Electrical Resistance Networks and Random Walks

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1 Introduction

Any network-like structure can be thought of as a graph. Thus, graphs have a wide variety of applications.A random walk on a graph can be thought of as starting at a vertex and stepping across a randomly chosen edge. Applications include modeling forest fires and there are a lot of algorithmic applications.The physical concept of a resistance network has deep connection with random walks. This paper explores some of these connections. Specifically, the concepts of resistance is related to the access times. The central results of this paper came from [1].

2 Graphs [3]

A *graph* G is comprised of two sets: a vertex set $V(G) = \{g_1, \ldots, g_n\}$ and an edge set E(G). The edge set consists of two-element subsets (unordered pairs) of the vertex set, and is usually used to represent connections between these pairs of vertices. (In a *simple graph*, the only type considered in this paper, these pairs of elements must consist of distinct elements, so no edge can connect a vertex to itself.) This structure can be drawn in a pretty way on a piece of paper that looks like a network. In such a visual representation, the specific positioning of the vertices and edges does not matter; only the connections between vertices do.

Example: The graph consisting of the vertex set {1, 2, 3, 4} and the edge set {{1, 2}, {1, 4}, {2, 3}, {3, 4}} can be represented as a square:

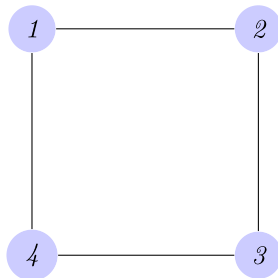
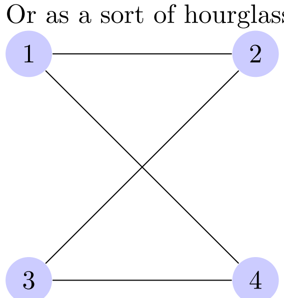


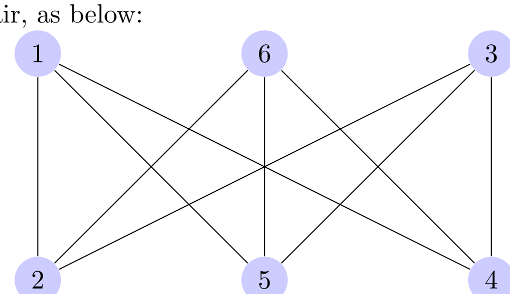
Figure 1.

Or as a sort of hourglass.



These represent, the same graph.

Graphs arise in many real-world applications. One simple example, often phrased as a puzzle, is the utility graph. There are three houses in a city, each of which needs to be connected to the gas, electric, and water utilities, the situation can be represented with a graph, with vertices corresponding to the three houses and the three utilities, and edges connecting every house-utility pair, as below:



Here is another, equally valid picture of the utility graph:

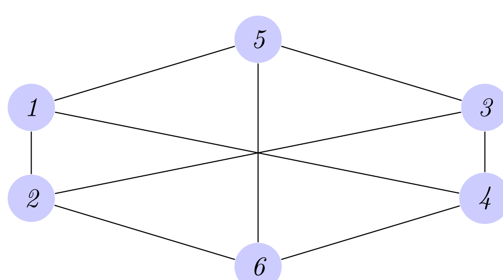


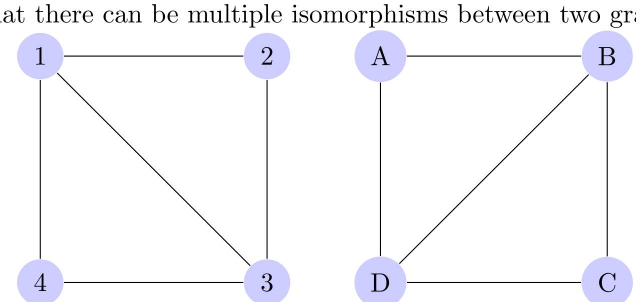
Figure 2. *utility graph*

The puzzle associated with the utility graph is to draw it in such a way that none of its edges cross, but there is a theorem that says it is in fact impossible.

As with many mathematical objects, it is helpful to define functions between graphs that preserve their fundamental structure. An *isomorphism* of graphs is a bijective map between their vertex sets that preserves edge relationships, in that two vertices are connected by an edge in the first graph if and only if their images under the isomorphism are connected in the second.

