Math 336 Final Paper

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1 Introduction

This paper will be a review of Gary Brookfield's paper "Factoring Forms" [1], which gives necessary and sufficient conditions for the factorization of polynomials in three variables where every term has degree 2 or every term has degree 3.

2 Definition of Terms

2.1 Forms

A form is a polynomial where each term has the same degree. The polynomial $x_1^2 + 2x_1x_2 + x_2^2$ is a form in two variables of degree two, the polynomial $x_1^4 + 3x_1x_2x_3^2 + 5x_1^2x_2^2 + 3x_2x_3^3$ is a form in three variables of degree four. The polynomial $x_1^3 + x_1^2 + x_1 + 1$ is not a form, because it has terms of degree varying from zero to three. We call forms of degree one linear, forms of degree two quadratic, forms of degree three cubic, and for the purposes of this paper we don't care about forms of higher degree. We call forms in two variables binary forms, and we call forms in three variables ternary forms.

When manipulating polynomials and forms, and discussing their degree, we run the danger of having terms we expected to exist accidentally canceling out. Therefore, before we discuss the subject of forms further, we need to show that if p and q are forms where $p \neq 0$ and $q \neq 0$, then $pq \neq 0$. This lemma was not mentioned in the paper I am reviewing, this is my own proof of the result.

Define a total ordering on terms of given degree as follows: Consider the first variable for which the two forms have different degrees. The form with a smaller degree of this variable is smaller. If there is no variable where the forms differ, the forms are equal. This is essentially the lexicographical order on the list of degrees.

Now consider three terms: a, b, c, where a is a term of degree n, and b and c are terms of degree m and $b \leq c$.

Consider the products ab and ac, which are both terms of degree $n + m$. We have some variable x_i where b and c first differ. For all the variables before x_i, ab and ac have the same degree for that variable. At x_i, ab has smaller degree than ac, so $ab < ac$. From this, we get if $a < b$ and $c < d$ then $ac < ad < bd$. Therefore, this order interacts nicely with multiplication.

Now, take two forms p and q not equal to 0. So they have at least one non-zero term, so they have a largest non-zero term, since the set of terms is finite and totally ordered. Let the largest term of p be b, and the largest term of q be d. For any other term a in p, $a < b$, and for any other term c in q, $c < d$. Therefore, for any pair of terms a, c which is not b, d, we have $ab < cd$, so there is only one pair of terms in p and q which comes out to bd. The coefficient of this term is therefore the product of the coefficients of b in p and d in q, which are non-zero by assumption, so pq has a term with non-zero coefficient, so $pq \neq 0$.

When we take the product of two forms, we multiply each term in the first form by each term in the second form, and add them all together. Therefore, if the first form has degree n, and the second form has degree m, then the product will be a form of degree $n+m$, and we know by the above zero product property that the terms will not accidentally cancel.

Furthermore, the product of two polynomials, at least one of which is not a form, cannot be a form. The proof of this was omitted from the paper I am reviewing. Assume that we have a polynomial p of degree n, that factors into one polynomial q, and another polynomial r, where r is not a form. Let a be the minimum degree of the terms of q, and let b be the maximum degree of the terms of q. Let c be the minimum degree of the terms of r, and let d be the maximum degree of the terms of r. Since we assumed r is not a form, c is strictly less than d. Split up the terms of q and r by their degree, that is, write

$$q = \sum_{i=a}^{b} q_i, \qquad r = \sum_{i=c}^{d} r_i,$$

where q_i and r_i are forms of degree i. By choice of a, b, c, and d, we know that $q_a, q_b, r_c, r_d \neq 0$. Therefore, $q_a r_c$ and $q_b r_d$ are both non-zero. Furthermore, notice that these make up all of the terms of lowest and highest degree of $p = qr$, and that the degree of $q_a r_c$ is $a+c$, which is strictly less than the degree of $q_b r_d$, which is $b + d = n$, since $c < d$. Therefore, p has terms of degree $a + c < n$, so p is not a form. Therefore, the factors of a form are themselves forms.

