

Math 336 Sample Problems

One notebook sized page of notes will be allowed on the test. The test will cover through section III.5 in Gamelin. You may assume that a holomorphic function is C^2 on its domain of definition.

1. Is it true that there is a holomorphic function f on $\mathbb{C} - 0$ with $Re(f) = \frac{x+y}{x^2+y^2}$? If so find it, if not prove that there is no such function.

2. Show that the linear fractional transform $w = Tz$ that maps z_1, z_2, z_3 to w_1, w_2, w_3 is determined by the equality of cross ratios

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

3. Let $T(z) = \frac{1}{2\cos(\frac{\pi}{n}) - z}$. Prove that $T^n(z) = z$. Does this contradict anything? T^n means T composed with itself n times. (The previous problem might help in solving this problem.)

4. Suppose $w \neq 1, w \in \mathbb{C}$ is an n^{th} root of unity, $w^n = 1$. Prove that

$$1 + 2w + 3w^2 + \dots + nw^{n-1} = \frac{n}{w-1}.$$

5. Suppose $Re(z_j) \geq 0$ for $j \geq 1$ and suppose the series

$$\begin{aligned} z_1 + z_2 + \dots + z_n + \dots \\ z_1^2 + z_2^2 + \dots + z_n^2 + \dots, \end{aligned}$$

both converge. Prove that

$$|z_1|^2 + |z_2|^2 + \dots + |z_n|^2 + \dots,$$

converges.

6. Let $f(z) = x^2 - y^2 + i \log(x^2 + y^2)$. Find the points at which f is complex differentiable. Find the points at which $g(z) = x - iy$ is complex analytic.

7. Let $f(z) = u(z) + iv(z), u = Re(f(z)), v = Imf((z))$ be analytic on an open connected set Ω . Suppose there are real numbers a, b, c with $a^2 + b^2 \neq 0$ and $au(z) + bv(z) = c$ for all $z \in \Omega$. Prove that f is constant.

8. Suppose that v is the harmonic conjugate of u and u is the harmonic conjugate of v . Show that u and v must be constant.
9. Let u be harmonic on W (assume u is twice continuously differentiable). Prove that $f(z) = u_x(z) - iu_y(z)$ is analytic.
10. Let a be a complex number and suppose $|a| < 1$. Let $f(z) = \frac{z - a}{1 - \bar{a}z}$. Prove the following statements.
- $|f(z)| < 1$, if $|z| < 1$.
 - $|f(z)| = 1$, if $|z| = 1$.
11. Suppose $f = u + iv$ is analytic on $\{Re(z) > 0\}$ and $u_x + v_y = 0$. Prove that there is a real number c and complex number d so that
- $$f(z) = icz + d.$$
12. Let $f(z) = e^{-z^{-4}}$ if $z \neq 0$, $f(0) = 0$. Prove that f is analytic at z if $z \neq 0$ and that the Cauchy-Riemann equations are satisfied at 0. Is f analytic at 0?
13. Let $z_j = e^{\frac{2\pi ij}{n}}$ denote the n roots of unity. Let $c_j = |1 - z_j|$ be the $n - 1$ chord lengths from 1 to the points $z_j, j = 1, \dots, n - 1$. Prove that the product $c_1 \cdot c_2 \cdots c_{n-1} = n$. *Hint:* Consider $z^n - 1$.
14. Find a sequence of complex numbers z_n such that $\sum_{n=1}^{\infty} z_n^k$ converges for every $k = 1, 2, \dots$ but $\sum_{n=1}^{\infty} |z_n|^k$ diverges for every $k = 1, 2, \dots$. *Hint:* Try $z_n = \frac{e^{2\pi i n s}}{\log(n+1)}$ for an appropriate real number s .
15. Suppose f is analytic on a connected open set. Assume $f^2 = \bar{f}$. Prove that f is constant. What are the possible values of the constant?
16. You will need to know the definitions of the following terms and statements of the following theorems.
- Modulus (absolute value) and argument of a complex number
 - Complex derivative
 - Complex analytic function
 - Cauchy-Riemann equations
 - Harmonic functions and harmonic conjugate
 - Complex exponential function and trigonometric functions
 - Complex logarithm and powers
 - Linear fractional transformations
 - Mean value property
 - Maximum principle

13. There may be homework problems or example problems from the text on the midterm.