Here is an intuitive way to think about this: given two graphs, if there is a way to relabel the vertices of one so that the two graphs are equal, they are isomorphic.

Example: Although these two graphs have different vertex sets, they are nonetheless isomorphic, with an isomorphism sending 1 to B, 2 to C, 3 to D, and 4 to A. This relabling turns the first graph into a copy of the second. Note that there can be multiple isomorphisms between two graphs.



In any graph, the *degree* of a specific vertex v, denoted $d(v)$, is the number of edges emanating from that vertex. For example, in the graph on the right above, the degree of vertex A is 2, whereas that of vertex B is 3. The degree of a vertex is preserved by isomorphisms, since the images of all vertices connected to a vertex in the domain graph will be exactly those connected to its image in the codomain graph.

Lemma 1.

$$\sum_{g \in V(G)} d(g) = 2m$$

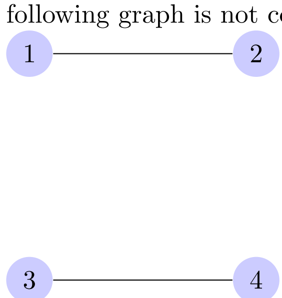
where m is the number of edges of the graph.

Proof. In the given sum, each edge will be counted twice: once for each of the vertices it connects. Every edge's endpoint will add 1 to the degree of each of these vertices, and thus add 2 to the total sum. Then the entire sum will be twice the number of edges. □

A graph $H(V_H, E_H)$ is a *subgraph* of $G(V_G, E_G)$ if $V_H \subset V_G$ and $\{u, v\} \in E_H \leftrightarrow u, v \in V_H$ and $\{u, v\} \in E_G$. Intuitively, H is a subgraph of G if erasing edges, vertices, and all the edges corresponding to a erased vertex results in H. For example, the square graph in figure 1 is a subgraph of the utility graph in Figure 2.

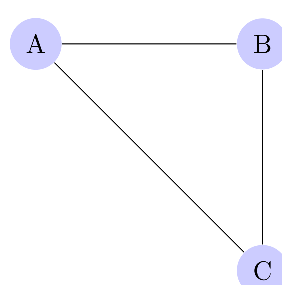
3 Access times

The *neighbors* of a vertex v, denoted $N(v)$ are the vertices that are connected to v by an edge. Formally, w is a neighbor of v if $\{w, v\} \in E(G)$.
A *walk* on a graph is a sequence of vertices in which consecutive pairs are neighbors. This process can be thought of as "walking" along the edges of a graph. For example, On the square graph in figure 1, the sequence of vertices *1, 2, 3* is a walk but *1,3,2* is not.
A graph is *connected* if there is a single walk that visits all the vertices. The following graph is not connected.



From now on, all the graphs in the paper will be connected graphs.
A *random walk* is a walk where the next vertex in the sequence is chosen uniformly randomly from the neighbors of the current vertex.
Later, I will consider two ways to complicate matters. First, the edges of of G can be assigned a weight. The walk will go from u to v with probability p_{uv} instead of probability $\frac{1}{d(u)}$
Second, a walk can be assigned a *cost*: define a function $f : V \times V \to \mathbb{R}$. Start the cost at 0. Each time the walk goes from u to v, add $f(\{u, v\})$ to the cost. Note that $f((u, v)) = f((v, u))$ is not a requirement.
The *access time* from u to v, denoted H_{uv}, is the expected number of steps required to reach vertex u from vertex v.[2] This is like computing the cost of a random walk with f((u,v))=1 for all $(u, v) \in V \times V$. The *commute time* from u to v, denoted κ_{uv} is the expected number of steps in a random walk starting at u, passing through v, and then returning back to u. Thus $\kappa_{uv} = H_{uv} + H_{vu}$.
Similarly, $H_{uv}f$ will denote the expected cost of a random walk relative to cost function f, and the *commute cost* will be defined $\kappa_{uv}f = H_{uv}f + H_{vu}f$.
Setting $f \equiv 1$, $H_{uv}f = H_uv$.

