2r}{3}$ is unique for the stock, the present value of the payoff to the holder of a call option is $E[return] = \frac{1+2r}{3}\frac{50}{1+r} - C$, so $C = \frac{50+100r}{3(1+r)}$.

Deriving the Black Scholes Formula for a Call Option

Each derivation uses the following assumptions, stated in Black and Scholes (1973, p. 640)):

- The short-term interest rate is known and constant through time.
- The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price.
- The stock pays no dividends or other distributions.
- The option is European, i.e. can only be exercised at maturity.
- There are no transaction costs in buying or selling the stock or option.
- It is possible to borrow any fraction of the price of a security to buy or hold it, at the short-term interest rate.
- There are no penalties to short selling.

Black and Scholes' Derivation

Let $w(x,t)$ be the value of the option as a function of x, the price of the underlying stock, and t, the time. Then for each share of underlying stock an investor owns, he must sell short $1/w_1(x,t)$ options to hedge his position, where $w_1(x,t) = \frac{\partial w}{\partial x}(x,t)$. The resulting portfolio value E is given by $E = x - \frac{w}{w_1}$.

(Note that for such a hedged position, for small Δx and a small time interval Δt, $\Delta E = \Delta x - \frac{\Delta w}{w_1} \approx 0$.)

Independent of whether the position is hedged continuously or not, the return on the above hedged position is certain, and by the arbitrage principle, the risk-free rate. If the position is hedged continuously, the return on the hedged position becomes certain. If the position is not hedged continuously, the portfolio's risk is small even for large changes in the underlying stock's price, and consists entirely of unsystematic risk. Hence, again by the arbitrage principle, the return on the hedged position is certain. Thus, the rate of return is given by the short term risk-free rate $r\Delta t$, and the total return is given by the equity position multiplied by this rate: $\Delta E = (x - w/w_1)r\Delta t$, where r is the risk-free rate of return.

By stochastic calculus,

$$\Delta w = w_1 \Delta x + \frac{1}{2} w_{11} v^2 x^2 \Delta t + w_2 \Delta t,$$

where v^2 is the variance of the stock return. Then

$$\Delta E = -\left(\frac{1}{2}w_{11}v^2x^2 + w_2\right)\frac{\Delta t}{w_1} = (x - w/w_1)r\Delta t.$$

The equivalent partial differential equation is

$$w_2 = rw - rxw_1 - \frac{1}{2}v^2x^2w_{11}.$$

Now consider the boundary conditions imposed by the structure of the call option. Let t^* be the maturity date of the option. Then

$$w(x, t^*) = \max\{x - c, 0\}.$$

By considering the substitution

$$w(x,t) = e^{r(t-t^*)}y[(2/v^2)\left(r - \frac{v^2}{2}\right)[\ln x/c - \left(r - \frac{v^2}{2}\right)(t - t^*)]$$
$$-(2/v^2)\left(r - \frac{v^2}{2}\right)^2(t - t^*)],$$

the differential equation becomes

$$y_2 = y_{11}$$

with boundary conditions

$$y(u,0) = \begin{cases} 0, & u < 0 \\ c\left[e^{u(v^2/2)/(r-v^2/2)} - 1\right], & u \geq 0. \end{cases}$$

Observe that the differential equation has reduced to the heat equation! Its solution is given by Churchill (1963, p.155). We obtain

$$w(x,t) = xN(d_1) - ce^{r(t-t^*)}N(d_2),$$
$$d_1 = \frac{\ln(x/c) + (r - v^2/2)(t^* - t)}{v\sqrt{t^* - t}},$$
$$d_2 = \frac{\ln(x/c) + (r - v^2/2)(t^* - t)}{v\sqrt{t^* - t}},$$

where $N(d)$ is the cumulative normal density function.

Binomial approximation to geometric Brownian motion

Sheldon Ross derives the Black Scholes formula for an option's price by a probabilistic argument. He first notes that a multiperiod binomial up/down model for stock prices yields a unique no-arbitrage price on a call option. Then he demonstrates that every geometric Brownian motion can be approximated by such a multiperiod binomial model, where the approximation becomes exact as the time increments shrink to zero. Thus, when stock prices are modeled by a geometric Brownian motion, a call option has a unique no-arbitrage price, given by a multiperiod binomial model's unique no-arbitrage price with infinintely small time increments.

Multiperiod binomial model has geometric Brownian motion as its limiting process (Ross p.32-35). Stock prices $S(y)$ follow a geometric Brownian motion with drift parameter μ and volatility parameter σ if, for all nonnegative values of y and t, the random variable

$$\frac{S(t+y)}{S(y)}$$

is independent of all prices up to time y, and

$$\log\left(\frac{S(t+y)}{S(y)}\right)$$

is a normal random variable with mean μt and variance $t\sigma^2$. Then the expected stock price grows at rate $\mu + \sigma^2/2$.