2.2 Homogenization

Now we show that there is a bijection between polynomials of degree less than or equal to n in k variables and forms of degree n in $k + 1$ variables, which process we will call homogenization. Let w be the name of the new variable. To go from a polynomial to a form, multiply each term (which will have degree less than or equal to n) by a power of w such that the term has degree exactly n. To go from a form back to a polynomial, set w equal to 1.

As an example, consider the third degree polynomial

$$x^3 + 2xy + y^2 + 3x + 2y + 1.$$

The homogenized version of this is

$$x^3 + 2xyw + y^2w + 3xw^2 + 2yw^2 + w^3.$$

Homogenizing polynomials to a fixed degree n has nice structure preserving properties. The sum of the homogenization of two polynomials (of degree less than or equal to n) is the homogenization of the sum. Additionally, if we have two terms of degree a and b homogenized to degree n and m respectively, then the product of the homogenized terms will have a factor of $w^{n+m-a-b}$, so the product of the homogenized terms will be the homogenization to degree $n + m$ of the product of the terms. Combining this with the lemma about sums, we can extend this to say that the product of two polynomials of degree a and b homogenized to degree n and m respectively is the homogenization to degree $n+m$ of the product. Therefore, if we have polynomials p, q, and r such that p factors as qr, then the homogenization of q to degree n and the homogenization of r to degree m factor the homogenization of p to degree $n+m$, and vice versa.

Since homogenization preserves factorization, we can answer questions about the factorization of polynomials by answering questions about the factorizations of forms in one more variable. Results for forms of three variables and low degree are presented in this paper.

2.3 Factorization

We will call a form *reducible* if the form can be factored at all (remember that these factors must of necessity be themselves forms), and *completely reducible* if the form can be factored into linear forms. Note that if a quadratic form is reducible at all, then it is completely reducible. To simplify the discussion, we allow the coefficients of our forms and factors to be complex numbers.

Note that any binary form is completely reducible. We can simply un-homogenize it by setting $x_2 = 1$, ending up with a polynomial in x_1. The fundamental theorem of algebra says that we can then factor it into linear factors, which when re-homogenized will factor the original binary form.

The question we pursue in this paper is which ternary forms are completely reducible.

2.4 Linearly independent

Remember that a set of vectors is linearly independent if the only time a linear combination of those vectors is 0 is when all the coefficients are 0.

Lemma 1. *If f is a quadratic ternary form, then there exist linearly independent $u, v \in \mathbb{C}^3$ such that $f(u) = f(v) = 0$.*

Proof. Let f be a quadratic ternary form. We write

$$f(x_1, x_2, x_3) = f_{11}x_1^2 + f_{22}x_2^2 + f_{33}x_3^2 + f_{12}x_1x_2 + f_{13}x_1x_3 + f_{23}x_2x_3.$$

Suppose $f_{11} \neq 0$. Then

$$f(z,1,0) = f_{11}z^2 + f_{12}z + f_{22},$$

so we can solve for z in the complex plane and obtain at least one 0 at $u = \langle z_0, 1, 0\rangle$. In the exact same manner, we find a zero of $f(z,0,1) = f_{11}z^2 + f_{13}z + f_{33}$ at $v = \langle z_1, 0, 1\rangle$. Any linear combination $au+bv$ will have second component a and third component v, so $au + bv$ is only 0 when $a = b = 0$, so u and v are linearly independent.

Symmetrically, the same holds if $f_{22} \neq 0$ or $f_{33} \neq 0$.

It remains to consider when $f_{11} = f_{22} = f_{33} = 0$. But then $f(1,0,0) = f(0,1,0) = 0$, so taking $u = \langle 1,0,0\rangle$ and $v = \langle 0,1,0\rangle$ works.

Therefore, for every quadratic form f, there exist linearly independent $u, v \in \mathbb{C}^3$ such that $f(u) = f(v) = 0$. Q.E.D.

