RIEMANN'S FUNCTIONAL EQUATION

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Abstract. This paper investigates the functional equation of the Riemann zeta function $\zeta(s)$. The functional equation is useful for a few reasons. It allows us to immediately conclude that the function has zeros at the negative even integers. In the process of a proof we shall follow, the analytic continuation of the zeta function also falls out as a consequence. Furthermore, having established the functional equation, we can find formulas for the derivatives of the zeta function. The functional equation has close ties with the Riemann hypothesis, playing a role in empirically searching for zeros on the critical line.

1. Deriving the Functional Equation

The aim of the first paper we shall review is to present a short proof of Riemann's functional equation, [3]

$$\zeta(1-s) = \frac{\Gamma(s)}{(2\pi)^s} 2\cos\frac{\pi s}{2}\zeta(s). \tag{1.1}$$

In fact, the method the paper undertakes is to prove the slightly more general functional equation on the Hurwitz zeta function. Riemann's functional equation follows directly from this relation. The proof is based upon the Lipschitz summation formula, which itself is proved using Poisson summation, a technique from Fourier analysis. The meromorphic continuation of both the periodized zeta function (another component of the proofs) and the Hurwitz zeta function to the whole complex plane is a corollary of the results in this paper. Riemann's original proofs for the relation, on the other hand, use either the theta function and its Mellin transform or contour integration. Comparatively, the proof we shall follow mostly gets away with manipulating infinite series and reasoning about the analytic functions that pop out. In the words of the authors, the paper is intended to provide guidance to readers "unfamiliar with the circle of ideas" related to $\zeta(s)$.

Now let us set the stage for establishing the functional equation. Recall that the Riemann zeta function is defined as the continuation of $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, initially defined on $\Re(s) > 1$, where the series converges absolutely by comparison to $\sum n^{-p}$ for $p > 1$. The Hurwitz zeta function is an extension of the Riemann zeta function, similarly defined as

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^{s}}.$$

Here we take $0<a\leq 1$ and $\Re(s)>1$. We have $\zeta(s,1)=\zeta(s)$. Another relevant extension of the Riemann zeta function is what we call the periodized zeta function

$$F(s,a)=\sum_{n=1}^{\infty}\frac{e^{2\pi ina}}{n^{s}},$$

again defined initially for $\Re(s)>1$, $a\in\mathbb{R}$. We also define the Fourier transform $\hat{f}$ of f to be $\hat{f}(m)=\int_{-\infty}^{\infty}f(x)\,e^{2\pi ixm}dx$, so that we can use Poisson summation, which says

$$\sum_{n\in\mathbb{Z}}f(n)=\sum_{m\in\mathbb{Z}}\hat{f}(m)$$

for "sufficiently nice" f. The specifics of this are not terribly important to the result we want. Poisson summation is the main tool in the first theorem we state, the Lipschitz summation formula.

Theorem 1.

$$\sum_{n=1}^{\infty}(n-\alpha)^{s-1}\,e^{2\pi i\tau(n-\alpha)}=\frac{\Gamma(s)}{(-2\pi i)^{s}}\sum_{m\in\mathbb{Z}}\frac{e^{2\pi i\alpha m}}{(\tau+m)^{s}},\tag{1.2}$$

where $\Re(s)>1$, $\Im(\tau)>0$, *and* $0\leq\alpha<1$.