Now consider a multiperiod binomial model with time increments Δ, in which the price of a stock increases by a factor of u with probability p or decreases by a factor of d with probability 1-p, with

$$u = e^{\sigma\sqrt{\Delta}},\ d = e^{-\sigma\sqrt{\Delta}},\ p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right).$$

Let Y_i be 1 if the price increases at time $i\Delta$, 0 if it decreases. Then in the first n time increments, the stock has increased $\sum_{i=1}^{n} Y_i$ times, and decreased $n - \sum_{i=1}^{n} Y_i$ times. So $S(n\Delta)$, the time at the end of the period, is

$$S(n\Delta) = S(0)u^{\sum_{i=1}^{n} Y_i} d^{n-\sum_{i=1}^{n} Y_i}$$

$$\Leftrightarrow S(n\Delta) = d^n S(0)\left(\frac{u}{d}\right)^{\sum_{i=1}^{n} Y_i}.$$

Let $t = n\Delta$ to obtain

$$\frac{S(t)}{S(0)} = d^{t/\Delta}\left(\frac{u}{d}\right)^{\sum_{i=1}^{t/\Delta} Y_i}.$$

Then

$$\log\left(\frac{S(t)}{S(0)}\right) = \frac{-t\sigma}{\sqrt{\Delta}} + 2\sigma\sqrt{\Delta}\sum_{i=1}^{t/\Delta} Y_i.$$

Take $\Delta \to 0$, and by the central limit theorem, $\sum_{i=1}^{t/\Delta} Y_i$ becomes normal, so $\log(S(t)/S(0))$ is a normal random variable. It follows from the above formula that the variable also has mean μt, variance $\sigma^2 t$. Hence, the binomial model exactly approximates a geometric Brownian motion as Δ grows arbitrarily small.

No-arbitrage option price under multiperiod binomial model (Ross p.76). Suppose that stock prices at time period t, $\{S_t\}_{t=0}^{\infty}$, evolve as follows: The initial stock price is $S(0)$, the interest rate is r, and $S(t) = uS(t-1)$ or $dS(t-1)$, with $d < 1 + r < u$. Let

$$X_i = \begin{cases} 1 \text{ if S(t)=uS(t-1)}, \\ 0 \text{ if S(t)=dS(t-1)}. \end{cases}$$

We want to find the probability vector $P(X_1 = x_1, ..., X_n = x_n),\ i = 1, ..., n$ so that all bets are fair. Let $\alpha = P(X_1 = x_1, ..., X_{i-1} = x_{i-1}),\ p = P(X_i = 1|\alpha)$. Then the expected gain on this bet in i-1 time units is $\alpha[p(1+r)^{-1}uS(i-1) + (1-p)(1+r)^{-1}dS(i-1) - S(i-1)]$, which is 0 only when $p = \frac{1+r-d}{u-d}$.

Suppose X_i are independent Bernoulli (0 or 1) random variables with success probability p, where success is defined to be an increase in stock price and indicated by $X_i = 1$. Then $Y = \sum_{i=1}^{n} X_i$ is a binomial random variable with parameters n, p. Then $S(n) = u^Y d^{n-Y} S(0)$, so the present value of owning a call option on the stock is

$$(1+r)^{-n} \max\{S(n) - K, 0\},$$

and the expected present value of owning it is

$$(1+r)^{-n} E[\max\{S(n) - K\}] = (1+r)^{-n} E[\max\{S(0)u^Y d^{n-Y} - K\}].$$

Then the unique no-arbitrage option cost C is

$$C = (1+r)^{-n} E[\max\{S(0)u^Y d^{n-Y} - K\}]$$

No-arbitrage option price under geometric Brownian motion (Ross p.85-88). Consider the multiperiod binomial model in which a stock price can increase by a factor of u with probability p, decrease by a factor of d with probability 1-p, where

$$u = e^{\sigma\sqrt{t/n}} \approx 1 + \sigma\sqrt{t/n} + \frac{\sigma^2 t}{2n},$$
$$d = e^{-\sigma\sqrt{t/n}} \approx 1 - \sigma\sqrt{t/n} + \frac{\sigma^2 t}{2n},$$

where n is large, and the approximations are given by the first three terms of the Taylor expansion of e^x about $x = 0$. Let K be the exercise price of a call option, and t be the exercise time. Then the payoff to a call option owner is $\max\{S(t) - K, 0\}$.

Recall that the no-arbitrage probability vector is given by $p = \frac{1+rt/n-d}{u-d} \approx \frac{1}{2} + \frac{r\sqrt{t/n}}{2\sigma} - \frac{\sigma\sqrt{t/n}}{4}$. Let Y be a binomial variable with parameters n, p as before. Again, as n grows large, by the central limit theorem, Y becomes normal. Then

$$C = (1+rt/n)^{-n} E[\max\{0, S(0)u^Y d^{n-Y} - K\}]$$
$$= (1+rt/n)^{-n} E[\max\{0, S(0)e^W - K\}],$$

where $W = 2\sigma\sqrt{t/n}Y - \sigma\sqrt{nt}$. It follows that $EW \approx (r - \sigma^2/2)t$, $Var(W) \approx \sigma^2 t$, where the approximates become exact as n grows large. Then the unique no-arbitrage price is given by

$$C = e^{-rT} E \max\{S(0)e^W - K, 0\},$$

where W is normal with mean $(r - \sigma^2/2)t$, variance $\sigma^2 t$. Then

$$C = S(0)\psi(w) - Ke^{-rt}\psi(w - \sigma\sqrt{t},$$

where

$$w = \frac{rt + \sigma^2 t/2 - \log(K/S(0))}{\sigma\sqrt{t}},$$

and $\psi(x)$ is the standard normal distribution function. So we have obtained the Black Scholes option pricing formula.

References

[Ross 1999] Ross, Sheldon M. An Introduction to Mathematical Finance: Options and Other Topics. Cambridge, U.K.: Cambridge UP, 1999. Print.

[Black and Scholes 1973] Black, Fischer, and Myron Scholes. "The Pricing of Options and Corporate Liabilities." Journal of Political Economy J POLIT ECON 81.3 (1973): 637-47. Web.