An essay on the Riemann Hypothesis

Alain Connes

Abstract The Riemann hypothesis is, and will hopefully remain for a long time, a great motivation to uncover and explore new parts of the mathematical world. After reviewing its impact on the development of algebraic geometry we discuss three strategies, working concretely at the level of the explicit formulas. The first strategy is "analytic" and is based on Riemannian spaces and Selberg's work on the trace formula and its comparison with the explicit formulas. The second is based on algebraic geometry and the Riemann-Roch theorem. We establish a framework in which one can transpose many of the ingredients of the Weil proof as reformulated by Mattuck, Tate and Grothendieck. This framework is elaborate and involves noncommutative geometry, Grothendieck toposes and tropical geometry. We point out the remaining difficulties and show that RH gives a strong motivation to develop algebraic geometry in the emerging world of characteristic one. Finally we briefly discuss a third strategy based on the development of a suitable "Weil cohomology", the role of Segal's Γ-rings and of topological cyclic homology as a model for "absolute algebra" and as a cohomological tool.

1 Introduction

Let $\pi(x) := \#\{p \mid p \in \mathscr{P}, p < x\}$ be the number of primes less than x with $\frac{1}{2}$ added when x is prime. Riemann [96] found for the counting function [1]

$$f(x) := \sum \frac{1}{n} \pi(x^{\frac{1}{n}}),$$

Alain Connes

Collège de France, 3 Rue d'Ulm, Paris, IHÉS, 35 Route de Chartres, Bures sur Yvette and Ohio State University, Columbus, Ohio. e-mail: alain@connes.org

[1] Similar counting functions were already present in Chebyshev's work

the following formula involving the integral logarithm function $\mathrm{Li}(x) = \int_0^x \frac{dt}{\log t}$,

$$f(x) = \mathrm{Li}(x) - \sum_\rho \mathrm{Li}(x^\rho) + \int_x^\infty \frac{1}{t^2 - 1} \frac{dt}{t \log t} - \log 2 \tag{1}$$

in terms[2] of the non-trivial zeros ρ of the analytic continuation (shown as well as two proofs of the functional equation by Riemann at the beginning of his paper) of the Euler zeta function

$$\zeta(s) = \sum \frac{1}{n^s}$$

Reading Riemann's original paper is surely still the best initiation to the subject. In his lecture given in Seattle in August 1996, on the occasion of the 100-th anniversary of the proof of the prime number theorem, Atle Selberg comments about Riemann's paper: [99]

> It is clearly a preliminary note and might not have been written if L. Kronecker had not urged him to write up something about this work (letter to Weierstrass, Oct. 26 1859). It is clear that there are holes that need to be filled in, but also clear that he had a lot more material than what is in the note[3]. What also seems clear : Riemann is not interested in an asymptotic formula, not in the prime number theorem, what he is after is an exact formula!

The Riemann hypothesis (RH) states that all the non-trivial zeros of ζ are on the line $\frac{1}{2} + i\mathbb{R}$. This hypothesis has become over the years and the many unsuccessful attempts at proving it, a kind of "Holy Grail" of mathematics. Its validity is indeed one of the deepest conjectures and besides its clear inference on the distribution of prime numbers, it admits relations with many parts of pure mathematics as well as of quantum physics.

It is, and will hopefully remain for a long time, a great motivation to uncover and explore new parts of the mathematical world. There are many excellent texts on RH, such as [12] which explain in great detail what is known about the problem, and the many implications of a positive answer to the conjecture. When asked by John Nash to write a text on RH[4], I realized that writing one more encyclopedic text would just add another layer to the psychological barrier that surrounds RH. Thus I have chosen deliberately to adopt another point of view, which is to navigate between the many forms of the explicit formulas (of which (1) is the prime example) and possible strategies to attack the problem, stressing the value of the elaboration of new concepts rather than "problem solving".

- *RH and algebraic geometry*

  We first explain the Riemann-Weil explicit formulas in the framework of adeles and global fields in §2.1. We then sketch in §2.3 the geometric proof of RH for

---

[2] More precisely Riemann writes $\sum_{\Re(\alpha)>0} \left(\mathrm{Li}(x^{\frac{1}{2}+\alpha i}) + \mathrm{Li}(x^{\frac{1}{2}-\alpha i}) \right)$ instead of $\sum_\rho \mathrm{Li}(x^\rho)$ using the symmetry $\rho \to 1 - \rho$ provided by the functional equation, to perform the summation.

[3] See [52] Chapter VII for detailed support to Selberg's comment

[4] My warmest thanks to Michael Th. Rassias for the communication

function fields as done by Weil, Mattuck, Tate and Grothendieck. We then turn to the role of RH in generating new mathematics, its role in the evolution of algebraic geometry in the XX-th century through the Weil conjectures, proved by Deligne, and the elaboration by Grothendieck of the notions of scheme and of topos.

- *Riemannian Geometry, Spectra and trace formulas*

  Besides the proof of analogues of RH such as the results of Weil and of Deligne, there is another family of results that come pretty close. They give another natural approach of RH using analysis, based on the pioneering work of Selberg on trace formulas. These will be reviewed in Section 3 where the difficulty arising from the minus sign in front of the oscillatory terms will be addressed.

- *The Riemann-Roch strategy: A Geometric Framework*

  In Section 4, we shall describe a geometric framework, established in our joint work with C. Consani, allowing us to transpose several of the key ingredients of the geometric proof of RH for function fields recalled in §2.3. It is yet unclear if this is the right set-up for the final Riemann-Roch step, but it will illustrate the power of RH as an incentive to explore new parts of mathematics since it gives a clear motivation for developing algebraic geometry in characteristic 1 along the line of tropical geometry. This will take us from the world of characteristic p to the world of characteristic 1, and give us an opportunity to describe its relation with semi-classical and idempotent analysis, optimization and game theory[5], through the Riemann-Roch theorem in tropical geometry [4, 56, 92].

- *Absolute Algebra and the sphere spectrum*

  The arithmetic and scaling sites which are the geometric spaces underlying the Riemann-Roch strategy of Section 4 are only the semiclassical shadows of a more mysterious structure underlying the compactification of $\operatorname{Spec}\mathbb{Z}$ that should give a cohomological interpretation of the explicit formulas. We describe in this last section an essential tool coming from algebraic topology: Segal's Γ-rings and the sphere spectrum, over which all previous attempts at developing an absolute algebra organize themselves. Moreover, thanks to the results of Hesselholt and Madsen in particular, topological cyclic homology gives a cohomology theory suitable to treat in a unified manner the local factors of L-functions.

---

[5] one of the topics in which John Nash made fundamental contributions

2 RH and algebraic geometry

I will briefly sketch here the way RH, once transposed in finite characteristic, has played a determining role in the upheaval of the very notion of geometric space in algebraic geometry culminating with the notions of scheme and topos due to Grothendieck, with the notion of topos offering a frame of thoughts of incomparable generality and breadth. It is a quite remarkable testimony to the unity of mathematics that the origin of this discovery lies in the greatest problem of analysis and arithmetic.

2.1 The Riemann-Weil explicit formulas, Adeles and global fields

Riemann's formula (1) is a special case of the "explicit formulas" which establish a duality between the primes and the zeros of zeta. This formula has been extended by Weil in the context of global fields which provides a perfect framework for a generalization of RH since it has been solved, by Weil, for all global fields except number fields.

2.1.1 The case of ζ

Let us start with the explicit formulas (*cf.* [113, 115, 11, 93]). We start with a function $F(u)$ defined for $u \in [1, \infty)$, continuous and continuously differentiable except for finitely many points at which both $F(u)$ and $F'(u)$ have at most a discontinuity of the first kind, [6] and such that, for some $\varepsilon > 0$, $F(u) = O(u^{-1/2 - \varepsilon})$. One then defines the Mellin transform of F as

$$\Phi(s) = \int_1^\infty F(u)\, u^{s-1} du \tag{2}$$

The explicit formula then takes the form

$$\Phi(\frac{1}{2}) + \Phi(-\frac{1}{2}) - \sum_{\rho \in \mathrm{Zeros}} \Phi(\rho - \frac{1}{2}) = \sum_p \sum_{m=1}^\infty \log p \; p^{-m/2} F(p^m) + \tag{3}$$

$$+ (\frac{\gamma}{2} + \frac{\log \pi}{2}) F(1) + \int_1^\infty \frac{t^{3/2} F(t) - F(1)}{t(t^2 - 1)} dt$$

where $\gamma = -\Gamma'(1)$ is the Euler constant, and the zeros are counted with their multiplicities *i.e.* $\sum_{\rho \in \mathrm{Zeros}} \Phi(\rho - \frac{1}{2})$ means $\sum_{\rho \in \mathrm{Zeros}} \mathrm{order}(\rho) \Phi(\rho - \frac{1}{2})$.

---

[6] and at which the value of $F(u)$ is defined as the average of the right and left limits there

2.1.2 Adeles and global fields

By a result of Iwasawa [76] a field $\mathbb{K}$ is a finite algebraic number field, or an algebraic function field of one variable over a finite constant field, if and only if there exists a semi-simple (*i.e.* with trivial Jacobson radical [78]) commutative ring R containing $\mathbb{K}$ such that R is locally compact, but neither compact nor discrete and $\mathbb{K}$ is discrete and cocompact in R. This result gives a conceptual definition of what is a "global field" and indicates that the arithmetic of such fields is intimately related to analysis on the parent ring R which is called the ring of adeles of $\mathbb{K}$ [114, 104]. It is the opening door to a whole world which is that of automorphic forms and representations, starting in the case of GL_1 with Tate's thesis [104] and Weil's book [114]. Given a global field $\mathbb{K}$, the ring $\mathbb{A}_{\mathbb{K}}$ of adeles of $\mathbb{K}$ is the restricted product of the locally compact fields $\mathbb{K}_v$ obtained as completions of $\mathbb{K}$ for the different places v of $\mathbb{K}$. The equality $dax = |a|dx$ for the additive Haar measure defines the module $\mathrm{Mod} : \mathbb{K}_v \to \mathbb{R}_+$, $\mathrm{Mod}(a) := |a|$ on the local fields $\mathbb{K}_v$ and also as a group homomorphism $\mathrm{Mod} : C_{\mathbb{K}} \to \mathbb{R}_+^*$ where $C_{\mathbb{K}} = \mathrm{GL}_1(\mathbb{A}_{\mathbb{K}})/\mathbb{K}^{\times}$ is the idele class group. The kernel of the module is a compact subgroup $C_{\mathbb{K},1} \subset C_{\mathbb{K}}$ and the range of the module is a cocompact subgroup $\mathrm{Mod}(\mathbb{K}) \subset \mathbb{R}_+^*$. On any locally compact modulated group, such as $C_{\mathbb{K}}$ or the multiplicative groups $\mathbb{K}_v^*$, one normalizes the Haar measure d^*u uniquely so that the measure of $\{u \mid 1 \leq |u| \leq \Lambda\}$ is equivalent to $\log \Lambda$ when $\Lambda \to \infty$.

2.1.3 Weil's explicit formulas

As shown by Weil, in [115], adeles and global fields give the natural framework for the explicit formulas. For each character $\chi \in \widehat{C_{\mathbb{K},1}}$ one chooses an extension $\tilde{\chi}$ to $C_{\mathbb{K}}$ and one lets $Z_{\tilde{\chi}}$ be the set (with multiplicities and taken modulo the orthogonal of $\mathrm{Mod}(\mathbb{K})$, *i.e.* $\{s \in \mathbb{C} \mid q^s = 1, \forall q \in \mathrm{Mod}(\mathbb{K})\}$) of zeros of the L-function associated to $\tilde{\chi}$. Let then α be a nontrivial character of $\mathbb{A}_{\mathbb{K}}/\mathbb{K}$ and $\alpha = \prod \alpha_v$ its local factors. The explicit formulas take the following form, with $h \in \mathscr{S}(C_{\mathbb{K}})$ a Schwartz function with compact support:

$$\hat{h}(0) + \hat{h}(1) - \sum_{\chi \in \widehat{C_{\mathbb{K},1}}} \sum_{Z_{\tilde{\chi}}} \hat{h}(\tilde{\chi}, \rho) = \sum_v {\int_{\mathbb{K}_v^*}}' \frac{h(u^{-1})}{|1 - u|} \, d^*u \tag{4}$$

where the principal value $\int_{\mathbb{K}_v^*}'$ is normalized by the additive character α_v (*cf.* [27] Chapter II, 8.5, Theorem 2.44 for the precise notations and normalizations) and for any character ω of $C_{\mathbb{K}}$ one lets

$$\hat{h}(\omega, z) := \int h(u) \, \omega(u) \, |u|^z \, d^*u, \ \ \hat{h}(t) := \hat{h}(1, t) \tag{5}$$

For later use in §4.1 we compare (3) with the Weil way (4) of writing the explicit formulas. Let the function h be the function on $C_{\mathbb{Q}}$ given by $h(u) := |u|^{-\frac{1}{2}} F(|u|)$ (with $F(v) = 0$ for $v < 1$). Then $\hat{h}(\omega, z) = 0$ for characters with non-trivial restriction

to $C_{\mathbb{Q},1} = \hat{\mathbb{Z}}^{\times}$, while $\hat{h}(1,z) = \Phi(z - \frac{1}{2})$. Moreover note that for the archimedean place v of $\mathbb{K} = \mathbb{Q}$ one has, disregarding the principal values for simplicity,

$$\int_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|} \, d^*u = \int_{\mathbb{R}^*} \frac{h(u)}{|1-u^{-1}|} \, d^*u$$

$$= \frac{1}{2} \int_1^\infty h(t) \left(\frac{1}{|1-t^{-1}|} + \frac{1}{|1+t^{-1}|} \right) \frac{dt}{t} = \int_1^\infty \frac{t^{3/2}F(t)}{t(t^2-1)} dt$$

where the $\frac{1}{2}$ comes from the normalization of the multiplicative Haar measure of $\mathbb{R}^*$ viewed as a modulated group. In a similar way, the normalization of the multiplicative Haar measure on $\mathbb{Q}_p^*$ shows that for the finite place associated to the prime p one gets the term $\sum_{m=1}^\infty \log p \, p^{-m/2} F(p^m)$.