2.5 Hessian

The Hessian is the determinant of the matrix of second partial derivatives of a function. For a ternary form such as we will be studying in this paper, the Hessian, denoted as $\mathcal{H}$, is

$$\mathcal{H}(f) = \begin{vmatrix} \partial_{11}^2 f & \partial_{12}^2 f & \partial_{13}^2 f \\ \partial_{21}^2 f & \partial_{22}^2 f & \partial_{23}^2 f \\ \partial_{31}^2 f & \partial_{32}^2 f & \partial_{33}^2 f \end{vmatrix}$$

If f is a quadratic form, then we have

$$f(x_1,x_2,x_3) = f_{11}x_1^2 + f_{22}x_2^2 + f_{33}x_3^2 + f_{12}x_1x_2 + f_{13}x_1x_3 + f_{23}x_2x_3.$$

Calculating the Hessian, we find

$$\mathcal{H}(f) = \begin{vmatrix} 2f_{11} & f_{12} & f_{13} \\ f_{12} & 2f_{22} & f_{23} \\ f_{13} & f_{23} & 2f_{33} \end{vmatrix}$$

$$\mathcal{H}(f) = 8f_{11}f_{22}f_{33} + f_{12}f_{23}f_{13} + f_{13}f_{12}f_{23} - 2f_{11}f_{23}^2 - 2f_{33}f_{12}^2 - 2f_{22}f_{13}^2$$

which is a constant.

Now, we consider how the Hessian changes with a linear change of variables:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = Ay,$$

where A is invertible, equivalently u, v, and w are linearly independent, equivalently $\det A \neq 0$.

That is, we consider

$$F(y) = f(Ay).$$

Note that this transformation preserves the degree of forms, and if f is a form of degree n, and $f = pq$, and p and q are forms of degree r and s respectively, then

$$p(Ay)q(Ay) = f(Ay) = F(y),$$

and $p(Ax)$ and $q(Ax)$ are again forms of degree r and s. This can be generalized to any finite number of factors, so if p is completely reducible to linear factors, then F is also completely reducible. Since A is invertible, we can just consider A^{-1} to show that if F is completely reducible, then f is completely reducible as well.

Now we ask what is $\mathcal{H}(F)$?

Well, we have

$$\partial_{y_1}[f(Ay)] = u_1\partial_1 f(Ay) + u_2\partial_2 f(Ay) + u_3\partial_3 f(Ay) = \sum_{i=1}^{3} u_i\partial_i f(Ay)$$

(similarly for ∂_{y_2} and ∂_{y_3}). In matrix notation (∇f is the gradient of f), we have $\nabla F = A^T\nabla f$.

Taking another partial derivative, we have

$$\partial_{y_2}\partial_{y_1}[f(Ay)] = \sum_{i=1}^{3}\sum_{j=1}^{3} u_i v_j \partial_{ji} f(Ay).$$

(Again, similarly for the other partial derivatives). In matrix notation,

$$\begin{pmatrix} \partial^2_{11}F & \partial^2_{12}F & \partial^2_{13}F \\ \partial^2_{21}F & \partial^2_{22}F & \partial^2_{23}F \\ \partial^2_{31}F & \partial^2_{32}F & \partial^2_{33}F \end{pmatrix} = A^T \begin{pmatrix} \partial^2_{11}f & \partial^2_{12}f & \partial^2_{13}f \\ \partial^2_{21}f & \partial^2_{22}f & \partial^2_{23}f \\ \partial^2_{31}f & \partial^2_{32}f & \partial^2_{33}f \end{pmatrix} A.$$

The multiplicative property of determinants gives

$$\mathcal{H}(F) = (\det A)^2\mathcal{H}(f).$$

Therefore, $\mathcal{H}(F)$ is 0 if and only if $\mathcal{H}(f)$ is 0.

3 Proof of the case of quadraic ternary forms

The main theorem in this paper, is that:

1. A quadratic ternary form f is completely reducible if and only if $\mathcal{H}(f) = 0$.
2. A cubic ternary form f is completely reducible if and only if $\mathcal{H}(f) = \lambda f$ for some constant λ.

We will reproduce the proof of the quadratic case of this theorem; the cubic case can be proved using the same methods.