The proof is to use the Poisson summation formula on the left-hand side of (1.2), defining the summand function $f(x)=(x-\alpha)^{s-1}\,e^{2\pi i\tau(x-\alpha)}$ to be 0 for $x\leq\alpha$. This function is nice enough to use Poisson summation on, and we end up with

$$\sum_{m\in\mathbb{Z}}e^{2\pi im\alpha}\int_{0}^{\infty}x^{s-1}e^{2\pi i(\tau+m)x}dx=\frac{1}{(-2\pi i)^{s}}\sum_{m\in\mathbb{Z}}\frac{e^{2\pi im\alpha}}{(\tau+m)^{s}}\int_{R}y^{s-1}e^{-y}dy,$$

where R is the complex ray $\{y:y=-2\pi i(m+\tau)\,x\text{ and }x\geq 0\}$. Then we employ a familiar integration technique using Cauchy's integral theorem. Consider the integral along a piece of pie formed by going along the real axis for some radius, then following a circular arc counterclockwise about the origin, then coming back to the origin along R. Since $y^{s-1}e^{-y}$ is analytic, this integral is 0, and as the radius grows, the integral along the arc decays to 0. Therefore, the improper integral may be taken along the real axis, and so the theorem follows by the definition of $\Gamma(s)$.

The next steps involve some results that are mostly bookkeeping to justify the steps we will take in proving the relation on the Hurwitz zeta function. The results are not terribly illuminating, but we state them for use in that proof. They are mainly about being able to interchange limiting operations.

Lemma. (a) *Suppose* $0 \le a < 1$ *and* $\Re(s) < 0$. *Then*

$$\lim_{\tau \to 0} \sum_{n=1}^{\infty} (n-a)^{s-1} e^{2\pi i n \tau} = \sum_{n=1}^{\infty} (n-a)^{s-1} = \zeta(1-s, 1-a).$$

(b) *Let* $0 \le a < 1$ *and* $y > 0$. *Write*

$$S_y(s) = \sum_{m \neq 0} e^{2\pi i a m} \left((m+iy)^{-s} - m^{-s} + siym^{-s-1} \right),$$

where the sum is taken over all nonzero integers and for the complex powers we use the principle branch of the argument, with range $(-\pi, \pi]$. *Then:*

(i) $S_y(s)$ *converges absolutely for* $\Re(s) > -1$;
(ii) $S_y(s)$ *is holomorphic in* s *for* $\Re(s) > -1$;
(iii) *if* $\Re(s) > -1$, *then* $\lim_{y \to 0^+} S_y(s) = 0$.

With these in hand, we can prove the Hurwitz relation, from which Riemann's functional equation will follow.

Theorem 2.

$$e^{-\pi i s/2} F(s, a) + e^{\pi i s/2} F(s, -a)$$

can be continued analytically into $\Re(s) > -1$. *When* $-1 < \Re(s) < 0$ *and* $0 < a \le 1$, *we have the Hurwitz relation*

$$\zeta(1-s, a) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-\pi i s/2} F(s, a) + e^{\pi i s/2} F(s, -a) \right). \tag{1.3}$$

Proof. In (1.2), there is a $(\tau + m)^{-s}$ factor. The idea is to subtract the first two terms of the binomial expansion of this factor, $m^{-s} + \tau s m^{-s-1}$. Moving the $\Gamma(s) / (-2\pi i)^s$ factor in (1.2) to the other side and then subtracting from $(\tau + m)^{-s}$ for nonzero indices of the sum, we get

$$\begin{aligned} &\frac{1}{\tau^s} + \sum_{m \neq 0} e^{2\pi i \alpha m} \left((\tau + m)^{-s} - m^{-s} + \tau s m^{-s-1} \right) \\ &\qquad = \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{n=1}^{\infty} (n-\alpha)^{s-1} e^{2\pi i \tau (n-\alpha)} - \sum_{m \neq 0} \frac{e^{2\pi i \alpha m}}{m^s} + \tau s \sum_{m \neq 0} \frac{e^{2\pi i \alpha m}}{m^{s+1}}. \end{aligned}$$

for $0 \le \alpha < 1$ and $\Re(s) > 1$. We note that

$$\begin{aligned} \sum_{m \neq 0} \frac{e^{2\pi i \alpha m}}{m^s} &= \sum_{m=1}^{\infty} \frac{e^{2\pi i \alpha m}}{m^s} + \sum_{m=1}^{\infty} \frac{e^{2\pi i \alpha (-m)}}{(-m)^s} \\ &= F(s, \alpha) + (-1)^{-s} \sum_{m=1}^{\infty} \frac{e^{-2\pi i \alpha m}}{m^s} \\ &= F(s, \alpha) + e^{-\pi i s} F(s, -\alpha) \end{aligned}$$