2.2 RH for function fields

When the module $\mathrm{Mod}(\mathbb{K})$ of a global field is a discrete subgroup of $\mathbb{R}_+^*$ it is of the form $\mathrm{Mod}(\mathbb{K}) = q^{\mathbb{Z}}$ where q is a prime power, and the field $\mathbb{K}$ is the function field of a smooth projective curve C over the finite field $\mathbb{F}_q$.

Already at the beginning of the XX-th century, Emil Artin and Friedrich Karl Schmidt have generalized RH to the case of function fields. We refer to the text of Cartier [18] where he explains how Weil's definition of the zeta function associated to a variety over a finite field slowly emerged, starting with the thesis of E. Artin where this zeta function was defined for quadratic extensions of $\mathbb{F}_q[T]$, explaining F. K. Schmidt's generalization to arbitrary extensions of $\mathbb{F}_q[T]$ and the work of Hasse on the Riemann hypothesis for elliptic curves over finite fields.

When the global field $\mathbb{K}$ is a function field, geometry comes to the rescue. The problem becomes intimately related to the geometric one of estimating the number $N(q^r) := \#C(\mathbb{F}_{q^r})$ of points of C rational over a finite extension $\mathbb{F}_{q^r}$ of the field of definition of C. The analogue of the Riemann zeta function is a generating function: the Hasse-Weil zeta function

$$\zeta_C(s) := Z(C, q^{-s}), \quad Z(C,T) := \exp\left(\sum_{r \geq 1} N(q^r) \frac{T^r}{r} \right) \tag{6}$$

The analogue of RH for ζ_C was proved by André Weil in 1940. Pressed by the circumstances (he was detained in jail) he sent a Comptes-Rendus note to E. Cartan announcing his result. Friedrich Karl Schmidt and Helmut Hasse had previously been able to transpose the Riemann-Roch theorem in the framework of geometry over finite fields and shown its implications for the zeta function: it is a rational fraction (of the variable T) and it satisfies a functional equation. But it took André Weil several years to put on solid ground a general theory of algebraic geometry in finite characteristic that would justify his geometric arguments and allow him to transpose

the Hodge index theorem in the form due to the Italian geometers Francesco Severi and Guido Castelnuovo at the beginning of the XX-th century.

2.3 The proof using Riemann-Roch on $\bar{C} \times \bar{C}$

Let C be a smooth projective curve over the finite field $\mathbb{F}_q$. The first step is to extend the scalars from $\mathbb{F}_q$ to an algebraic closure $\bar{\mathbb{F}}_q$. Thus one lets

$$\bar{C} := C \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q \qquad (7)$$

This operation of extension of scalars does not change the points over $\bar{\mathbb{F}}_q$, *i.e.* one has $\bar{C}(\bar{\mathbb{F}}_q) = C(\bar{\mathbb{F}}_q)$. The Galois action of the Frobenius automorphism of $\bar{\mathbb{F}}_q$ raises the coordinates of any point $x \in C(\bar{\mathbb{F}}_q)$ to the q-th power and this transformation of $C(\bar{\mathbb{F}}_q)$ coincides with the *relative Frobenius* $\mathrm{Fr}_r := \mathrm{Fr}_C \times \mathrm{Id}$ of $\bar{C}$, where Fr_C is the *absolute Frobenius* of C (which is the identity on points of the scheme and the q-th power map in the structure sheaf). The relative Frobenius Fr_r is $\bar{\mathbb{F}}_q$-linear by construction and one can consider its graph in the surface $X = \bar{C} \times_{\bar{\mathbb{F}}_q} \bar{C}$ which is the square of $\bar{C}$. This graph is the Frobenius correspondence Ψ. It is important to work over an algebraically closed field in order to have a good intersection theory. This allows one to express the right hand side of the explicit formula (4) for the zeta function ζ_C as an intersection number $D.\Delta$, where Δ is the diagonal in the square and $D = \sum a_k \Psi^k$ is the divisor given by a finite integral linear combination of powers of the Frobenius correspondence. The terms $\hat{h}(0)$, $\hat{h}(1)$ in the explicit formula are also given by intersection numbers $D.\xi_j$, where

$$\xi_0 = e_0 \times \bar{C}, \ \xi_1 = \bar{C} \times e_1 \qquad (8)$$

where the e_j are points of $\bar{C}$. One then considers divisors on X up to the additive subgroup of principal divisors *i.e.* those corresponding to an element $f \in \mathscr{K}$ of the function field of X. The problem is then reduced to proving the negativity of $D.D$ (the self-intersection pairing) for divisors of degree zero. The Riemann-Roch theorem on the surface X gives the answer. To each divisor D on X corresponds an index problem and one has a finite dimensional vector space of solutions $H^0(X, \mathscr{O}(D))$ over $\bar{\mathbb{F}}_q$. Let

$$\ell(D) = \dim H^0(X, \mathscr{O}(D)) \qquad (9)$$

The best way to think of the sheaf $\mathscr{O}(D)$ is in terms of Cartier divisors, *i.e.* a global section of the quotient sheaf $\mathscr{K}^\times / \mathscr{O}_X^\times$, where $\mathscr{K}$ is the constant sheaf corresponding to the function field of X and $\mathscr{O}_X$ is the structure sheaf. The sheaf $\mathscr{O}(D)$ associated to a Cartier divisor is obtained by taking the sub-sheaf of $\mathscr{K}$ whose sections on U_i form the sub $\mathscr{O}_X$-module generated by $f_i^{-1} \in \Gamma(U_i, \mathscr{K}^\times)$ where the f_i represent D locally. One has a "canonical" divisor K and Serre duality

$$\dim H^2(X, \mathscr{O}(D)) = \dim H^0(X, \mathscr{O}(K-D)) \qquad (10)$$

Moreover the following Riemann-Roch formula holds

$$\sum_{0}^{2}(-1)^j \dim H^j(X, \mathscr{O}(D)) = \frac{1}{2}D.(D-K) + \chi(X) \qquad (11)$$

where $\chi(X)$ is the arithmetic genus. All this yields the Riemann-Roch inequality

$$\ell(D) + \ell(K-D) \geq \frac{1}{2}D.(D-K) + \chi(X) \qquad (12)$$

One then applies Lemma 1 to the quadratic form $\mathfrak{s}(D,D') = D.D'$ using the ξ_j of (8). One needs three basic facts ([62])

1. If $\ell(D) > 1$ then D is equivalent to a strictly positive divisor.
2. If D is a strictly positive divisor then

$$D.\xi_0 + D.\xi_1 > 0$$

3. One has $\xi_0.\xi_1 = 1$ and $\xi_j.\xi_j = 0$.

One then uses (12) to show (see [62]) that if $D.D > 0$ then after a suitable rescaling by $n > 0$ or $n < 0$ one gets $\ell(nD) > 1$ which shows that the hypothesis (2) of the following simple Lemma 1 is fulfilled, and hence that RH holds for ζ_C,

Lemma 1. *Let $\mathfrak{s}(x,y)$ be a symmetric bilinear form on a vector space E (over $\mathbb{Q}$ or $\mathbb{R}$). Let $\xi_j \in E$, $j \in \{0,1\}$, be such that*

1. $\mathfrak{s}(\xi_j, \xi_j) = 0$ and $\mathfrak{s}(\xi_0, \xi_1) = 1$.
2. For any $x \in E$ such that $\mathfrak{s}(x,x) > 0$ one has $\mathfrak{s}(x, \xi_0) \neq 0$ or $\mathfrak{s}(x, \xi_1) \neq 0$.

Then one has the inequality

$$\mathfrak{s}(x,x) \leq 2\mathfrak{s}(x, \xi_0)\mathfrak{s}(x, \xi_1), \quad \forall x \in E \qquad (13)$$

The proof takes one line but the meaning of this lemma is to reconcile the "naive positivity" of the right hand side of the explicit formula (4) (which is positive when $h \geq 0$ vanishes near $u = 1$) with the negativity of the left hand side needed to prove RH (*cf.* §3.1 (17) below).

At this point we see that it is highly desirable to find a geometric framework for the Riemann zeta function itself, in which the Hasse-Weil formula (6), the geometric interpretation of the explicit formulas, the Frobenius correspondences, the divisors, principal divisors, Riemann-Roch problem on the curve and the square of the curve all make sense.

Such a tentative framework will be explained in Section 4. It involves in particular the refinement of the notion of geometric space which was uncovered by Grothendieck and to which we now briefly turn.

2.4 Grothendieck and the notion of topos

The essential ingredients of the proof explained in §2.3 are the intersection theory for divisors on $\bar{C} \times \bar{C}$, sheaf cohomology and Serre duality, which give the formulation of the Riemann-Roch theorem. Both owe to the discovery of sheaf theory by J. Leray and the pioneering work of J. P. Serre on the use of sheaves for the Zariski topology in the algebraic context, with his fundamental theorem comparing the algebraic and analytic frameworks. The next revolution came from the elaboration by A. Grothendieck and M. Artin of etale ℓ-adic cohomology. It allows one to express the Weil zeta function of a smooth projective variety X defined over a finite field $\mathbb{F}_q$ *i.e.* the function $Z(X,t)$ given by (6) with $t = q^{-s}$ which continues to make sense in general, as an alternate product of the form

$$Z(X,t) = \prod_{j=0}^{2\dim X} \det(1 - tF^* \mid H^j(\bar{X}_{\mathrm{et}}, \mathbb{Q}_\ell))^{(-1)^{j+1}} \qquad (14)$$

where F^* corresponds to the action of the Frobenius on the ℓ-adic cohomology and ℓ is a prime which is prime to q. This equality follows from a Lefschetz formula for the number $N(q^r)$ of fixed points of the r-th power of the Frobenius and when $X = C$ is a curve the explicit formulas reduce to the Lefschetz formula. The construction of the cohomology groups $H^j(\bar{X}_{\mathrm{et}}, \mathbb{Q}_\ell)$ is indirect and they are defined as :

$$H^j(\bar{X}_{\mathrm{et}}, \mathbb{Q}_\ell) = \varprojlim_n \left(H^j(\bar{X}_{\mathrm{et}}, \mathbb{Z}/\ell^n \mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

where $\bar{X}_{\mathrm{et}}$ is the etale site of $\bar{X}$. Recently the etale site of a scheme has been refined [5] to the *pro-etale* site whose objects no longer satisfy any finiteness condition. The cohomology groups $H^j(\bar{X}_{\mathrm{proet}}, \bar{\mathbb{Q}}_\ell)$ are then directly obtained using the naive interpretation (without torsion coefficients). One needs to pay attention in (14) to the precise definition of F, it is either the relative Frobenius Fr_r or the *Geometric Frobenius* Fr_g which is the inverse of the *Arithmetic Frobenius* Fr_a. The product $\mathrm{Fr}_a \circ \mathrm{Fr}_r = \mathrm{Fr}_r \circ \mathrm{Fr}_a$ is the absolute Frobenius Fr which acts trivially on the ℓ-adic cohomology. To understand the four different incarnations of "the Frobenius" it is best to make them explicit in the simplest example of the scheme $\mathrm{Spec}\, R$ where $R = \bar{\mathbb{F}}_q[T]$ is the ring of polynomials $P(T) = \sum a_j T^j$, $a_j \in \bar{\mathbb{F}}_q$

- Geometric Frobenius: $\sum a_j T^j \mapsto \sum a_j^{1/q} T^j$
- Relative Frobenius: $P(T) \mapsto P(T^q)$
- Absolute Frobenius: $P(T) \mapsto P(T)^q$
- Arithmetic Frobenius: $\sum a_j T^j \mapsto \sum a_j^q T^j$

The motivation of Grothendieck for developing etale cohomology came from the search of a Weil cohomology and the Weil conjectures which were solved by Deligne in 1973 ([45]).