Example Here is a triangle graph:



I will find the access time from A to B. Different access times in the same graph can be related to each other.From vertex A, the probability of stepping directly to vertex B is $\frac{1}{2}$ and this takes 1 step. However, there is also a probability of $\frac{1}{2}$ of stepping to vertex C. From vertex C, the expected amount of time to reach B is H_{CB}. Hence, we reach B in 1 step with probability $\frac{1}{2}$ and in $1 + H_{BC}$ steps with probability $frac12$. Thus, from these considerations we get the equation

$$H_{AB} = \frac{1}{2} * 1 + \frac{1}{2}(1 + H_{CB})$$

Intuitively, because this graph is very symmetrical, H_{AB} should be the same as any other access time. Formally speaking, the graph is symmetrical because every mapping from V onto V produces an automorphism of the graph. For example, defining $f : V \to V$ by $A \to B, B \to C$, and $C \to A$ is an automorphism. Because automorphisms "relabel" the vertices of the graph, $H_{AB} = H_{f(A)f(B)} = H_{BC}$. By looking at other automorphisms, one can show that all the access times in this graph are the same. Making this substitution, we get the equation

$$H_{AB} = \frac{1}{2} * 1 + \frac{1}{2}(1 + H_{AB})$$

Solving this equation results in $H_{AB} = 2$
The equation I used to express the access time can be generalized to any graph.

Lemma 2. *For $u \neq v$,*

$$H_{uv} = 1 + \sum_{w \in N(u)} \frac{1}{d(u)} * H_{uw}$$

Where we take the convention $H_{uu} = 0$
This system of equations has a unique solution.

Proof. As before, this equation can be obtained by writing the H_{uv} in terms of the other access times in the graph.
For each $w \in N(u)$,there is a $\frac{1}{d(u)}$ chance of stepping from u to w. Stepping from u to w takes 1 step. From w,the amount of time to reach v is H_{wv}. Thus, there is a $\frac{1}{d(u)}$ probability of the commute time being $1 + H_{wu}$. Thus we get the equation

$$H_{uv} = \sum_{w \in N(u)} \frac{1}{d(u)} * (H_{uw} + 1) = 1 + \sum_{w \in N(u)} \frac{1}{d(u)} * H_{uw}$$

There are $n(n-1)$ ordered pairs $\{u, v\}$ with $u \neq v$. Thus we have $n(n-1)$ equations in $n(n-1)$ variables. If we include the equations $H_{uu} = 0$ we get n^2 equations in n^2 variables.
I claim this system of equations has a unique solution. The equation under Lemma 2 can be rewritten as

$$H_{uv} - \sum_{w\in N(u)} \frac{1}{d(u)} * H_{uw} = 1$$

Using matrix notation, all these equations can be written as

$$AH = v$$

where **H** is a n^2 long vector holding all the H_{uv}'s, **A** is the coefficient matrix, and **v** is the corresponding vector of zeroes and 1's. Proving that the vector H is unique is equivalent to proving that the only solution to **AH=0** is **0**. The statement **AH=0** is equivalent to the equations

$$H_{uv} - \sum_{w\in N(u)} \frac{1}{d(u)} * H_{wu} = 0$$

if $v \neq u$

$$H_{uu} = 0$$

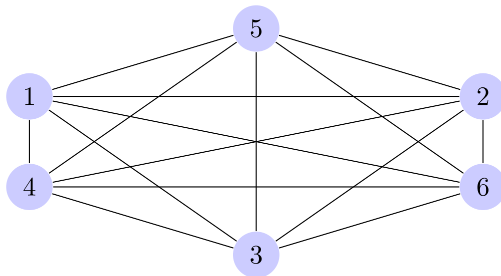
Let H_{uv} be the largest access time; $H_{uv} \geq H_{xy} : (x, y) \in V \times V$. If **AH=0**, H_{uv} is the average of d(u) terms. Keep in mind that it is possible that one of these terms is zero.

$$H_{uv} = \frac{1}{d(u)} \sum_{w\in N(u)} H_{wu}$$

If some $H_{wv} < H_{uv}$ then $H_{wv} > H_{uv}$ for some other $w \in N(u)$ because H_{uv} is the average of these d(u) numbers.However, $H_{wv} > H_{uv}$ would contradict that H_{uv} is the maximum access time, so for all $w \in N(u)$, $H_{wv} \geq H_{uv}$. Because we also have $H_{wv} \leq H_{uv}$, $H_{wv} = H_{uv} = max_{(x,y)\in VxV} Hxy$ for every neighbor w of u.
We could repeat this calculation for every element in N(u) to show that for every vertex within two steps of u, $H_{wv} = max_{(x,y)\in VxV} Hxy$. Because our graph is connected, we could repeat to show that for all $w \in V$,
$H_{wv} = max_{(x,y)\in VxV} Hxy$. Because $H_{vv} = 0$, $max_{(x,y)\in VxV} Hxy = 0$.
Using a similar argument, we can show $min_{(x,y)\in VxV} Hxy = 0$. Thus every H_{uv} is zero. Thus the coefficient matrix **A** is invertible, So the system of equations **AH=v** has a unique solution.