and take $\tau = iy$, $y > 0$ to rewrite what we have as

$$\begin{aligned}(1.4) \quad \frac{1}{(iy)^s} + \sum_{m \neq 0} e^{2\pi i \alpha m} \left((m+iy)^{-s} - m^{-s} + siym^{-s-1} \right) \\ = \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{n=1}^{\infty} (n-\alpha)^{s-1} e^{-2\pi y(n-\alpha)} \\ - \left(F(s,\alpha) + e^{-\pi i s} F(s,-\alpha) \right) + iys \left(F(s+1,\alpha) - e^{-\pi i s} F(s+1,-\alpha) \right).\end{aligned}$$

The sum $\sum_{n=1}^{\infty} (n-\alpha)^{s-1} e^{-2\pi y(n-\alpha)}$ is entire in s the exponential term causes the series to converge normally on $\mathbb{C}$. If we move the $(-2\pi i)^s / \Gamma(s)$ factor back to the left-hand side, the Lemma tells us that the left-hand side is meromorphic in $\Re(s) > -1$ with at most a simple pole at $s = 0$ from $\Gamma(s)$. This tells us that the remaining terms

$$-\frac{\Gamma(s)}{(-2\pi i)^s} \left(F(s,\alpha) + e^{-\pi i s} F(s,-\alpha) \right) + \frac{iys\Gamma(s)}{(-2\pi i)^s} \left(F(s+1,\alpha) - e^{-\pi i s} F(s+1,-\alpha) \right)$$

are meromorphic in $\Re(s) > -1$ with at most a simple pole at $s = 0$. Since $y > 0$ is arbitrary, we find that

$$-\frac{\Gamma(s)}{(-2\pi i)^s} \left(F(s,\alpha) + e^{-\pi i s} F(s,-\alpha) \right)$$

is meromorphic in $\Re(s) > -1$ with at most a simple pole at $s = 0$ by letting $y \to 0^+$, since the other term vanishes uniformly on compact sets. This establishes the first part of the theorem.

Now for $-1 < \Re(s) < 0$, let $y \to 0^+$ in (1.4). The left-hand side vanishes by the Lemma and the restriction of $\Re(s)$ and the sum on the right-hand side tends to $\zeta(1-s, 1-\alpha)$. Hence, for $0 \leq \alpha < 1$ we have

$$\zeta(1-s, 1-a) - \frac{\Gamma(s)}{(-2\pi i)^s} \left(F(s,\alpha) + e^{-\pi i s} F(s,-\alpha) \right) = 0.$$

in $-1 < \Re(s) < 0$. Letting $a = 1 - \alpha$ gives (1.3) since $F(s, 1-a) = F(s,-a)$ and $F(s,-1+a) = F(s,a)$. □

Since $F(s,1) = F(s,-1) = \zeta(s)$, 1.3 yields

$$\zeta(1-s) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-\pi i s/2} + e^{\pi i s/2} \right) \zeta(s) = \frac{\Gamma(s)}{(2\pi)^s} 2\cos\frac{\pi s}{2} \zeta(s),$$

which is the functional equation (1.1) that we want.

2. Using the Functional Equation

[3] goes on to show that we can analytically continue $\zeta(s,a)$ and $F(s,a)$ using the work done so far. The first supplementary result is that $F(0,-a) + F(0,a) = -1$, which we obtain by taking $s \to 0^-$ in 1.4 and using the Lemma. We also note that

$\lim_{s\to 0^-} F(s,a)$ is finite for this to work, because a bit of manipulation shows that $\lim_{s\to 0^-}\left(e^{\pi is}-e^{-\pi is}\right)F(s,a)=0$. Since $e^{\pi is}-e^{-\pi is}$ has a simple zero at $s=0$, the order of $F(s,a)$ at $s=0$ cannot be negative (because the order of the product is the sum of the orders); i.e. it is finite at $s=0$. We use this to prove the following.