In his quest Grothendieck uncovered several key concepts such as those of schemes and above all that of topos, in his own words:

C'est le thème du topos, et non celui des schémas, qui est ce "lit", ou cette "rivière pro-
fonde", où viennent s'épouser la géométrie et l'algèbre, la topologie et l'arithmétique,
la logique mathématique et la théorie des catégories, le monde du continu et celui des
structures "discontinues" ou "discrètes". Si le thème des schémas est comme le *cœur*
de la géométrie nouvelle, le thème du topos en est l'enveloppe, ou la *demeure*. Il est
ce que j'ai conçu de plus vaste, pour saisir avec finesse, par un même langage riche en
résonances géométriques, une "essence" commune à des situations des plus éloignées les
unes des autres, provenant de telle région ou de telle autre du vaste univers des choses
mathématiques.

3 Riemannian Geometry, Spectra and trace formulas

Riemannian Geometry gives a wealth of "spectra" of fundamental operators associ-
ated to a geometric space, such as the Laplacian and the Dirac operators.

3.1 The Selberg trace formula

In the case of compact Riemann surfaces X with constant negative curvature -1,
the Selberg trace formula [97], takes the following form where the eigenvalues of
the Laplacian are written in the form[7] $\lambda_n = -(\frac{1}{4} + r_n^2)$. Let $\delta > 0$, $h(r)$ be an an-
alytic function in the strip $|\Im(r)| \leq \frac{1}{2} + \delta$ and such that $h(r) = h(-r)$ and with
$(1 + r^2)^{1+\delta}|h(r)|$ being bounded. Then [97, 98, 69], with A the area of X,

$$\sum h(r_n) = \frac{A}{4\pi} \int_{-\infty}^{\infty} \tanh(\pi r) h(r) r dr + \sum_{\{T\}} \frac{\log N(T_0)}{N(T)^{\frac{1}{2}} - N(T)^{-\frac{1}{2}}} g(\log N(T)) \quad (15)$$

where g is the Fourier transform of h, *i.e.* more precisely $g(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-irs} dr$.
The $\log N(T)$ are the lengths of the periodic orbits of the geodesic flow with
$\log N(T_0)$ being the length of the primitive one. Already in 1950-51, Selberg saw
the striking similarity of his formula with (3) which (*cf.* [69]) can be rewritten in the
following form, with h and g as above and the non-trivial zeros of zeta expressed in
the form $\rho = \frac{1}{2} + i\gamma$,

$$\sum_{\gamma} h(\gamma) = h(\frac{i}{2}) + h(-\frac{i}{2}) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega(r) h(r) dr - 2 \sum \Lambda(n) n^{-\frac{1}{2}} g(\log n) \quad (16)$$

where

$$\omega(r) = \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + i\frac{r}{2} \right) - \log \pi, \quad \frac{\Gamma'}{\Gamma}(s) = \int_0^1 \frac{1 - t^{s-1}}{1 - t} dt - \gamma, \quad \forall s, \Re(s) > 0$$

---

[7] where the argument of r_n is either 0 or $-\pi/2$

and $\Lambda(n)$ is the von-Mangoldt function with value $\log p$ for powers p^ℓ of primes and zero otherwise. Moreover Selberg found that there is a zeta function which corresponds to (15) in the same way that $\zeta(s)$ corresponds to (16). The role of Hilbert space is crucial in the work of Selberg to ensure that the zeros of his zeta function satisfy the analogue of RH. This role of Hilbert space is implicit as well in RH which has been reformulated by Weil as the positivity of the functional $W(g)$ defined as both sides of (16). More precisely the equivalent formulation is that $W(g \star g^*) \geq 0$ on functions g which correspond to Fourier transforms of analytic functions h as above (*i.e.* even and analytic in a strip $|\Im z| \leq \frac{1}{2} + \delta$) where for even functions one has $g^*(s) := \overline{g(-s)} = \overline{g(s)}$. Moreover by [13, 17], it is enough, using Li's criterion (*cf.* [83, 13]), to check the positivity on a small class of explicit real valued functions with compact support. In fact for later purposes it is better to write this criterion as

$$RH \iff \mathfrak{s}(f,f) \leq 0, \ \forall f \mid \int f(u)d^*u = \int f(u)du = 0 \tag{17}$$

where for real compactly supported functions on $\mathbb{R}_+^*$, we let $\mathfrak{s}(f,g) := N(f \star \tilde{g})$ where $\star$ is the convolution product on $\mathbb{R}_+^*$, $\tilde{g}(u) := u^{-1}g(u^{-1})$, and

$$N(h) := \sum_{n=1}^\infty \Lambda(n)h(n) + \int_1^\infty \frac{u^2 h(u) - h(1)}{u^2 - 1} d^*u + c\,h(1), \ c = \frac{1}{2}(\log \pi + \gamma) \tag{18}$$

The Selberg trace formula has been considerably extended by J. Arthur and plays a key role in the Langland's program. We refer to [1] for an introduction to this vast topic.

3.2 The minus sign and absorption spectra

The Selberg trace formula [97, 98] for Riemann surfaces of finite area, acquires additional terms which make it look *e.g.* in the case of $X = H/PSL(2, \mathbb{Z})$ (where H is the upper half plane with the Poincaré metric) even more similar to the explicit formulas, since the parabolic terms now involve explicitly the sum

$$2\sum_{n=1}^\infty \frac{\Lambda(n)}{n} g(2\log n)$$

Besides the square root in the $\Lambda(n)$ terms in the explicit formulas (16)

$$-2\sum_{n=1}^\infty \frac{\Lambda(n)}{n^{\frac{1}{2}}} g(\log n)$$

there is however a striking difference which is that these terms occur with a positive sign instead of the negative sign in (16), as discussed in [69] §12. This discussion of the minus sign was extended to the case of the semiclassical limit of Hamiltonian

systems in physics in [6]. In order to get some intuition of what this reveals, it is relevant to go back to the origin of spectra in physics, *i.e.* to the very beginning of spectroscopy. It occurred when Joseph Von Fraunhofer (1787-1826) could identify, using self-designed instruments, about 500 dark lines in the light coming from the sun, decomposed using the dispersive power of a spectroscope such as a prism (*cf.* Figure 1). These dark lines constitute the "absorption spectrum" and it took about 45 years before Kirchhoff and Bunsen noticed that several of these Fraunhofer lines coincide (*i.e.* have the same wave length) with the bright lines of the "emission" spectrum of heated elements, and showed that they could be reobtained by letting white light traverse a cold gas. In his work on the trace formula in the finite covolume case, Selberg had to take care of a superposed continuous spectrum due to the presence of the non-compact cusps of the Riemann surface.

Fig. 1 The three kinds of spectra occuring in spectroscopy: 1) The top one is the "continuous spectrum" which occurs when white light is decomposed by passing through a prism. 2) The middle one is the "emission spectrum" which occurs when the light emitted by a heated gas is decomposed by passing through a prism and gives shining lines-a signature of the gas-over a dark background. 3) The third one is the "absorption spectrum" which occurs when white light traverses a cold gas and is then decomposed by passing through a prism. It appears as dark lines in a background continuous spectrum. The absorption lines occur at the same place as the emission lines.
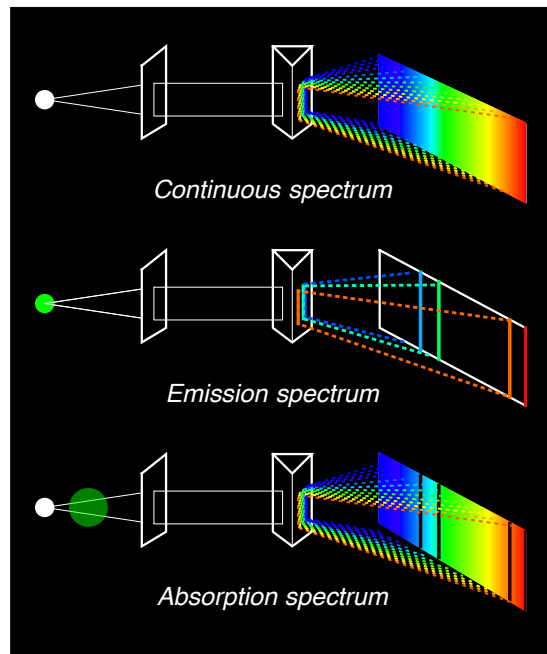
3.3 The adele class space and the explicit formulas

I had the chance to be invited at the Seattle meeting in 1996 for the celebration of the proof of the prime number theorem. The reason was the paper [14] (inspired from [79]) in which the Riemann zeta function appeared naturally as the partition function of a quantum mechanical system (BC system) exhibiting phase transitions. The RH had been at the center of discussions in the meeting and I knew the analogy

between the BC-system and the set-up that V. Guillemin proposed in [64] to explain the Selberg trace formula using the action of the geodesic flow on the horocycle foliation. To a foliation is associated a von Neumann algebra [22], and the horocycle foliation on the sphere bundle of a compact Riemann surface gives a factor of type II_∞ on which the geodesic flow acts by scaling the trace. An entirely similar situation comes canonically from the BC-system at critical temperature and after interpreting the dual system in terms of adeles, I was led by this analogy to consider the action of the idele class group of $\mathbb{Q}$ on the adele class space, *i.e.* the quotient $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}$ of the adeles $\mathbb{A}_\mathbb{Q}$ of $\mathbb{Q}$ by the action of $\mathbb{Q}^\times$. I knew from the BC-system that the action of $\mathbb{Q}^\times$, which preserves the additive Haar measure, is ergodic for this measure and gives the same factor of type II_∞ as the horocycle foliation. Moreover the dual action scales the trace in the same manner.

Let $\mathbb{K}$ be a global field and $C_\mathbb{K} = GL_1(\mathbb{A}_\mathbb{K})/\mathbb{K}^\times$ the idele class group. The module $\mathrm{Mod}: C_\mathbb{K} \to \mathbb{R}_+^*$ being proper with cocompact range, one sees that the Haar measure on the Pontrjagin dual group of $C_\mathbb{K}$ is diffuse. Since a point is of measure 0 in a diffuse measure space there is no way one can see the absorption spectrum without introducing some smoothness on this dual which is done using a Sobolev space $L_\delta^2(C_\mathbb{K})$ of functions on $C_\mathbb{K}$ which (for fixed $\delta > 1$) is defined as

$$||\xi||^2 = \int_{C_\mathbb{K}} |\xi(x)|^2 \rho(x)\, d^*x, \quad \rho(x) := (1 + \log|x|^2)^{\delta/2} \tag{19}$$

Definition 1. Let $\mathbb{K}$ be a global field, the adele class space of $\mathbb{K}$ is the quotient $X_\mathbb{K} = \mathbb{A}_\mathbb{K}/\mathbb{K}^\times$ of the adeles of $\mathbb{K}$ by the action of $\mathbb{K}^\times$ by multiplication.

We then consider the codimension 2 subspace $\mathscr{S}(\mathbb{A}_\mathbb{K})_0$ of the Bruhat-Schwartz space $\mathscr{S}(\mathbb{A}_\mathbb{K})$ (*cf.* [15]) given by the conditions $f(0) = 0$, $\int f\, dx = 0$ The Sobolev space $L_\delta^2(X_\mathbb{K})_0$ is the separated completion of $\mathscr{S}(\mathbb{A}_\mathbb{K})_0$ for the norm with square

$$||f||^2 = \int_{C_\mathbb{K}} |\sum_{q \in \mathbb{K}^*} f(qx)|^2 \rho(x)\, |x| d^*x \tag{20}$$

Note that by construction all functions of the form $f(x) = g(x) - g(qx)$ for some $q \in \mathbb{K}^\times$ belong to the radical of the norm (20), which corresponds to the operation of quotient of Definition 1. In particular the representation of ideles on $\mathscr{S}(\mathbb{A}_\mathbb{K})$ given by

$$(\vartheta(\alpha)\xi)(x) = \xi(\alpha^{-1}x) \ \ \forall \alpha \in GL_1(\mathbb{A}_\mathbb{K}),\ x \in \mathbb{A}_\mathbb{K} \tag{21}$$

induces a representation ϑ_a of $C_\mathbb{K}$ on $L_\delta^2(X_\mathbb{K})_0$. One has by construction a natural isometry $\mathfrak{E}: L_\delta^2(X_\mathbb{K})_0 \to L_\delta^2(C_\mathbb{K})$ which intertwines the representation ϑ_a with the regular representation of $C_\mathbb{K}$ in $L_\delta^2(C_\mathbb{K})$ multiplied by the square root of the module. This representation restricts to the cokernel of the map $\mathfrak{E}$, which splits as a direct sum of subspaces labeled by the characters of the compact group $C_{\mathbb{K},1} = \mathrm{Ker\,Mod}$ and its spectrum in each sector gives the zeros of L-functions with Grössencharakter. The shortcoming of this construction is in the artificial weight $\rho(x)$, which is needed to see this absorption spectrum but only sees the zeros which are on the critical line and where the value of δ artificially cuts the multiplicities of the zeros (*cf.* [24]).