□

Now I will generalize the situation in the previous example to a graph with n vertices. A graph where each vertex is connected to every other vertex is called a *complete graph* or a *clique*. It is denoted K_n The triangle graph used in the previous example is the complete graph on 3 vertices. Here is the complete graph on 6 vertices:



Because complete graphs are very symmetric $H_{uw} = H_{yx}$ where $u \neq w$ and $x \neq y$ are any vertices in V. This can be proven using automorphisms as in the previous example.
Thus, using this symmetry and the equation in Lemma 2, we get

$$H_{uv} = 1 + \sum_{w \in N(u)} \frac{1}{d(u)} * H_{uw} = 1 + \sum_{w \in N(u)} \frac{1}{n-1} * H_{uv}$$

$$= 1 + \sum_{w \in N(u), w \neq v} \frac{1}{n-1} * H_{uv} + 0 * \frac{1}{n-1} = 1 + \frac{n-2}{n-1} H_{uv}$$

Solving this equation results in $H_{uv} = n - 1$
Another illustrative example is a random walk on a *path graph*: if $\{v_1 \dots v_n\}$ is the edge set, the edge set is $\{\{v_i, v_{i+1}\} : 1 \leq i \leq n-1\}\}$. Figure 3 is the path graph on three vertices.

Figure 3.

The access time between the two vertices on the end is $\Omega(n^2)$. A *lollipop graph* is complete graph on n/2 vertices attached to a path with n/2 vertices. The access time between the end vertex of the path and a vertex in the clique is $\Omega(n^3)$[1]. Notice that a lollipop graph on n verices can be constructed from a path on n vertices. This example shows that adding more vertices to a graph doesn't necessarily decrease the access time.

4 Electrical Resistance on Graphs

Resistance networks from physics turns out to be a really useful concept in studying random walks. In this section I will define the electrical resistance rules we have from physics.

Resistance Define a function $r : E \to \mathbb{R}$. $r(\{uv\})$, which I will denote r_{uv} is the resistance of that edge. For graphs with unweighted edges, I will use the function $r \equiv 1$.

Current The source of current will be the nodes of the graph. At each node a function $S : V \to \mathbb{R}$ is defined. The current emanating from a vertex will be denoted S_v. The total current injected and removed from the nodes must sum to zero. Physically, this implies charge can't build up in the graph.
Now we want to define *current* and *voltage* so that they satisfy the circuit laws from physics. Current is a function $I : V \times V \to \mathbb{R}$ and similarly voltage is a function $\phi : V \times V \to \mathbb{R}$. I will denote $I(\{u, v\})$ by I_{uv} and ϕ_{uv} analogously. Here are the physical laws that govern resistance circuits.

Kirchhoff's First Law $\sum_{w \in N(v)} I_{uw} = S_u$

Ohm's Law $\phi_{uv} = r_{uv} I_{uv}$

Kirchhoff's Second Law If $v_1, v_2 \dots v_n, v_1$ is a sequence of vertices starting and ending at v_1, $\sum_{i=1}^{n-1} \phi_{v_i v_{i+1}} + \phi_{v_n v_1} = 0$
Note that Ohm's law and Kirchhoff's second law imply that $I_{uv} = -I_{vu}$ and $\phi_{uv} = -\phi_{vu}$.
Now the goal is to show that given functions S and r the current along each edge can be uniquely defined. By Ohm's law, this will also uniquely determine the voltages. In the following analysis, I will assume $I_{uv} = -I_{vu}$ and $\phi_{uv} = -\phi_{vu}$. Note that this cuts the number of variables by half. For each vertex, by Kirchhoff's first law we have the equation $\sum_{w \in N(v)} I_{uw} = S_u$. This results in $|V|$ equations.

Claim Any $|V| - 1$ of these equations are linearly independent.