Corollary. (a) *For $0<a\leq 1$, the Hurwitz zeta function $\zeta(s,a)$ is holomorphic in $\mathbb{C}$ except for a simple pole of residue 1 at $s=1$.*

(b) *The relation 1.3 holds in all of $\mathbb{C}$ and has the alternative form*

$$F(s,a)=\frac{(2\pi)^s e^{-\pi is/2}}{2i\Gamma(s)\sin(\pi s)}\left(\zeta(1-s,a)-e^{\pi is}\zeta(1-s,1-a)\right).$$

(c) *For $a\notin\mathbb{Z}$, the periodized zeta function $F(s,a)$ is entire in s.*

We get (a) by looking at 1.3 and seeing that the left-hand side is holomorphic for $\Re(s)<0$ by the initial definition of $\zeta(s,a)$ and using Theorem 2 to see that the right-hand side is holomorphic for $\Re(s)>-1$, except for a possible simple pole at $s=0$ due to $\Gamma(s)$. Since these functions agree on the strip $-1<\Re(s)<0$, we can use each side to extend the other side analytically to the whole plane. By the identity theorem, these functions are the same. Using the previous result that $F(0,-a)+F(0,a)=-1$ and the fact that $\Gamma(s)$ has a simple pole of residue -1 at $s=0$, the pole of $\zeta(s,a)$ follows. Now that the relation holds for all of $\mathbb{C}$, (b) follows from a simple manipulation of the relation by looking at $\zeta(1-s,a)$ and $\zeta(1-s,1-a)$ and solving for $F(s,a)$. This alternative form shows that $F(s,a)$ is holomorphic in $\mathbb{C}$ except possibly at $\{0,1\}$, where it might have simple poles. Why? We already know by the original definition that it is holomorphic for $\Re(s)>1$, so the only problems at integers $s\leq 1$ due to the sine term in the denominator of the relation and the poles of the $\zeta(1-s,a)$ and $\zeta(1-s,1-a)$ terms. At nonnegative integers, the denominator is not an issue because the simple zero of $\sin(\pi s)$ and the simple pole of $\Gamma(s)$ cancel each other out. However, the Hurwitz zeta function terms have poles at $s=0$. For $s=1$, the sine term creates a pole. Hence, $\{0,1\}$ are the possible singularities. For $a\notin\mathbb{Z}$, we can directly confirm that these poles do not exist by seeing that $F(0,a)$ and $F(1,a)$ are finite, which establishes (c).

We now look to [1] for further applications of the functional equation. This paper gives formulas for $\zeta^{(k)}(s)$ (in terms of integrals) and closed form evaluations of $\zeta^{(k)}(0)$, the latter of which we shall overview. The first result is the following.[1]

Theorem 3. *For each integer $k\geq 1$ and $s\in\mathbb{C}$ we have*

$$(-1)^k\zeta^{(k)}(1-s)=\sum_{m=0}^{k}\binom{k}{m}\left(e^{sz}z^{k-m}+e^{s\overline{z}}(\overline{z})^{k-m}\right)(\Gamma(s)\zeta(s))^{(m)} \tag{2.1}$$

[1] This paper uses $z=-\log 2\pi - i\pi/2$ more or less for its entirety, so anywhere we use z without prior notice, this is what it refers to. It is a bit unusual, so there are some reminders in the theorem statements.

where $z = -\log 2\pi - i\pi/2$.

The proof is to rewrite (1.1) as

$$\zeta(1-s) = \Gamma(s)\,\zeta(s)\left(e^{sz} + e^{s\overline{z}}\right), \tag{2.2}$$

and then differentiate both sides k times. The summation comes from applying Leibniz's rule (the extension of the product rule) to the right-hand side. With some manipulation, the right-hand side of (2.1) can be rewritten in the following way.