This state of affairs is greatly improved if one gives up trying to prove RH but retreats to an interpretation of the explicit formulas as a trace formula. One simply replaces the above Hilbert space set-up by a softer one involving nuclear spaces [90]. The spectral side now involves all non-trivial zeros and, using the preliminary results of [24, 25, 16] one gets that the geometric side is given by:

$$\text{Tr}_{\text{distr}} \left(\int h(w)\vartheta(w)d^*w \right) = \sum_v \int_{\mathbb{K}_v^\times} \frac{h(w^{-1})}{|1-w|} \, d^*w \qquad (22)$$

We refer to [24, 90, 27] for a detailed treatment. The subgroups $\mathbb{K}_v^\times \subset C_{\mathbb{K}} = \text{GL}_1(\mathbb{A}_{\mathbb{K}})/\text{GL}_1(\mathbb{K})$ arise as isotropy groups. One can understand why the terms $\dfrac{h(w^{-1})}{|1-w|}$ occur in the trace formula by computing, formally as follows, the trace of the scaling operator $T = \vartheta_{w^{-1}}$ when working on the local field $\mathbb{K}_v$ completion of the global field $\mathbb{K}$ at the place v, one has

$$T\xi(x) = \xi(wx) = \int k(x,y)\xi(y)dy$$

so that T is given by the distribution kernel $k(x,y) = \delta(wx-y)$ and its trace is

$$\text{Tr}_{\text{distr}}(T) = \int k(x,x)\,dx = \int \delta(wx-x)\,dx = \frac{1}{|w-1|} \int \delta(z)\,dz = \frac{1}{|w-1|}$$

When working at the level of adeles one treats all places on the same footing and thus there is an overall minus sign in front of the spectral contribution. Thus the Riemann spectrum appears naturally as an absorption spectrum from the adele class space. As such, it is difficult to show that it is "real". While this solves the problem of giving a trace formula interpretation of the explicit formulas, there is of course still room for an interpretation as an emission spectrum. However from the adelic point of view it is unnatural to separate the contribution of the archimedean place.

4 The Riemann-Roch strategy: A Geometric Framework

In this section we shall present a geometric framework which has emerged over the years in our joint work with C. Consani and seems suitable in order to transpose the geometric proof of Weil to the case of RH. The aim is to apply the Riemann-Roch strategy of §2.3. The geometry involved will be of elaborate nature inasmuch as it relies on the following three theories:

1. Noncommutative Geometry.
2. Grothendieck topoi.
3. Tropical Geometry.

4.1 The limit $q \to 1$ and the Hasse-Weil formula

In [102] (*cf.* §6), C. Soulé, motivated by [89] (*cf.* §1.5) and [103, 106, 81, 46, 47, 80], introduced the zeta function of a variety X over $\mathbb{F}_1$ using the *polynomial* counting function $N(x) \in \mathbb{Z}[x]$ associated to X. The definition of the zeta function is as follows

$$\zeta_X(s) := \lim_{q \to 1} Z(X, q^{-s})(q-1)^{N(1)}, \qquad s \in \mathbb{R} \tag{23}$$

where $Z(X, q^{-s})$ denotes the evaluation at $T = q^{-s}$ of the Hasse-Weil exponential series

$$Z(X, T) := \exp\left(\sum_{r \geq 1} N(q^r) \frac{T^r}{r}\right) \tag{24}$$

For instance, for a projective space $\mathbb{P}^n$ one has $N(q) = 1 + q + \ldots + q^n$ and

$$\zeta_{\mathbb{P}^n(\mathbb{F}_1)}(s) = \lim_{q \to 1} (q-1)^{n+1} \zeta_{\mathbb{P}^n(\mathbb{F}_q)}(s) = \frac{1}{\prod_0^n (s-k)}$$

It is natural to wonder on the existence of a "curve" C suitably defined over $\mathbb{F}_1$, whose zeta function $\zeta_C(s)$ is the complete Riemann zeta function $\zeta_{\mathbb{Q}}(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$ (*cf.* also [89]). The first step is to find a counting function $N(q)$ defined for $q \in [1, \infty)$ and such that (23) gives $\zeta_{\mathbb{Q}}(s)$. But there is an obvious difficulty since as $N(1)$ represents the Euler characteristic one should expect that $N(1) = -\infty$ (since the dimension of H^1 is infinite). This precludes the use of (23) and also seems to contradict the expectation that $N(q) \geq 0$ for $q \in (1, \infty)$. As shown in [30, 31] there is a simple way to solve the first difficulty by passing to the logarithmic derivatives of both terms in equation (23) and observing that the Riemann sums of an integral appear from the right hand side. One then gets instead of (23) the equation:

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = -\int_1^\infty N(u) \, u^{-s} d^* u \tag{25}$$

Thus the integral equation (25) produces a precise equation for the counting function $N_C(q) = N(q)$ associated to C:

$$\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = -\int_1^\infty N(u) \, u^{-s} d^* u \tag{26}$$

One finds that this equation admits a solution which is a *distribution* and is given with $\varphi(u) := \sum_{n<u} n \Lambda(n)$, by the equality

$$N(u) = \frac{d}{du} \varphi(u) + \kappa(u) \tag{27}$$

where $\kappa(u)$ is the distribution which appears in the explicit formula (3),

$$\int_1^\infty \kappa(u)f(u)d^*u = \int_1^\infty \frac{u^2 f(u) - f(1)}{u^2 - 1}d^*u + cf(1), \qquad c = \frac{1}{2}(\log\pi + \gamma)$$

The conclusion is that the distribution $N(u)$ is positive on $(1,\infty)$ and is given by

$$N(u) = u - \frac{d}{du}\left(\sum_{\rho\in Z}\text{order}(\rho)\frac{u^{\rho+1}}{\rho+1}\right) + 1 \qquad (28)$$

where the derivative is taken in the sense of distributions, and the value at $u = 1$ of the term $\omega(u) = \sum_{\rho\in Z}\text{order}(\rho)\frac{u^{\rho+1}}{\rho+1}$ is given by $\frac{1}{2} + \frac{\gamma}{2} + \frac{\log 4\pi}{2} - \frac{\zeta'(-1)}{\zeta(-1)}$.

Fig. 2 This represents a function $J(u)$ which is a primitive of the counting distribution $N(u)$. This function is increasing and tends to $-\infty$ when $u \to 1$. The wiggly graph represents the approximation of $J(u)$ obtained using the symmetric set Z_m of the first $2m$ zeros, by

$$J_m(u) = \frac{u^2}{2} - \sum_{Z_m}\text{order}(\rho)\frac{u^{\rho+1}}{\rho+1} + u$$

Note that $J(u) \to -\infty$ when $u \to 1+$
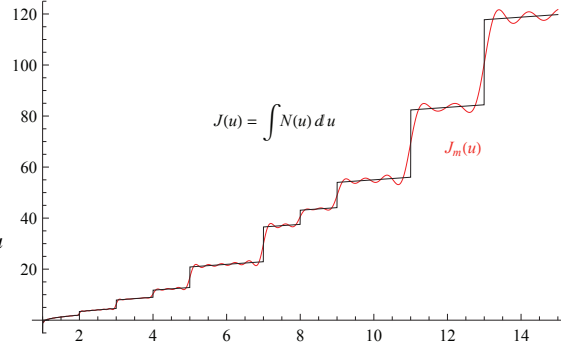
The primitive $J(u) = \frac{u^2}{2} - \omega(u) + u$ of $N(u)$ is an increasing function on $(1,\infty)$, but tends to $-\infty$ when $u \to 1+$ while its value $J(1)$ is finite. The tension between the positivity of the distribution $N(q)$ for $q > 1$ and the expectation that its value $N(1)$ should be $N(1) = -\infty$ is resolved by the theory of distributions: N is *finite* as a distribution, but when one looks at it as a function its value at $q = 1$ is formally given by

$$N(1) = 2 - \lim_{\varepsilon\to 0}\frac{\omega(1+\varepsilon) - \omega(1)}{\varepsilon} \sim -\frac{1}{2}E\log E, \qquad E = \frac{1}{\varepsilon}$$

which is $-\infty$ and in fact reflects, when $\varepsilon \to 0$, the density of the zeros. Note that this holds independently of the choice of the principal value in the explicit formulas. This subtlety does not occur for function fields $\mathbb{K}$ since their module $\text{Mod}(\mathbb{K})$ is discrete so that distributions and functions are the same thing. There is one more crucial nuance between the case $\mathbb{K} = \mathbb{Q}$ and the function fields: the distribution $\kappa(u)$ which is the archimedean contribution to $N(u)$ in (27), does not fulfill the natural inequality $N(q) \le N(q^r)$ expected of a counting function. This is due to the terms $|1 - u|^{-1}$ in the Weil explicit formula, which as explained in §2.1.3 contribute non-trivially at

the archimedean place, and indicate that the counting needs to take into account an ambient larger space and transversality factors as in [64]. In fact, we have seen in Section 3.3 that the noncommutative space of adele classes of a global field provides a framework to interpret the explicit formulas of Riemann-Weil in number theory as a trace formula, and that the geometric contributions give the right answer. In [31], we showed that the quotient

$$X_{\mathbb{Q}} := \mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}} / \hat{\mathbb{Z}}^{\times} \tag{29}$$

of the adele class space $\mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}}$ of the rational numbers by the maximal compact subgroup $\hat{\mathbb{Z}}^{\times}$ of the idele class group, gives by considering the induced action of $\mathbb{R}_{+}^{\times}$, the above counting distribution $N(u)$, $u \in [1, \infty)$, which determines, using the Hasse-Weil formula in the limit $q \to 1$, the complete Riemann zeta function. The next step is to understand that the action of $\mathbb{R}_{+}^{\times}$ on the space $X_{\mathbb{Q}}$ is in fact the action of the Frobenius automorphisms Fr_{λ} on the points of the arithmetic site– an object of algebraic geometry–over $\mathbb{R}_{+}^{\max}$. To explain this we first need to take an excursion in the exotic world of "characteristic one".

4.2 The world of characteristic 1

The key words here are: Newton polygons, Thermodynamics, Legendre transform, Game theory, Optimization, Dequantization, Tropical geometry. One alters the basic operation of addition of positive real numbers, replacing $x + y$ by $x \vee y := \max(x, y)$. When endowed with this operation as addition and with the usual multiplication, the positive real numbers become a semifield $\mathbb{R}_{+}^{\max}$. It is of characteristic 1, *i.e.* $1 \vee 1 = 1$ and contains the smallest semifield of characteristic 1, namely the Boolean semifield $\mathbb{B} = \{0, 1\}$. Moreover, $\mathbb{R}_{+}^{\max}$ admits non-trivial automorphisms and one has

$$\mathrm{Gal}_{\mathbb{B}}(\mathbb{R}_{+}^{\max}) := \mathrm{Aut}_{\mathbb{B}}(\mathbb{R}_{+}^{\max}) = \mathbb{R}_{+}^{*}, \ \ \mathrm{Fr}_{\lambda}(x) = x^{\lambda}, \ \forall x \in \mathbb{R}_{+}^{\max}, \ \lambda \in \mathbb{R}_{+}^{*}$$

thus providing a first glimpse of an answer to Weil's query in [112] of an algebraic framework in which the connected component of the idele class group would appear as a Galois group. More generally, for any abelian ordered group H we let $H_{\max} = H \cup \{-\infty\}$ be the semifield obtained from H by the max-plus construction, *i.e.* the addition is given by the max, and the multiplication by $+$. In particular $\mathbb{R}_{\max}$ is isomorphic to $\mathbb{R}_{+}^{\max}$ by the exponential map (*cf.* [57]). Historically, and besides the uses of $\mathbb{R}_{\max}$ in idempotent analysis and tropical geometry which are discussed below, an early use of $\mathbb{R}_{\max}$ occurred in the late fifties in the work of R. Cuninghame-Green in Birmingham, who established the spectral theory of irreducible matrices with entries in $\mathbb{R}_{\max}$ (*cf.* [41]) and in the sixties, in Leningrad, where Vorobyev used the $\mathbb{R}_{\max}$ formalism in his work motivated by combinatorial optimization, and proved a fundamental covering theorem. A systematic use of the $\mathbb{R}_{\max}$ algebra was developed by the INRIA group at the beginning of the 80's in their

work on the modelization of discrete event systems [20]. We refer to [57, 58] for a more detailed history of the subject, and for overwhelming evidence of its relevance in mathematics. We shall just give here a sample of this evidence starting by a really early occurrence in the work of C.G.J. Jacobi[8] and hoping to convince the reader that it would be a mistake to dismiss this algebraic formalism and the analogy with ordinary algebra as trivial.