Proof. I will denote $L_u = \sum_{w \in N(v)} I_{uw}$. The variable $I_u w$ appears only in the equations corresponding to u and w. Assume that$\sum a_v L_v = 0$ for some constants $a_v$not all zero. Take L_u to be one of the variables in this expression with nonzero coefficient. Then for all $w \in N(u)$, $a_w = a_v$ because I_{uv} must cancel I_{vu}. We can repeat this analysis for the neighbors of u. Because the graph is connected, this extends to the whole graph. Thus if we want $\sum a_v L_v = 0$, we must use all $|V|$ of the equations. Thus, $|V| - 1$ of the equations are linearly independent. □

I will get more equations via Kirchhoff's second law. To get these equations, build the graph inductively: at step 0 start with a *spanning tree*, a connected subgraph of G that contains no cycles. Such a tree will contain $|V| - 1$ edges. Define E_0 as the set of edges in this subgraph.
At step n, add an edge in $E - E_{n-1}$ to get the next subgraph. The edge just added to the graph is part of some cyclic subgraph; otherwise this edge would have been part of the spanning tree in step 0. Call this cycle $C_{(n)}$ and the new edge $\{v_{1n}, v_{2n}\}$. Let $v_1, v_2 \dots v_n, v_{n+1} = v_1$ be the vertices of the $C_{(n)}$ cycle. Thus by Kirchoff's second law we have the equation $\sum_{i=1}^{n} \phi_{v_i v_{i+1}} = 0$. Define E_n to be the edges in this current subgraph.

The process stops once E has been exhausted. Since the initial tree had $|V|-1$ edges, there are $|E|-|V|+1$ edges added. Every additional edge resulted in another equation, we received $|E|-|V|+1$ equations from this process. At every step, a new equation is added to the system of equations. Further, this new equation involves the variable corresponding to the edge just added; $\phi_{v_{1n}v_{2n}}$. This variable did not appear in any of the previous equations; thus the equation at step k is linearly independent from the equations gotten from steps 1 through k-1.

Claim Further, any equation obtained via Kirchhoff's first law is independent of the ones obtained via his second law.

Proof. This will be a proof by contradiction. If C is some cyclic subgraph and $v_1, v_2 \dots v_n, v_{n+1} = v_1$ its vertices, then define $P_C = \sum_{i=1}^{n} \phi_{v_i v_{i+1}}$. In the first law, all currents flow out of a vertex, while equations derived from the second law involve a one voltage drop starting at v and another ending at v. Looking at cycles passing through u, each involves the expression $\phi_{uw1} - \phi_{uw2}$ where the w's are neighbors of u. If $\sum a_i P_{Ci} = L_u$ then for each $w \in N(u)$, the coefficient of ϕ_{uw} is $\sum_{i:w\in C_i} a_i$ and this coefficient must be larger than zero.If a_k is the coefficient of ϕ_{uw1}, $-a_k$ is the coefficient of some other neighbor of u. Thus $\sum_{w\in N(u)} \sum_{i:w\in C_i} a_i = 0$. This contradicts $\sum_{i:w\in C_i} a_i > 0$. A similar calculation shows that a linear combination of L_u's cannot be written as a linear combination of P_{Ci}'s. □

Note that I did not necessarily assume that the P_{Ci}'s were the specific equations I obtained when inductively constructing the graph; the C_i's could be any cycles. This implies also that any P_C can not be written as a linear combination involving L_u's.
There are $|V|-1$ equations from Kirchhoff's first law and $|E|-|V|+1$ from Kirchhoff's second law in $|E|$ variables. These equations are linearly independent so the solution to them is unique. However, the $|E|-|V|+1$ equations obtained from Kirchhoff's first law clearly do not represent every possible cycle in the graph. However, for any cycle, P_C can not be written as a linear combination involving L_u's. Because the set of $|E|$ equations spans the solution space, it follows that any P_C can be written as a linear combination of the equations for the $|E|-|V|+1$ cycles considered in the inductive construction of the graph. Because $P_{Ci} = 0$ for these cycles, $P_C = 0$ for all other cycles.
Now the effective resistance between two vertices can be defined by Ohm's law: If I_{uv} units of current flow from u to v and no other vertices produce or absorb current the effective resistance is defined via $R_{uv} = \frac{\phi_{uv}}{I_{uv}}$. The usual rules of computing effective resistance can be derived from these considerations.