Theorem 4. *For each integer* $k \geq 1$ *and* $s \in \mathbb{C}$ *we have*

$$\begin{aligned}
&(-1)^k \zeta^{(k)}(1-s) \\
&\quad = 2(2\pi)^{-s} \sum_{m=0}^{k} \binom{k}{m} \left(\Re\left(z^{k-m}\right)\cos\frac{\pi s}{2} + \Im\left(z^{k-m}\right)\sin\frac{\pi s}{2}\right)\left(\Gamma(s)\,\zeta(s)\right)^{(m)} \\
&= 2(2\pi)^{-s} \sum_{m=0}^{k}\sum_{r=0}^{m} \binom{k}{m}\binom{m}{r} \left(\Re\left(z^{k-m}\right)\cos\frac{\pi s}{2} + \Im\left(z^{k-m}\right)\sin\frac{\pi s}{2}\right)\Gamma^{(r)}(s)\,\zeta^{(m-r)}(s).
\end{aligned}$$

where again $z = -\log 2\pi - i\pi/2$.

The trigonometric terms make this relation rather interesting for integer-valued inputs. If $s = 2n+1$ for $n = 1, 2, 3, \ldots$, the cosine term vanishes and the sine term becomes $(-1)^n$, so we get

$$\begin{aligned}
&(-1)^k \zeta^{(k)}(-2n) \\
&\qquad = \frac{2(-1)^n}{(2\pi)^{2n+1}} \sum_{m=0}^{k}\sum_{r=0}^{m} \binom{k}{m}\binom{m}{r} \Im\left(z^{k-m}\right)\Gamma^{(r)}(2n+1)\,\zeta^{(m-r)}(2n+1).
\end{aligned}$$

This shows that $\zeta^{(k)}(-2n)$ is a linear combination of

$$\zeta(2n+1),\, \zeta'(2n+1), \ldots, \zeta^{(k)}(2n+1).$$

Similarly, letting $s = 2n$ shows that $\zeta^{(k)}(1-2n)$ is a linear combination of

$$\zeta(2n),\, \zeta'(2n), \ldots, \zeta^{(k)}(2n).$$

The most interesting result in my opinion is the closed form for $\zeta^{(k)}(0)$, because the proof is rather clever.

The idea is to equate coefficients of power series. Since $\zeta(1-s)$ is analytic at $s = 1$, it has an expansion

$$\zeta(1-s) = \sum_{n=0}^{\infty} \frac{(-1)^n \zeta^{(n)}(0)}{n!}(s-1)^n.$$

Now we expand the right-hand side of (2.2) in its power series. The product $\Gamma(s)\zeta(s)$ has one simple pole of residue 1 at $s=1$ due to ζ, so it has a Laurent series expansion

$$\Gamma(s)\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} a_n (s-1)^n . \tag{2.3}$$

Then we write (again, with $z = -\log 2\pi - i\pi/2$) $e^{sz} = e^z e^{(s-1)z} = \sum_0^\infty e_n(z)(s-1)^n$, where $e_n(z) = e^z z^n/n!$, so that the product $\Gamma(s)\zeta(s)e^{sz}$ has the expansion

$$\begin{aligned}\left(\frac{1}{s-1} + \sum_{n=0}^{\infty} a_n (s-1)^n\right)&\left(\sum_{n=0}^{\infty} e_n(z)(s-1)^n\right)\\ &= \sum_{n=0}^{\infty} e_n(z)(s-1)^{n-1} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k e_{n-k}(z)\right)(s-1)^n\\ &= \frac{e^z}{s-1} + \sum_{n=0}^{\infty}\left(e_{n+1}(z) + \sum_{k=0}^{n} a_k e_{n-k}(z)\right)(s-1)^n .\end{aligned}$$