4.2.1 Optimization, Jacobi

One of the early instances, around 1840, of the use of matrices over $\mathbb{R}_{max}$ is the work of C.G.J. Jacobi [77] on optimal assignment problems, where he states

<div align="center">Problema</div>

Disponantur nn quantitates $h_k^{(i)}$ quaecunque in schema Quadrati, ita ut habeantur n series horizontales et n series verticales, quarum quaeque est n terminorum. Ex illis quantitatibus eligantur n transversales, *i.e.* in seriebus horizontalibus simul atque verticalibus diversis positae, quod fieri potest n! modis; ex omnibus illis modis quaerendus est is, qui summam n numerorum electorum suppeditet maximam.

In other words, given a square matrix $m_{ik} = h_k^{(i)}$ he looks for the maximum over all permutations σ of the quantity $\sum m_{j\sigma(j)}$. Using the algebraic rules of $\mathbb{R}_{max}$ one checks that he is in fact computing the analogue of the determinant for the matrix m_{ik}. In fact the perfect definition of the determinant is more subtle and was obtained in the work of Gondran-Minoux [61], instead of $\max \sum m_{j\sigma(j)}$ where σ runs over all permutations, one uses the signature of permutations and considers the pair

$$(\det_+(m_{ik}), \det_-(m_{ik})), \quad \det_\pm(m_{ik}) = \max_{\text{sign}(\sigma)=\pm} \sum m_{j\sigma(j)}$$

The remarkable fact is that the Cayley-Hamilton theorem now holds, as the equality of two terms $P_+(m) = P_-(m)$ corresponding to the characteristic polynomial $P = (P_+, P_-)$. Each of the terms $P_\pm(m) \in M_n(\mathbb{R}_{max})$ is computed from the original matrix $m \in M_n(\mathbb{R}_{max})$ using the rules of matrices with entries in $\mathbb{R}_{max}$ which turn $M_n(\mathbb{R}_{max})$ into a semiring.

4.2.2 Idempotent analysis

The essence of the theory of semiclassical analysis in physics rests in the comparison of quantum systems with their semiclassical counterpart, [63, 65, 66, 54, 6]. In the eighties V. P. Maslov and his collaborators developed a satisfactory algebraic framework which encodes the semiclassical limit of quantum mechanics. They called it idempotent analysis. We refer to [82, 84] for a detailed account and just mention briefly some salient features here. The source of the variational formulations of mechanics in the classical limit is the behavior of sums of exponentials

---

[8] I am grateful to S. Gaubert for pointing out this early occurrence

$$\sum e^{-\frac{S_j}{\hbar}} \sim e^{-\frac{\inf S_j}{\hbar}}, \quad \text{when } \hbar \to 0$$

which are, when $\hbar \to 0$, dominated by the contribution of the minimum of S. The starting observation is that one can encode this fundamental principle by simply conjugating the addition of numbers by the power operation $x \mapsto x^{\varepsilon}$ and passing to the limit when $\varepsilon \to 0$. The new addition of positive real numbers is

$$\lim_{\varepsilon \to 0} \left(x^{\frac{1}{\varepsilon}} + y^{\frac{1}{\varepsilon}} \right)^{\varepsilon} = \max\{x, y\} = x \vee y$$

and one recovers $\mathbb{R}_+^{\max}$ as the natural home for semiclassical analysis. The superposition principle of quantum mechanics, *i.e.* addition of vectors in Hilbert space, now makes sense in the limit and moreover the "fixed point argument" proof of the Perron-Frobenius theorem works over $\mathbb{R}_+^{\max}$ and shows that irreducible compact operators have one and only one eigenvalue[9], thus reconciling classical determinism with the quantum variability. But the most striking discovery of this school of Maslov, Kolokolstov and Litvinov [82, 84] is that the Legendre transform which plays a fundamental role in all of physics and in particular in thermodynamics in the nineteenth century, is simply the Fourier transform in the framework of idempotent analysis!

The contact between the INRIA school and the Maslov school was established in 92 when Maslov was invited in the Seminar of Jacques Louis Lions in College de France. At the BRIMS HP-Labs workshop on Idempotency in Bristol (1994) organized by J. Gunawardena, several of the early groups of researchers in the field were there, and an animated discussion took place on how the field should be named. The names max-plus, exotic, tropical, idempotent were considered, each one having its defaults.

4.2.3 Tropical geometry, Riemann-Roch theorems and the chip firing game

The tropical semiring $\mathbb{N}_{\min} = \mathbb{N} \cup \{\infty\}$ with the operations min and $+$ was introduced by Imre Simon in [101] to solve a decidability problem in rational language theory. His work is at the origin of the term "tropical" used in tropical geometry which is a vast subject, see *e.g.* [59, 53, 91, 87]. We refer to [108] for an excellent introduction starting from the sixteenth Hilbert problem. In its simplest form (*cf.* [56]) a tropical curve is given by a metric graph Γ (*i.e.* a graph with a usual line metric on its edges). The natural structure sheaf on Γ is the sheaf $\mathscr{O}$ of real valued functions which are continuous, convex, piecewise affine with integral slopes. The operations on such functions are given by the pointwise operations of $\mathbb{R}_{\max}$-valued functions, *i.e.* $(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in \Gamma$ and similar for the product which is given by pointwise addition. One also adjoins the constant $-\infty$ which plays the role of the zero element in the semirings of sections. One proceeds as in the classical case with

---

[9] as mentioned above, this result was obtained already for matrices in 1962 by R. Cuninghame-Green

the construction of the sheaf $\mathscr{K}$ of semifields of quotients and finds the same type of functions as above but no longer convex. Cartier divisors make sense and one finds that the order of a section f of $\mathscr{K}$ at a point $x \in \Gamma$ is given by the sum of the (integer valued) outgoing slopes. The conceptual explanation of why the discontinuities of the derivative should be interpreted as zeros or poles is due to Viro, [109] who showed that it follows automatically if one understands that[10] the sum $x \vee x$ of two equal terms in $\mathbb{R}_{\max}$ should be viewed as ambiguous with all values in the interval $[-\infty, x]$ on equal footing. In their work Baker and Norine [4] proved in the discrete set-up of graphs (where g is the genus and K the canonical divisor) the Riemann-Roch equality in the form

$$r(D) - r(K - D) = \mathrm{Deg}(D) - g + 1 \tag{30}$$

where by definition $r(D) := \max\{k \mid H^0(D - \tau) \neq \{-\infty\}, \ \forall \tau \geq 0, \ \mathrm{Deg}(\tau) = k\}$ and $H^0(D)$ is the $\mathbb{R}_{\max}$-module of global sections f of the associated sheaf $\mathscr{O}_D$ *i.e.* sections of $\mathscr{K}$ such that $D + (f) \geq 0$. The essence of the proof of [4] is that the inequality $\mathrm{Deg}(D) \geq g$ for a divisor implies $H^0(D) \neq \{-\infty\}$. Once translated in the language of the chip firing game (*op.cit.*), this fact is equivalent to the existence of a winning strategy if one assumes that the total sum of dollars attributed to the vertices of the graph is $\geq g$ where g is the genus. We refer to [56, 92] for variants of the above Riemann-Roch theorem, and to [48, 10, 94] for early occurrences of these ideas in a different context (including sandpile models and parking functions!).

4.3 The arithmetic and scaling sites

4.3.1 The arithmetic site and Frobenius correspondences

The *arithmetic site* [37, 38] is an object of algebraic geometry involving two elaborate mathematical concepts: the notion of topos and of (structures of) characteristic 1 in algebra. A nice fact (*cf.* [60]) in characteristic 1 is that, provided the semiring R is multiplicatively cancellative (*i.e.* equivalently if it injects in its semifield of fractions) the map $x \mapsto x^n = \mathrm{Fr}_n(x)$ is, for any integer $n \in \mathbb{N}^{\times}$, an injective endomorphism Fr_n of R. One thus obtains a canonical action of the semigroup $\mathbb{N}^{\times}$ on any such R and it is thus natural to work in the topos $\widehat{\mathbb{N}^{\times}}$ of sets endowed with an action of $\mathbb{N}^{\times}$.

Definition 2. The arithmetic site $\mathscr{A} = (\widehat{\mathbb{N}^{\times}}, \mathbb{Z}_{\max})$ is the topos $\widehat{\mathbb{N}^{\times}}$ endowed with the *structure sheaf* $\mathscr{O} := \mathbb{Z}_{\max}$ viewed as a semiring in the topos using the action of $\mathbb{N}^{\times}$ by the Frobenius endomorphisms.

The topological space underlying the arithmetic site is the Grothendieck topos of sets endowed with an action of the multiplicative monoïd $\mathbb{N}^{\times}$ of non-zero positive integers. As we have seen above the semifield $\mathbb{R}_+^{\max}$ of tropical real numbers admits

---

[10] as seen when using $\mathbb{R}_{\max}$ as the target of a valuation

a one parameter group of Frobenius automorphisms Fr_λ, $\lambda \in \mathbb{R}_+^\times$, given by $\mathrm{Fr}_\lambda(x) = x^\lambda \ \forall x \in \mathbb{R}_+^{\max}$. Using a straightforward extension in the context of semi-ringed topos of the classical notion of algebraic geometry of a point over a ring, one then gets the following result which gives the bridge between the noncommutative geometry and topos points of view:

Theorem 1. *[37, 38] The set of points of the arithmetic site $\mathscr{A}$ over $\mathbb{R}_+^{\max}$ is canonically isomorphic with $X_\mathbb{Q} = \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q} / \hat{\mathbb{Z}}^\times$. The action of the Frobenius automorphisms Fr_λ of $\mathbb{R}_+^{\max}$ on these points corresponds to the action of the idele class group on $X_\mathbb{Q} = \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q} / \hat{\mathbb{Z}}^\times$.*

The square of the arithmetic site is the topos $\widehat{\mathbb{N}^{\times 2}}$ endowed with the structure sheaf defined globally by the multiplicatively cancellative semiring associated to the tensor square $\mathbb{Z}_{\min} \otimes_\mathbb{B} \mathbb{Z}_{\min}$ over the smallest Boolean semifield of characteristic one. In this way one obtains the semiring whose elements are Newton polygons and whose operations are given by the convex hull of the union and the sum. The points of the square of the arithmetic site over $\mathbb{R}_+^{\max}$ coincide with the product of the points of the arithmetic site over $\mathbb{R}_+^{\max}$. Then, we describe the Frobenius correspondences $\Psi(\lambda)$ as congruences on the square parametrized by positive real numbers $\lambda \in \mathbb{R}_+^\times$.

The remarkable fact at this point is that while the arithmetic site is constructed as a combinatorial object of countable nature it possesses nonetheless a one parameter semigroup of "correspondences" which can be viewed as congruences in the square of the site.

In the context of semirings, the congruences *i.e.* the equivalence relations compatible with addition and product, play the role of the ideals in ring theory. The Frobenius correspondences $\Psi(\lambda)$, for a rational value of λ, are deduced from the diagonal of the square, which is described by the product structure of the semiring, by composition with the Frobenius endomorphisms. We interpret these correspondences geometrically, in terms of the congruence relation on Newton polygons corresponding to their belonging to the same half planes with rational slope λ. These congruences continue to make sense also for irrational values of λ and are described using the best rational approximations of λ, while different values of the parameter give rise to distinct congruences. The composition of the Frobenius correspondences is given for $\lambda, \lambda' \in \mathbb{R}_+^\times$ such that $\lambda \lambda' \notin \mathbb{Q}$ by the rule [37, 38]

$$\Psi(\lambda) \circ \Psi(\lambda') = \Psi(\lambda \lambda') \tag{31}$$

The same equality still holds if λ and λ' are rational numbers. When λ, λ' are irrational and $\lambda \lambda' \in \mathbb{Q}$ one has

$$\Psi(\lambda) \circ \Psi(\lambda') = \mathrm{Id}_\varepsilon \circ \Psi(\lambda \lambda') \tag{32}$$

where Id_ε is the tangential deformation of the identity correspondence.