Resistors in sequence

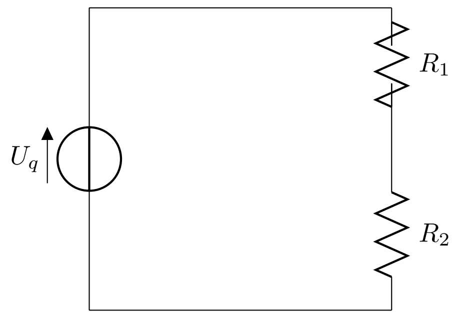


Figure 4.

Imagine that u and v are attached to a larger graph. Assume $S_w = 0$, so the net current moving through this component of the graph is equal to I_T by Kirchoff's law. We know $\phi_{uv} = \phi_{uw} + \phi_{wv}$ so $R_{uv}I_T = R_{uw}I_T + R_{wv}I_T$ so $R_{uv} = R_{uw} + R_{wv}$

Parallel Resistors

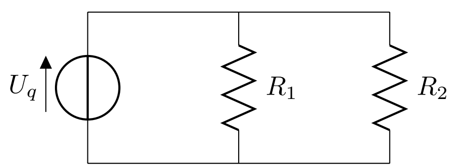


Figure 5.

Once again, by Kirchoff's first law, $I_T = I1 + I_{e2}$, where I_1 and I_2 are the currents flowing through the resistors 1 and 2 and I_T is the total current through the battery. Using Ohm's law, this equation becomes $\frac{V_T}{R_T} = \frac{V_1}{R_1} + \frac{V_2}{R_{edge2}}$. Since resistor 1 and resistor 2 together form a loop and resistor 1 and the rest of the circuit form a loop, $V_T = V_1 + V_2$. Thus $\frac{V_T}{R_T} = \frac{V_T}{R} + \frac{V_T}{R_2}$, which implies $\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_{edge2}}$. This law can be extended to n resistors in parallel.

5 Electrical Resistance and Access Times[1]

We will use the laws mentioned in the previous section to find relationships between access times and resistance.

Theorem 1. $\kappa_{uv} = 2mR_{uv}$

Proof. For the graph G, set $I_w = d(w)$ for $v \in V - v$ and $I_v = d(v) - 2m$. Define ϕ_{uv} to be the potential between u and v in this case. For $u \neq v$ By Kirchoff's first law:

$$d(u) = \sum_{w \in N(u)} I_{wu}$$

By Kirchoff's second law, Ohm's law, and that $r_{uw} = 1$:

$$\sum_{w \in N(u)} \phi_{uv} - \phi_{uw} = \sum_{w \in N(u)} \phi_{uw} = \sum_{w \in N(u)} I_{uw} r_{uw} = \sum_{w \in N(u)} I_{uw} * 1 = d(u)$$

Further, $\phi_{vv} = 0$ Thus, we have the equation $d(u) = \sum_{w \in N(u)} \phi_{uw} - \phi_{wv}$ Simplifying, $d(u) = \sum_{w \in N(u)} \phi_{uv} - \phi_{wv} = d(u)\phi_{uv} - \sum_{w \in N(u)} \phi_{wv}$ This is equivalent to

$$\phi_{uv} = 1 + \frac{1}{d(u)} \sum_{w \in N(u)} \phi_{wv}$$

Notice that substituting $\phi_{uv} = H_{uv}$ results in the equation for access times! (Compare with Lemma 2). Thus $\phi_{uv} = H_{uv}$.
Now consider the reverse process. The circuit will run backwards; 2m-d(v) units of current will come out of v and d(u) units of current will be absorbed at all the other vertices. Since the potential between u and v in the forward process is ϕ_{uv}, the potential between u and v in the reverse process is $-\phi_{uv}$. Equivalently, the potential between v and u in the reverse process is ϕ_{uv}.
Now we can superimpose these two processes. One process is 2m-d(v) units of current coming out of v and d(u) units of current absorbed at all the other vertices; and the second process is 2m-d(u) units of current absorbed by u and d(v) units of current coming out of all other vertices. Superimposing these two networks results in a graph with 2m units of current are injected into v, 2m units of current removed from u, and at all other nodes the flow of charge cancels out. Thus, $\phi_{uv} + \phi_{vu} = 2mR_{vu}$

□

The following is another nice interpretation of the quantity R_{uv}. It can be shown that in a unweighted random walk between u and v, the expected number of times crossing every edge is the same. I will not show this here.Since there are m edges, the expected number of times that an arbitrary edge is crossed is $2R_{uv}$.
The idea behind the previous proof can be generalized to looking at graphs with non uniform resistance and costs assigned to edges.