Now we can equate coefficients through (2.2) to get that

$$(-1)^n \frac{\zeta^{(n)}(0)}{n!} = e_{n+1}(z) + e_{n+1}(\overline{z}) + \sum_{k=0}^{n} a_k \left(e_{n-k}(z) + e_{n-k}(\overline{z})\right),$$

since $e^z + e^{\overline{z}} = 0$, so the leading $(s-1)^{-1}$ term drops out (as it very well should, if our identity was to hold). In particular, $e^z = -i/(2\pi)$ and $e^{\overline{z}} = i/(2\pi)$, so

$$e_n(z) + e_n(\overline{z}) = \frac{-iz^n + i\overline{z}^n}{2\pi n!} = \frac{1}{\pi}\frac{\Im(z^n)}{n!}.$$

Hence we find that

$$(-1)^n \frac{\zeta^{(n)}(0)}{n!} = \frac{1}{\pi}\frac{\Im\left(z^{n+1}\right)}{(n+1)!} + \frac{1}{\pi}\sum_{k=0}^{n} a_k \frac{\Im\left(z^{n-k}\right)}{(n-k)!}.$$

We have that $\Im\left(z^0\right) = 0$ and it turns out that $a_0 = 0$, which lets us delete the first and last terms of the sum to get the following theorem.

Theorem 5. *If $z = -\log 2\pi - i\pi/2$ and $n \geq 0$, we have*

$$(-1)^n \frac{\zeta^{(n)}(0)}{n!} = \frac{1}{\pi}\frac{\Im\left(z^{n+1}\right)}{(n+1)!} + \frac{1}{\pi}\sum_{k=1}^{n-1} a_k \frac{\Im\left(z^{n-k}\right)}{(n-k)!}, \tag{2.4}$$

where the a_n are determined by 2.3.

The paper goes on to develop some machinery to compute the a_n. (2.4) lets us compute the derivatives of ζ at 0; for example, $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{1}{2\pi}\Im\left(z^2\right) = -\frac{1}{2}\log 2\pi$ (easy since the sum does not have any terms). Another interesting result that is readily observed through decimal approximations of the values is that $\zeta^{(n)}(0)/n! \to -1$ (which aligns with the fact that the radius of convergence for the power series at 0 is 1). Here is an except of the table in the paper that shows this.

n	$\zeta^{(n)}(0)$	$\zeta^{(n)}(0)/n!$
0	-0.50000000000000	-0.50000000000000
1	-0.918938533204672	-0.918938533204672
2	-2.0063564559085	-1.003178227954292
3	-6.0047111668622	-1.000785194477042
4	-23.99710318801370	-0.9998792995005709
5	-120.0002329075584	-1.000001940896320
6	-720.0009368251297	-1.000001301146014
7	-5039.999150176233	-0.9999998313841731
8	-40320.00023243172	-1.000000005764676
9	-362880.0003305895	-1.000000000911016

3. Application to the Riemann Hypothesis

Of course, no paper discussing the Riemann zeta function is complete without at least a cursory mention of the Riemann hypothesis. Recall that the Euler product formula

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

for $\Re(s) > 1$ implies that $\zeta(s) \neq 0$ for $\Re(s) > 1$. We seen that the zeta function has no zeros along the line $\Re(s) = 1$ using the techniques of Zagier [6]. Then by applying the functional equation 1.1 and the fact that $\Gamma(s)$ has no zeros, we see that the zeta function has no zeros in the left half-plane except at the zeros of $\cos\frac{\pi s}{2}$, which are of course at the negative even integers. These are referred to as the trivial zeros of the zeta function; the remaining nontrivial zeros are therefore within the strip $0 < \Re(s) < 1$. This is the basis for the Riemann hypothesis, which is the assertion that s is a nontrivial zero of the zeta function only if $\Re(s) = 1/2$ (often referred to as the critical line). Seeing as $\Re(s) = 1/2$ if and only if $\Re(1-s) = 1/2$, it is not hard to believe that the functional equation is closely tied to the Riemann hypothesis. Indeed, it has been known for a while that the functional equation allows for a fairly simple restatement of the Riemann hypothesis. A paper by Spira [5] gives the following result concerning the function multiplied by $\zeta(s)$ in the functional equation. Recall that we commonly denote $s = \sigma + it$.