4.3.2 The scaling site and Riemann-Roch theorems

The Scaling Site $\hat{\mathscr{A}}$, [40], is the algebraic geometric space obtained from the arithmetic site $\mathscr{A}$ of [37, 38] by extension of scalars from the Boolean semifield $\mathbb{B}$ to the tropical semifield $\mathbb{R}_+^{\max}$. The points of $\hat{\mathscr{A}}$ are the same as the points $\mathscr{A}(\mathbb{R}_+^{\max})$ of the arithmetic site over $\mathbb{R}_+^{\max}$. But $\hat{\mathscr{A}}$ inherits from its structural sheaf a natural structure of tropical curve, in a generalized sense, allowing one to define the sheaf of rational functions and to investigate an adequate version of the Riemann-Roch theorem in characteristic 1. In [40], we tested this structure by restricting it to the periodic orbits of the scaling flow, *i.e.* the points over the image of $\operatorname{Spec}\mathbb{Z}$ under the canonical morphism of toposes $\Theta : \operatorname{Spec}\mathbb{Z} \to \mathscr{A}$ (*cf.* [38], §5.1). We found that for each prime p the corresponding circle of length $\log p$ is endowed with a quasi-tropical structure which turns this orbit into the analogue $C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$ of a classical elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}}$. In particular rational functions, divisors, etc all make sense. A new feature is that the degree of a divisor can now be any real number. The Jacobian of C_p (*i.e.* the quotient $J(C_p)$ of the group of divisors of degree 0 by principal divisors) is a cyclic group of order $p-1$. For each divisor D there is a corresponding Riemann-Roch problem with solution space $H^0(D)$ and the continuous dimension $\operatorname{Dim}_{\mathbb{R}}(H^0(D))$ of this $\mathbb{R}_{\max}$-module is defined as the limit

$$\operatorname{Dim}_{\mathbb{R}}(H^0(D)) := \lim_{n\to\infty} p^{-n}\dim_{\text{top}}(H^0(D)^{p^n}) \tag{33}$$

where $H^0(D)^{p^n}$ is a natural filtration and $\dim_{\text{top}}(\mathscr{E})$ is the topological dimension of an $\mathbb{R}_{\max}$-module $\mathscr{E}$. One has the following Riemann-Roch formula [40],

Theorem 2. (*i*) *Let $D \in \operatorname{Div}(C_p)$ be a divisor with $\deg(D) \geq 0$. Then the limit in* (33) *converges and one has $Dim_{\mathbb{R}}(H^0(D)) = \deg(D)$.*
(*ii*) *The following Riemann-Roch formula holds*

$$Dim_{\mathbb{R}}(H^0(D)) - Dim_{\mathbb{R}}(H^0(-D)) = \deg(D), \ \forall D \in \operatorname{Div}(C_p)$$

The appearance of arbitrary positive real numbers as continuous dimensions in the Riemann-Roch formula is due to the density in $\mathbb{R}$ of the subgroup $H_p \subset \mathbb{Q}$ of fractions with denominators a power of p. This outcome is the analogue in characteristic 1 of what happens for modules over matroid C^*-algebras and the type II normalized dimensions as in [49].

At this point, what is missing is an intersection theory and a Riemann-Roch theorem on the square of the arithmetic site. One expects that the right hand side of the Riemann-Roch formula will be of the form $\frac{1}{2}D.D = \mathfrak{s}(f,f)$ when the divisor D is of the form

$$D(f) = \int \Psi(\lambda)f(\lambda)d^*\lambda$$

Here $f(\lambda)$ is a real valued function with compact support of the variable $\lambda \in \mathbb{R}_+^*$ and $\mathfrak{s}(f,f)$ is as in (17). More precisely $D.D$ should be obtained as the intersection number of $D \circ \tilde{D}$ (defined using composition of correspondences) with the diagonal Δ and hence as a suitably defined distributional trace as for the counting function $N(u)$ of §4.1 so that $\frac{1}{2}D(f).D(f) = \mathfrak{s}(f,f)$ with the notations of (17). So far the Riemann-Roch formula in tropical geometry is limited to curves and there is no Serre duality or good cohomological version of H^j for $j \neq 0$, but in the above context one can hope that a Riemann-Roch inequality of the type (12), *i.e.* of the form

$$\mathrm{Dim}_{\mathbb{R}}(H^0(D)) + \mathrm{Dim}_{\mathbb{R}}(H^0(-D)) \geq \frac{1}{2}D.D$$

would suffice to apply the strategy of Section 2.3 to prove the key inequality (17).

Table 1 Here are a few entries in the analogy:

C curve over $\mathbb{F}_q$	Arithmetic Site $\mathscr{A} = (\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$ over $\mathbb{B}$
Structure sheaf $\mathscr{O}_C$	Structure sheaf $\mathbb{Z}_{\max}$
$\bar{C} = C \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$	Scaling Site $\hat{\mathscr{A}} = ([0,\infty) \rtimes \mathbb{N}^\times, \mathscr{O})$ over $\mathbb{R}_+^{\max}$
$C(\bar{\mathbb{F}}_q) = \bar{C}(\bar{\mathbb{F}}_q)$	$\mathscr{A}(\mathbb{R}_+^{\max}) = \hat{\mathscr{A}}(\mathbb{R}_+^{\max})$
Galois action on $C(\bar{\mathbb{F}}_q)$	Galois action on $\mathscr{A}(\mathbb{R}_+^{\max})$
Structure sheaf $\mathscr{O}_{\bar{C}}$ of $\bar{C} = C \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$	Structure sheaf $\mathscr{O} = \mathbb{Z}_{\max} \hat{\otimes}_{\mathbb{B}} \mathbb{R}_+^{\max}$ piecewise affine convex functions, integral slopes
Sheaf $\mathscr{K}$ of rational functions $\bar{C}$ on $\bar{C} = C \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$	Sheaf $\mathscr{K}$ of piecewise affine functions with integral slopes
Cartier divisors = sections of $\mathscr{K}/\mathscr{O}^*$	Sections of $\mathscr{K}/\mathscr{O}^*$
$X = \bar{C} \times \bar{C}$	$\hat{\mathscr{A}} \times \hat{\mathscr{A}}$
$D = \sum a_k \Psi^k$ Frobenius correspondence Ψ	$D = \int \Psi(\lambda) f(\lambda) d^*\lambda$ Correspondences $\Psi(\lambda)$

5 Absolute Algebra and the sphere spectrum

Even if the Riemann-Roch strategy of Section 4 happened to be successful, one should not view the arithmetic and scaling sites for more than what they are, namely a semiclassical shadow of a still mysterious structure dealing with compactifications of $\operatorname{Spec}\mathbb{Z}$. An essential role in the unveiling of this structure should be played, for the reasons briefly explained below, by the discovery made by algebraic topologists in the 80's (see [50]) that in their world of "spectra" (in their sense) the sphere spectrum is a generalized ring $\mathbb{S}$ which is more fundamental than the ring $\mathbb{Z}$ of integers, while the latter becomes an $\mathbb{S}$-algebra. Over the years the technical complications of dealing with spaces "up to homotopy" have greatly been simplified, in particular for the smash product of spectra. For the purpose of arithmetic applications, Segal's Γ-rings provide a very simple algebraic framework which succeeds to unify several attempts pursued in recent times in order to define the meaning of "absolute algebra". In particular it contains the following three possible categories that had been considered previously to handle this unification: namely the category $\mathscr{M}$ of monoïds as in [42, 43, 31, 33], the category $\mathscr{H}$ of hyperrings of [32, 34, 35] and finally the category $\mathscr{S}$ of semirings as in [26, 37, 38, 40]. Thanks to the work of L. Hesselholt and I. Madsen briefly explained below in §5.2 one now has at disposal a candidate cohomology theory in the arithmetic context: topological cyclic homology.

5.1 Segal's Γ-rings

Let Γ^{op} be the small, full subcategory of the category of finite pointed sets whose objects are the the pointed finite sets[11] $k_+ := \{0,\dots,k\}$, for $k \geq 0$. The object 0_+ is both initial and final so that Γ^{op} is a *pointed category*. The notion of a discrete Γ-space, *i.e.* of a Γ-set is as follows:

Definition 3. A Γ-set F is a functor $F : \Gamma^{\mathrm{op}} \longrightarrow \mathfrak{Sets}_*$ between pointed categories from Γ^{op} to the category of pointed sets.

The morphisms $\operatorname{Hom}_{\Gamma^{\mathrm{op}}}(M,N)$ between two Γ-sets are natural transformations of functors. The category $\Gamma\mathfrak{Sets}_*$ of Γ-sets is a symmetric closed monoidal category (*cf.* [50], Chapter II). The monoidal structure is given by the smash product (denoted $X \wedge Y$) of Γ-sets which is a Day product. The closed structure property is shown in [86] (*cf.* also [50] Theorem 2.1.2.4). The specialization of Definition 2.1.4.1. of [50] to the case of Γ-sets yields the following

Definition 4. A Γ-ring $\mathscr{A}$ is a Γ-set $\mathscr{A} : \Gamma^{\mathrm{op}} \longrightarrow \mathfrak{Sets}_*$ endowed with an associative multiplication $\mu : \mathscr{A} \wedge \mathscr{A} \to \mathscr{A}$ and a unit $1 : \mathbb{S} \to \mathscr{A}$, where $\mathbb{S} : \Gamma^{\mathrm{op}} \longrightarrow \mathfrak{Sets}_*$ is the inclusion functor.

---

[11] where 0 is the base point.

Thus Γ-rings[12] make sense and the sphere spectrum corresponds to the simplest possible Γ-ring: $\mathbb{S}$. One can then easily identify the category $\Gamma\mathfrak{Sets}_*$ of Γ-sets with the category $\mathfrak{Mod}(\mathbb{S})$ of $\mathbb{S}$-modules. In [51], N. Durov developed a geometry over $\mathbb{F}_1$ intended for Arakelov theory applications by using monads as generalizations of classical rings. While in the context of [51] the tensor product $\mathbb{Z}\otimes_{\mathbb{F}_1}\mathbb{Z}$ produces an uninteresting output isomorphic to $\mathbb{Z}$, we showed in [39] that the same tensor square, re-understood in the theory of $\mathbb{S}$-algebras, provides a highly non-trivial object. The Arakelov compactification of $\mathrm{Spec}\,\mathbb{Z}$ is endowed naturally with a structure sheaf of $\mathbb{S}$-algebras and each Arakelov divisor provides a natural sheaf of modules over the structure sheaf. This new structure of $\overline{\mathrm{Spec}\,\mathbb{Z}}$ over $\mathbb{S}$ endorses a one parameter group of weakly invertible sheaves whose tensor product rules are the same as the composition rules (31), (32) of the Frobenius correspondences over the arithmetic site [37, 38]. The category $\mathfrak{Mod}(\mathbb{S})$ of $\mathbb{S}$-modules is not an abelian category and thus the tools of homological algebra need to be replaced along the line of the Dold-Kan correspondence, which for an abelian category $\mathscr{A}$ gives the correspondence between chain complexes in ≥ 0 degrees and simplicial objects *i.e.* objects of $\mathscr{A}^{\Delta^{\mathrm{op}}}$.

At this point one has the following simple but very important observation that Γ-spaces should be viewed as simplicial objects in $\Gamma\mathfrak{Sets}_* \equiv \mathfrak{Mod}(\mathbb{S})$, so that homotopy theory should be considered as the homological algebra corresponding to the "absolute algebra" taking place over $\mathbb{S}$.

Table 2 Short dictionary homology–homotopy

$X \in Ch_{\geq 0}(\mathscr{A})$	$M \in \mathfrak{Mod}(\mathbb{S})^{\Delta^{\mathrm{op}}}$
$H_q(X)$	$\pi_q(M)$
$H_q(f) : H_q(X) \simeq H_q(Y)$ quasi-isomorphism	$\pi_q(f) : \pi_q(M) \simeq \pi_q(N)$ weak equivalence
$f_n : X_n \overset{\subseteq}{\to} Y_n$ + projective cokernel	cofibration (stable)
$f_n : X_n \to Y_n$ surjective if $n > 0$	stable fibration

---

[12] equivalently $\mathbb{S}$-algebras

We refer to Table 2 for a short dictionary. The category of Γ-spaces is the central tool of [50], while the relations between algebraic K-theory and topological cyclic homology is the main topic.