Theorem 2. *Consider a graph with nonuniform resistances on the edges. The transition probabilities between two neighboring nodes u and v will be defined by*

$$p_{uv} = \frac{1/r_{uv}}{\sum_{w in N(u)} 1/r_{uw}}$$

. Set $F = \sum_{\{u,v\} \in E} \frac{f(xy)+f(yx)}{r_{xy}}$*. Then the commute cost is* FR_{uv}*.*

Proof. First, expressing H_{uv} in terms of other access times, there is a p_{uw} chance that the walk will transition from u to w and this transition will incur f(uw) cost. Thus,these considerations result in the equation $H_{uv}f = \sum_{w \in N(v)} p_{uw}(H_{uw} + f(uw))$. The reasoning used to obtain these equations is the same to Lemma 2.
Now I will denote $\frac{1}{\sum_{w \in N(u)} 1/r_{uw}}$ by R_{Tu}. To give this quantity a physical interpretation, consider V-{u} and the edges between these vertices as a single component in a circuit connected to u. The edges connecting to u are parallel resistors in the circuit, so R_{Tu} is the effective resistance between u and the

rest of the graph. Note that $p_{uv} = R_{Tu}/r_{uv}$. This expression also has a nice physical interpretation; p_{uv} is the proportion of the total resistance between u and the rest of the graph contributed by r_{uv}.
Similar to the proof of the previous theorem, inject $S_u = \sum_{v \in N(u)} f(uv)/r_{uv}$ into every vertex of the graph and remove F current from vertex v. By Kirchhoff's first law and Ohm's law,
$S_u = I_net = \sum_{w \in N(u)} \frac{\phi_{uw}}{r_{uw}} = \sum_{w \in N(u)} \frac{\phi_{uv} - \phi wv}{r_{uw}} =$
$\phi_{uv} \sum_{w \in N(u)} 1/r_{uw} + \sum_{w \in N(u)} \phi_{wv}/r_{uw} = \frac{\phi_{uv}}{R_{Tu}} + \sum_{w \in N(u)} \phi_{wv}/r_{uw}$
Multiplying this equation by R_{Tu} results in

$$S_u R_{Tu} = \phi_{uv} + \sum_{w \in N(u)} R_{Tu} \phi_{wv}/r_{uw} = \phi_{uv} + \sum_{w \in N(u)} \phi_{wv} p_{uw}$$

By the definition of the value of S_u at the beginning of the proof $S_u R_{Tu} = \sum_{w \in N(v)} p_{uw} f(uw)$. Rearranging terms, the equation is $\phi_{uv} f = \sum_{w \in N(v)} p_{uw}(\phi_{uw} + f(uw))$. Once again, substituting $H_{uv} = \phi_{uv}$ solves the system of equations. As before, when running this circuit backwards, the voltage drop between v and u is $H_{uv} f$. In this situation, $F - S_v$ current comes out of node v and S_x current is removed from every other node x. In the forward process for H_{vu}, d(v) units of current are injected into all vertices in V-{u} and $F - S_u$ units of are removed from u. Superimposing these two processes results in F units of current emanating from v and then being removed from u. Thus $\kappa_{uv} f = H_{uv} f + H_{vu} f = F R_{uv}$
□

In the above construction, the transition probabilities were defined in terms of the resistances. It is possible to define the resistances in terms of the desired transition probabilities, but I will not show that here.

6 Conclusion

This paper discussed access time between two vertices in a graph. The methods and results of this paper could be extended to find specific expressions for finding the commute times between a vertex and an edge.
To summarize, two main results were discussed in this paper. First, in an unweighted random walk, the commute time can be written in terms of the resistances: $\kappa_{uv} = 2mR_{uv}$. Second, in a weighted random walk with costs assigned to the edges the commute time between u and v is FR_{uv} where $F = \sum_{\{u,v\} \in E} \frac{f(xy)+f(yx)}{r_{xy}}$.

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