Theorem 6. *For $t \geq 10$, $1/2 < \sigma < 1$, $|g(s)| > 1$ where $g(s) = \frac{\Gamma(s)}{(2\pi)^s} 2\cos\frac{\pi s}{2}$.*

The value $t = 10$ is only chosen for convenience in the estimates; a later paper [2] optimizes the hypotheses to only require that $|t| \geq 6.8$ and $\sigma > 1/2$. The proofs of these results use Stirling's formula for the gamma function and mainly involve calculating partial derivatives with respect to σ. In any case, this result implies that $|\zeta(1-s)| > |\zeta(s)|$ for $t \geq 10$ and $1/2 < \sigma < 1$, except possibly at points where $\zeta(s) = 0$, where the functional equation implies $\zeta(1-s) = \zeta(s) = 0$. Hence the Riemann hypothesis would that the inequality holds in the entire range $t \geq 10$, $1/2 < \sigma < 1$. The inequality holding for the whole range would also imply the Riemann hypothesis. This is because the zeros of the zeta function come in conjugate pairs, so this inequality would confirm that there are no zeros in the whole strip except possibly for $-10 \leq t \leq 10$, which can be checked directly. In fact, [4] notes that the functional equation provides a basis for counting the zeros of the zeta function in a bounded strip and confirming that they lie on the line $\Re(s) = 1/2$. We have already seen that the argument principle allows us to count the zeros of an analytic function. We then introduce another form of the functional equation:

$$\xi(s) = \xi(1-s), \tag{3.1}$$

where

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

This equation can be obtained from our original with some functional equations for $\Gamma(s)$, specifically the Euler reflection formula $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$ and the duplication formula $\Gamma(2s) = \pi^{-1/2}2^{2s-1}\Gamma(s)\Gamma(s+1/2)$. We also note that $\xi(1/2+it) \in \mathbb{R}$ for $t \in \mathbb{R}$. To see this, first note that $\overline{\xi(1/2+it)} = \xi(1/2-it)$; the $\frac{1}{2}s(s-1)$ term is unchanged by direct computation and we can push conjugation down to s for the remaining terms. Then the functional equation tells us that

$$\overline{\xi\left(\frac{1}{2}+it\right)} = \xi\left(\frac{1}{2}-it\right) = \xi\left(1-\left(\frac{1}{2}+it\right)\right) = \xi\left(\frac{1}{2}+it\right)$$

for $t \in \mathbb{R}$. We conclude that sign changes of ξ on the critical line correspond precisely to zeros of the zeta function, given that the rest of the terms are nonzero. This is the promised method of counting zeros. If we count n sign changes of $\xi(1/2+it)$ in a certain range of $0 < t < T$ and also use the argument principle to confirm that there are exactly n zeros in $0 < t < T$, $0 < \sigma < 1$, we verify that all the zeros of the zeta function lie on the critical line up to that bound. This works as long as we have a rigorous bound on an error term for computing $\zeta(s)$ and also as long as the zeros are simple, so that we can actually pair up sign changes with zeros. This has been the case for all the zeros examined so far, but the question of whether all the zeros of the zeta function are simple is also open.

The Riemann zeta function and its cousins arguably characterize the appeal and utility of complex analysis. We have seen that the functional equation is a defining trait of the zeta function and paves the way to things such as its analytic continuation to the entire complex plane (except at 1), the values of its derivatives, and the location of its zeros. While there are many aspects of the zeta function that we did not explore in this paper, it is hopefully evident that the functional equation still lends us great insight into the behavior of the function and why it is so fascinating.

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