5.2 Topological cyclic homology

As shown in [39] the various attempts done in recent times to develop "absolute algebra" are all unified by means of the well established concept of $\mathbb{S}$-algebra, *i.e.* of Γ-rings. Moreover (*cf.* [50]) this latter notion is at the root of the theory of topological cyclic homology which can be understood as cyclic homology over the absolute base $\mathbb{S}$, provided one uses the appropriate Quillen model category. In particular, topological cyclic homology is now available to understand the new structure of $\overline{\mathrm{Spec}\,\mathbb{Z}}$ using its structure sheaf and modules. The use of cyclic homology in the arithmetic context is backed up by the following two results:

- At the archimedean places, and after the initial work of Deninger [46, 47] to recast the archimedean local factors of arithmetic varieties [100] as regularized determinants, we showed in [36] that cyclic homology in fact gives the correct infinite dimensional (co)homological theory for arithmetic varieties. The key operator Θ in this context is the generator of the λ-operations $\Lambda(k)$ [85, 110, 111] in cyclic theory. More precisely, the action u^{Θ} of the multiplicative group $\mathbb{R}_+^{\times}$ generated by Θ on cyclic homology, is uniquely determined by its restriction to the dense subgroup $\mathbb{Q}_+^{\times} \subset \mathbb{R}_+^{\times}$ where it is given by the formula

$$k^{\Theta}|_{HC_n} = \Lambda(k)\,k^{-n}, \ \forall n \geq 0, \ k \in \mathbb{N}^{\times} \subset \mathbb{R}_+^{\times} \tag{34}$$

Let X be a smooth, projective variety of dimension d over an algebraic number field $\mathbb{K}$ and let $v|\infty$ be an archimedean place of $\mathbb{K}$. Then, the action of the operator Θ on the archimedean cyclic homology HC^{ar} (*cf.* [36]) of X_v satisfies

$$\prod_{0 \leq w \leq 2d} L_v(H^w(X), s)^{(-1)^w} = \frac{det_{\infty}(\frac{1}{2\pi}(s-\Theta)|_{HC_{\mathrm{od}}^{\mathrm{ar}}(X_v)})}{det_{\infty}(\frac{1}{2\pi}(s-\Theta)|_{HC_{\mathrm{ev}}^{\mathrm{ar}}(X_v)})} \tag{35}$$

The left-hand side of (35) is the product of Serre's archimedean local factors of the complex L-function of X (*cf.*[100]). On the right-hand side, det_{∞} denotes the regularized determinant and one sets

$$HC_{\mathrm{ev}}^{\mathrm{ar}}(X_v) = \bigoplus_{n=2k \geq 0} HC_n^{\mathrm{ar}}(X_v), \ HC_{\mathrm{od}}^{\mathrm{ar}}(X_v) = \bigoplus_{n=2k+1 \geq 1} HC_n^{\mathrm{ar}}(X_v)$$

- L. Hesselholt and I. Madsen have shown (*cf. e.g.* [72, 70, 71]) that the de Rham-Witt complex, an essential ingredient of crystalline cohomology (*cf.* [8, 74]), arises naturally when one studies the topological cyclic homology of smooth algebras over a perfect field of finite characteristic. One of the remarkable features

in their work is that the arithmetic ingredients such as the Frobenius and restriction maps are naturally present in the framework of topological cyclic homology. Moreover L. Hesselholt has shown [73] how topological periodic cyclic homology with its inverse Frobenius operator may be used to give a cohomological interpretation of the Hasse-Weil zeta function of a scheme smooth and proper over a finite field in the form (*cf.* [73]):

$$\zeta(X,s) = \frac{det_\infty(\frac{1}{2\pi}(s-\Theta)|_{TP_{od}(X)})}{det_\infty(\frac{1}{2\pi}(s-\Theta)|_{TP_{ev}(X)})} \tag{36}$$

The similarity between (35) and (36) (applied to a place of good reduction) suggests the existence of a global formula for the L-functions of arithmetic varieties, involving cyclic homology of $\mathbb{S}$-algebras, and of a Lefschetz formula in which the local factors appear from the periodic orbits of the action of $\mathbb{R}_+^*$.

One of the stumbling blocks in order to reach a satisfactory cohomology theory is the problem of coefficients. Indeed, the natural coefficients at a prime p for crystalline cohomology are an extension of $\mathbb{Q}_p$ and it is traditional to relate them with complex numbers by an embedding of fields. Similarly, (36) uses an embedding of the Witt ring $W(\mathbb{F}_q) \to \mathbb{C}$. To an analyst it is clear that since such embeddings cannot be measurable[13] they will never be effectively constructed. This begs for a better construction, along the lines of Quillen's computation of the algebraic K-theory of finite fields, which instead would only involve the ingredient of the Brauer lifting, *i.e.* a group injection of the multiplicative group of $\bar{\mathbb{F}}_p$ as roots of unity in $\mathbb{C}$.

5.3 Final remarks

The Riemann hypothesis has been extended far beyond its original formulation to the question of localization of the zeros of L-functions. There are a number of constructions of L-functions coming from three different sources, Galois representations, automorphic forms and arithmetic varieties. André Weil liked to compare (*cf.* [12] §12 and also [117] vol. 1, p. 244–255 and vol. 2, p. 408–412), the puzzle of these three different writings to the task of deciphering hieroglyphics with the help of the Rosetta Stone. In some sense the L-functions play a role in modern mathematics similar to the role of polynomials in ancient mathematics, while the explicit formulas play the role of the expression of the symmetric functions of the roots in terms of the coefficients of the polynomial. If one follows this line of thought, the RH should be seen only as a first step since in the case of polynomials there is no way one should feel to have understood the zeros once one proves that they are, say, real numbers. In fact Galois formulated precisely the problem as that

---

[13] A measurable group homomorphism from $\mathbb{Z}_p^\times$ to $\mathbb{C}^\times$ cannot be injective

of finding all numerical relations between the roots of an equation, with the trivial ones being given by the symmetric functions, while the others, when determined, will reveal a complete understanding of the zeros as obtained, in the case of polynomials, by Galois theory. In a fragment, page 103, of the complete works of Galois [55] concerning the memoir of February 1830, he delivers the essence of his theory:

> Remarquons que tout ce qu'une équation numérique peut avoir de particulier, doit provenir de certaines relations entre les racines. Ces relations seront rationnelles c'est-à-dire qu'elles ne contiendront d'irrationnelles que les coefficients de l'équation et les quantités adjointes. De plus ces relations ne devront pas être invariables par toute substitution opérée sur les racines, sans quoi on n'aurait rien de plus que dans les équations littérales. Ce qu'il importe donc de connaître, c'est par quelles substitutions peuvent être invariables des relations entre les racines, ou ce qui revient au même, des fonctions des racines dont la valeur numérique est déterminable rationnellement.[14]

References

1. J. Arthur, *An introduction to the trace formula*. Harmonic analysis, the trace formula, and Shimura varieties, 1263, Clay Math. Proc., 4, Amer. Math. Soc., Providence, RI, 2005.
2. M. Artin, A. Grothendieck, J-L. Verdier, eds. (1972), SGA_4, LNM 269-270-305, Berlin, New York, Springer-Verlag.
3. M.F. Atiyah, R. Bott, *A Lefschetz fixed point formula for elliptic complexes: I*, Annals of Math. Vol. 86 (1967) 374–407.
4. M. Baker, S. Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph*, Advances in Mathematics 215 (2007), 766–788.
5. B. Bhatt, P. Scholze, *The pro-etale topology for schemes* Preprint (2013), arXiv:1309.1198
6. M. Berry, *Riemann's zeta function: a model of quantum chaos*, Lecture Notes in Physics, Vol.263, Springer-Verlag, 1986.
7. M. Berry and J. Keating, *H = qp and the Riemann zeros*, "Supersymmetry and Trace Formulae: Chaos and Disorder", edited by J.P. Keating, D.E. Khmelnitskii and I.V. Lerner, Plenum Press.
8. P. Berthelot, *Cohomologie cristalline des schémas de caractéristique p > 0*, Lecture Notes in Math., vol. 407, Springer-Verlag, New York, 1974.
9. A. Beurling, *A closure problem related to the zeta function*, Proc. Nat. Ac. Sci. **41** (1955), 312-314.
10. A. Bjorner, L. Lovasz, P. W. Shor, *Chip-firing games on graphs*, European J. Combin., 12(4), (1991), 283–291.
11. E. Bombieri, *Problems of the Millenium: The Riemann Hypothesis*, Clay mathematical Institute (2000).

---

[14] In 2012 I had to give, in the French academy of Sciences, the talk devoted to the 200-th anniversary of the birth of Evariste Galois. On that occasion I read for the $n + 1$-th time the book of his collected works and was struck by the pertinence of the above quote in the analogy with L-functions. In the case of function fields one is dealing with Weil numbers and one knows a lot on their Galois theory using results such as those of Honda and Tate *cf.* [105].

12. E. Bombieri, *The classical theory of Zeta and L-functions*, Milan J. Math. Vol. 78 (2010) 1159.
13. E. Bombieri, J. Lagarias *Complements to Li's criterion for the Riemann hypothesis*. J. Number Theory 77 (1999), no. 2, 274287.
14. J.B. Bost, A. Connes, *Hecke algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series) Vol.1 (1995) N.3, 411–457.
15. F. Bruhat, *Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques*. Bull. Soc. Math. France, 89 (1961), 43–75.
16. J.F. Burnol, *The explicit formula and the conductor operator*, math.NT/9902080.
17. J.F. Burnol, *Sur les formules explicites. I. Analyse invariante*. C. R. Acad. Sci. Paris Sér. I Math., 331 (2000), 423–428.
18. P. Cartier, *Des nombres premiers à la géométrie algébrique (une brève histoire de la fonction zêta)*. Analyse diophantienne et géométrie algébrique, 5177, Cahiers Sém. Hist. Math. Sér. 2, 3, Univ. Paris VI, Paris, 1993.
19. P. Cartier, A Voros, *Une nouvelle interprétation de la formule des traces de Selberg*. The Grothendieck Festschrift, Vol. II, 167, Progr. Math., 87, Birkhauser Boston, Boston, MA, 1990.
20. G. Cohen, S. Gaubert, R. Nikoukhah, J.P. Quadrat, *Convex analysis and spectral analysis of timed event graphs*, Decision and Control, 1989, Proceedings of the 28th IEEE Conference.
21. A. Connes, *Spectral sequence and homology of currents for operator algebras*, Mathematisches Forschunginstitut oberwolfach, Tagungsbericht 42/81.
22. A. Connes, *A survey of foliations and operator algebras*. In "Operator algebras and applications", Part I (Kingston, Ont., 1980), pp. 521–628, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I., 1982.
23. A. Connes, *Cohomologie cyclique et foncteurs* Ext^n. C. R. Acad. Sci. Paris Sér. I Math. 296 (1983), no. 23, 953–958.
24. A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*. Selecta Math. (N.S.) 5 (1999), no. 1, 29–106.
25. A. Connes, *Formules explicites, formules de trace et réalisation spectrale des zéros de la fonction zéta*, Course at Collège de France, 1999.
26. A. Connes, *The Witt construction in characteristic one and Quantization*. Noncommutative geometry and global analysis, 83–113, Contemp. Math., 546, Amer. Math. Soc., Providence, RI, 2011.
27. A. Connes, M. Marcolli, *Noncommutative Geometry, Quantum Fields, and Motives*, Colloquium Publications, Vol.55, American Mathematical Society, 2008.
28. A. Connes, C. Consani, M. Marcolli, *Noncommutative geometry and motives: the thermodynamics of endomotives*, Advances in Mathematics, Vol.214 (2007) N.2, 761–831.
29. A. Connes, C. Consani, M. Marcolli, *Weil's proof and the geometry of the adeles class space*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, 339–405, Progr. Math., 269, Birkhauser Boston, Inc., Boston, MA, 2009.
30. A. Connes, C. Consani, *Schemes over* $\mathbb{F}_1$ *and zeta functions*, Compositio Mathematica 146 (6), (2010) 1383–1415.
31. A. Connes, C. Consani, *From monoids to hyperstructures: in search of an absolute arithmetic*, in Casimir Force, Casimir Operators and the Riemann Hypothesis, de Gruyter (2010), 147–198.
32. A. Connes, C. Consani, *The hyperring of adèle classes*, Journal of Number Theory 131 (2011) 159–194.
33. A. Connes, C. Consani *On the arithmetic of the BC-system*, J. Noncommut. Geom. 8 (2014), no. 3, 873–945.
34. A. Connes, C. Consani, *Characteristic one, entropy and the absolute point*, " Noncommutative Geometry, Arithmetic, and Related Topics", the Twenty-First Meeting of the Japan-U.S. Mathematics Institute, Baltimore 2009, JHUP (2012), 75–139.
35. A. Connes, C. Consani, *The universal thickening of the field of real numbers*, Advances in the Theory of Numbers, Fields Institute Communications 77 (2015).
36. A. Connes, C. Consani, *Cyclic homology, Serre's local factors and the λ-operations*; J. K-Theory 14 (2014), no. 1, 1–45.