Lemma 2. *If f is a quadratic ternary form that can be completely reduced, then $\mathcal{H}(f) = 0$.*

Proof. First, assume that f is a quadratic ternary form that can be completely reduced:

$$f = (ax_1 + bx_2 + cx_3)(dx_1 + ex_2 + fx_3),$$

$$f = adx_1^2 + bex_2^2 + cfx_3^2 + (ae + bd)x_1x_2 + (af + cd)x_1x_3 + (bf + ce)x_2x_3.$$

Calculating the Hessian of such an f, we find that

$$\begin{aligned}\mathcal{H}(f) =& 8abcdef + 2(ae + bd)(af + cd)(bf + ce)\\ &- 2ad(bf + ce)^2 - 2be(af + cd)^2 - 2cf(ae + bd)^2\\ =& 12abcdef - 12abcdef\\ &+ 2adb^2f^2 - 2adb^2f^2\\ &+ 2adc^2e^2 - 2adc^2e^2\\ &+ 2bea^2f^2 - 2bea^2f^2\\ &+ 2bec^2d^2 - 2bec^2d^2\\ &+ 2cfa^2e^2 - 2cfa^2e^2\\ &+ 2cfb^2d^2 - 2cfb^2d^2\\ =& 0.\end{aligned}$$

Therefore, if f is completely reducible, then $\mathcal{H}(f) = 0$. Q.E.D.

Lemma 3. *If f is a quadratic ternary form with no x_1^2 or x_2^2 term, and $\mathcal{H}(f) = 0$, then f is completely reducible*

Proof. We assume f has the form

$$f = f_{33}x_3^2 + f_{12}x_1x_2 + f_{13}x_1x_3 + f_{23}x_2x_3.$$

Computing the Hessian of f, we find

$$\mathcal{H}(f) = 2f_{12}f_{13}f_{23} - 2f_{33}f_{12}^2 = 2f_{12}(f_{13}f_{23} - f_{33}f_{12}) = 0.$$

We have two cases.

If $f_{12} = 0$, then

$$f = x_3(f_{13}x_1 + f_{23}x_2 + f_{33}x_3),$$

so f is completely reducible.

If $f_{12} \neq 0$, then $f_{13}f_{23} = f_{33}f_{12}$. Notice that

$$(f_{12}x_2 + f_{13}x_3)(f_{12}x_1 + f_{23}x_3) - (f_{13}f_{23} - f_{33}f_{12})x_3^2 = f_{12}f,$$

but the second term drops, so

$$f = \frac{1}{f_{12}}(f_{12}x_2 + f_{13}x_3)(f_{12}x_1 + f_{23}x_3),$$

so f is completely reducible. Q.E.D.

Theorem 1. *If f is a quadratic ternary form with $\mathcal{H}(f) = 0$, then f is completely reducible.*

Proof. Assume f is a quadratic ternary form with $\mathcal{H}(f) = 0$. By Lemma 1 there are linearly independent $u, v \in \mathbb{C}^3$ such that $f(u) = f(v) = 0$, and that we can choose a third vector $w \in \mathbb{C}^3$ so that $\{u, v, w\}$ is linearly independent. We let

$$A = \begin{pmatrix} | & | & | \\ u & v & w \\ | & | & | \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

and take

$$F(y) = f(Ay) = F_{11}y_1^2 + F_{22}y_2^2 + F_{33}y_3^2 + F_{12}y_1y_2 + F_{13}y_1y_3 + F_{23}y_2y_3.$$

Note that $F(1,0,0) = F_{11} = f(u) = 0$ and $F(0,1,0) = F_{22} = f(v) = 0$, so the y_1^2 and y_2^2 terms drop. Additionally, $\mathcal{H}(F) = (\det A)^2 \cdot 0 = 0$. Therefore, F is completely reducible by the special case Lemma 3 we proved above, and so f is completely reducible as well. Q.E.D.

Thus, a quadratic ternary form f is completely reducible if and only if $\mathcal{H}(f) = 0$.

References

[1] Gary Brookfield. "Factoring Forms". *The American Mathematical Monthly* 123.4 (2016): 347-362.