37. A. Connes, C. Consani, *The Arithmetic Site*, Comptes Rendus Mathématiques Ser. I 352 (2014), 971–975.
38. A. Connes, C. Consani, *Geometry of the Arithmetic Site*, ArXiv: 1502.05580.
39. A. Connes, C. Consani, *Absolute algebra and Segal's Γ-rings* ArXiv: 1502.05585
40. A. Connes, C. Consani, *The Scaling Site*, ArXiv: 1507.05818
41. R. Cuninghame-Green, *Minimax algebra*, Lecture Notes in Economics and Mathematical Systems, Volume 166, Springer, 1979.
42. A. Deitmar, *Schemes over* $\mathbb{F}_1$, in Number Fields and Function Fields Two Parallel Worlds. Ed. by G. van der Geer, B. Moonen, R. Schoof. Progr. in Math, vol. 239, 2005.
43. A. Deitmar, $\mathbb{F}_1$-*schemes and toric varieties*, Contributions to Algebra and Geometry Vol. 49, No. 2, pp. 517-525 (2008).
44. P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Symp. Pure Math. Vol.33 (1979) part II, 313–346.
45. P. Deligne, *La conjecture de Weil. I*. Publ. Math. Inst. Hautes Études Sci. No. 43 (1974), 273–307.
46. C. Deninger, *On the Γ-factors attached to motives*, Invent. Math. 104 (1991) 245–261.
47. C. Deninger, *Motivic L-functions and regularized determinants*, in "Motives", Proceedings of Symposia in Pure Mathematics, Vol. 55 (1994) Part I, 707–743.
48. D. Dhar, *Self-organized critical state of sandpile automaton models*. Phys. Rev. Lett., 64(14):1613 1616, Apr 1990.
49. J. Dixmier, *On some C*-algebras considered by Glimm*. J. Functional Analysis, 1, (1967), 182–203.
50. B. Dundas, T. Goodwillie, R. McCarthy, *The local structure of algebraic K-theory*. Algebra and Applications, 18. Springer-Verlag London, Ltd., London, 2013.
51. N. Durov, *New approach to Arakelov Geometry*. arXiv:0704.2030.
52. H.M. Edward, *Riemann's zeta function*, Dover, 2001.
53. M. Einsiedler, M. Kapranov, D. Lind, *Non-Archimedean amoebas and tropical varieties*. (English summary) J. Reine Angew. Math. 601 (2006), 139157.
54. M. V. Fedoriuk, V. P. Maslov, *Semiclassical approximation in quantum mechanics*. Translated from the Russian by J. Niederle and J. Tolar. Mathematical Physics and Applied Mathematics, 7. Contemporary Mathematics, 5. D. Reidel Publishing Co., Dordrecht-Boston, Mass., 1981.
55. E. Galois, Oeuvres de Galois, Gauthier-Villars, Paris (1962).
56. A. Gathmann and M. Kerber, *A Riemann-Roch theorem in tropical geometry*. Math. Z., 259(1):217–230, 2008.
57. S. Gaubert, *Methods and applications of (max, +) linear algebra*, STACS 97 (Lubek), Lecture Notes in Comput. Sci., vol. 1200, Springer, Berlin, (1997), 261–282.
58. S. Gaubert, *Two lectures on the max-plus algebra*. Proceedings of the 26th Spring School of Theoretical Computer Science, (1998), 83–147.
59. I. Gelfand, M. Kapranov, A Zelevinsky, *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory and Applications. Birkhauser Boston, Inc., Boston, MA, 1994.
60. J. Golan, *Semi-rings and their applications*, Updated and expanded version of The theory of semi-rings, with applications to mathematics and theoretical computer science [Longman Sci. Tech., Harlow, 1992. Kluwer Academic Publishers, Dordrecht, 1999.
61. M. Gondran, M. Minoux, *L'indépendance linéaire dans les dioïdes*. (French) Bull. Direction Etudes Rech. Sér. C Math. Inform. 1978, no. 1, 6790.
62. A. Grothendieck, *Sur une note de Mattuck-Tate* J. reine angew. Math. 200, 208-215 (1958).
63. V. Guillemin, S. Sternberg, *Geometric asymptotics*, Math. Surveys Vol. 14, American Mathematical Society, 1977.
64. V. Guillemin, *Lectures on spectral theory of elliptic operators*, Duke Math. J., Vol. 44, 3 (1977), 485-517.
65. M. Gutzwiller, *Classical Quantization of a Hamiltonian with Ergodic Behavior*, Physical Review Letters 45 (1980) 150-153.
66. M. Gutzwiller, *Chaos in classical and quantum mechanics*, Interdisciplinary Applied Mathematics, 1. Springer-Verlag, New York, 1990.

67. S. Haran, *Riesz potentials and explicit sums in arithmetic*, Invent. Math., 101 (1990), 697-703.
68. R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, New York Heidelberg Berlin 1977.
69. D. Hejhal, *The Selberg trace formula and the Riemann zeta function*. Duke Math. J. 43 (1976), no. 3, 441–482.
70. L. Hesselholt, *On the p-typical curves in Quillen's K-theory*. Acta Math. 177 (1996), no. 1, 1–53.
71. L. Hesselholt, *On the topological cyclic homology of the algebraic closure of a local field*, An Alpine Anthology of Homotopy Theory: Proceedings of the Second Arolla Conference on Algebraic Topology (Arolla, Switzerland, 2004), Contemp. Math., vol. 399, Amer. Math. Soc., Providence, RI, 2006, pp. 133–162.
72. L. Hesselholt, I. Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*. Topology 36 (1997), no. 1, 29–102.
73. L. Hesselholt, *Periodic topological cyclic homology and the Hasse-Weil zeta function*
74. L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Scient. Ec. Norm. Sup. (4) 12 (1979), 501–661.
75. A. Ingham, *The distribution of prime numbers* With a foreword by R. C. Vaughan. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1990. xx+114 pp.
76. K. Iwasawa, *On the rings of valuation vectors* Ann. of Math. (2) 57, (1953). 331356.
77. C.G.J. Jacobi, *De investigando ordine systatis aequationum differentialium vulgarium cujuscunque* C.G.J. Jacobi's gesammelte Werke, funfter Band, herausgegeben von K. Weierstrass, Berlin, Bruck und Verlag von Georg Reimer, 1890, p. 193-216.
78. N. Jacobson, *The radical and semi-simplicity for arbitrary rings*. Amer. J. Math. 67, (1945). 300–320.
79. B. Julia, Statistical theory of numbers, *Number Theory and Physics, Springer Proceedings in Physics,* **47** (1990).
80. M. Kapranov and A. Smirnov, *Cohomology determinants and reciprocity laws* Prepublication.
81. N. Kurokawa, *Multiple zeta functions: an example in Zeta functions in geometry* (Tokyo, 1990) Adv. Stud. Pure Math. Vol. 21 (1992), 219–226
82. V. Kolokoltsov, V. P. Maslov, *Idempotent analysis and its applications*. Mathematics and its Applications, 401. Kluwer Academic Publishers Group, Dordrecht, 1997.
83. X. J. Li, *The positivity of a sequence of numbers and the Riemann hypothesis*, J. Number Theory 65 (1997), 325–333.
84. G. Litvinov, *Tropical Mathematics, Idempotent Analysis, Classical Mechanics and Geometry*. Spectral theory and geometric analysis, 159–186, Contemp. Math., 535, Amer. Math. Soc., Providence, RI, 2011.
85. J.L. Loday, *Cyclic homology*. Grundlehren der Mathematischen Wissenschaften, 301. Springer-Verlag, Berlin, 1998.
86. M. Lydakis, *Smash products and Γ-spaces*, Math. Proc. Cambridge Philos. Soc. 126 (1999) 311–328.
87. D Maclagan, B. Sturmfels, *Introduction to tropical geometry*. Graduate Studies in Mathematics, 161. American Mathematical Society, Providence, RI, 2015.
88. S. Mac Lane, I Moerdijk, *Sheaves in geometry and logic. A first introduction to topos theory*. Corrected reprint of the 1992 edition. Universitext. Springer-Verlag, New York, 1994.
89. Yu.I. Manin, *Lectures on zeta functions and motives (according to Deninger and Kurokawa)*. Columbia University Number Theory Seminar (New York, 1992). Astérisque No. 228 (1995), 4, 121–163.
90. R. Meyer, *On a representation of the idele class group related to primes and zeros of L-functions*. Duke Math. J. Vol.127 (2005), N.3, 519–595.
91. G. Mikhalkin, *Enumerative tropical algebraic geometry in $\mathbb{R}^2$*. J. Amer. Math. Soc. 18 (2005), no. 2, 313377.
92. G. Mikhalkin and I. Zharkov, *Tropical curves, their Jacobians and theta functions*. In Curves and abelian varieties, volume 465 of Contemp. Math., p 203–230. Amer. Math. Soc., Providence, RI, 2008.

93. S. Patterson, *An introduction to the theory of the Riemann Zeta-function*, Cambridge Univ. Press, 1988.
94. A. Postnikov and B. Shapiro, *Trees, parking functions, syzygies, and deformations of monomial ideals*. Trans. Amer. Math. Soc., 356(8):31093142 (electronic), 2004.
95. B. Riemann, *Mathematical Werke*, Dover, New York (1953).
96. B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monat der Königl. Preuss. Akad. der Wissen. zu Berlin aus der Jahre 1859 (1860) 671–680. (English translation in M.H.Edwards "Riemann's zeta function", Dover 2001.)
97. A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, Journal of the Indian Mathematical Society 20 (1956) 47-87.
98. A. Selberg, *Collected Papers*, Springer, 1989.
99. A. Selberg, *The history of the prime number theorem*. Lecture given in Seattle, Monday August 12, 1996, 5-6pm. Prime Number Theorem, A SYMPOSIUM on the Riemann Hypothesis, Seattle, Washington, August 12   15, 1996. In "publications.ias.edu/sites/default/files/seattle.pdf"
100. J. P. Serre, *Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures)*. Sém. Delange-Pisot-Poitou, exp. 19, 1969/70.
101. I. Simon, *Limited subsets of the free monoid*. Proc. of the 19-th Annual Symposium on computer Science (1978), 143–150.
102. C. Soulé, *Les variétés sur le corps à un élément*. Mosc. Math. J. 4 (2004), no. 1, 217–244.
103. R. Steinberg, *A geometric approach to the representations of the full linear group over a Galois field*, Transactions of the AMS, Vol. 71, No. 2 (1951), pp. 274–282.
104. J. Tate, *Fourier analysis in number fields and Hecke's zeta-function*, Ph.D. Thesis, Princeton, 1950. Reprinted in J.W.S. Cassels and A. Frölich (Eds.) "Algebraic Number Theory", Academic Press, 1967.
105. J. Tate, *Classes d'isogénie des variététs abéliennes sur un corps fini, d'après T. Honda*, Séminaire Bourbaki, 21-ème année, 352, 1968-1969.
106. J. Tits, *Sur les analogues algébriques des groupes semi-simples complexes*. Colloque d'algèbre supérieure, Bruxelles 19–22 décembre 1956, Centre Belge de Recherches Mathématiques Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris (1957), 261–289.
107. B. Toën, M. Vaquié, *Au-dessous de* Spec $\mathbb{Z}$. (French) [Under SpecZ] J. K-Theory 3 (2009), no. 3, 437–500.
108. O. Viro, *From the sixteenth Hilbert problem to tropical geometry*. Jpn. J. Math. 3 (2008), no. 2, 185214.
109. O. Viro, *On basic concepts of tropical geometry*. (Russian) Tr. Mat. Inst. Steklova 273 (2011), Sovremennye Problemy Matematiki, 271–303; translation in Proc. Steklov Inst. Math. 273 (2011), no. 1, 252282
110. C. Weibel, *Cyclic Homology for schemes*. Proc. Amer. Math. Soc. 124 (1996), no. 6, 1655-1662.
111. C. Weibel, *The Hodge filtration and cyclic homology*. K-Theory 12 (1997), no. 2, 145-164.
112. A. Weil, *Sur la théorie du corps de classes* J. math. Soc. Japan, t. 3, 1951, p. 1-35.
113. A. Weil, *Sur les "formules explicites" de la théorie des nombres premiers*. (French) Comm. Sém. Math. Univ. Lund, (1952). Tome Supplémentaire, 252–265.
114. A. Weil, *Basic Number Theory*, Reprint of the second (1973) edition. Classics in Mathematics. Springer-Verlag, 1995.
115. A. Weil, *Sur les formules explicites de la théorie des nombres premiers*, Izv. Mat. Nauk., (Ser. Mat.) Vol.36 (1972) 3–18. (in Oeuvres compltes, Vol. 2, 48–62.)
116. A. Weil, *Fonction zeta et distributions*, Séminaire Bourbaki, Vol. 9 (1966), Exp. No. 312, 523–531.
117. A. Weil, *Oeuvres scientifiques/Collected papers*, I. II. III. Springer Collected Works in Mathematics. Springer, Heidelberg, 2014.
118. D. Zagier, Eisenstein series and the Riemannian zeta function, *Automorphic Forms, Representation Theory and Arithmetic,* Tata, Bombay (1979), 275-301.