TILING GROUPS

KASPER LINDBERG

1. Introduction

Tiling is the fundamental problem of tessellating a region, the attempt to fit some alphabet of tiles within that region such that there is no overlapping of the tiles and no empty space is left in the region. These mathematical tilings provide nice analogies to real life natural patterns that optimize space efficiency, like crystal structures or honeycombs. Although this problem is highly geometric, we can translate certain tiles and regions into an algebraic sense such we decompose the boundaries into certain combinations of elements of groups. These are known as Conway's Tiling Groups and by so called boundary invariants of these groups, we can determine whether or not we can solve the tiling problem.

In this paper, we examine the notion of translating in between viewing tiling as a geometric and algebraic problem and focus on some key examples where we can derive necessary and sufficient conditions for the tiling problems. We follow alongside Thurston [1] and use examples that will focus on tilings of so-called lozenges, essentially rhombuses inside the lattice regions of $\mathbb{Z}^2$.

2. Definitions

We will start with a quick exposition of the necessary algebraic and geometric definitions to understand a more rigorous approach to tiling, although high level intuition of the concepts is enough to follow along. We begin by outlining algebraic terms and then transitioning into the more abstract geometric terms that base themselves off the algebra.

2.1. Algebra.

Definition 2.1. A group $(G, \cdot)$ is a set G equipped with an operator $\cdot\colon G \times G \to G$, denoting $x \cdot y$ as just xy and xx as x^2 and so forth with higher powers, which satisfy the following group axioms

1. *Closure*: If x and y are in G then xy is in G
2. *Associativity*: For x, y, z in G , $x(yz) = (xy)z$
3. *Identity*: There exists an identity element 1 in G such that for each x in G, $x = 1x = x1$
4. *Inverse*: For each x in G, there is an inverse x^{-1} such that $xx^{-1} = x^{-1}x = 1$

We will mostly denote $(G, \cdot)$ as just G

Definition 2.2. A homomorphism from $(G, \cdot)$ to $(H, \star)$ is a map $\phi : G \to H$ such that $\phi(x \cdot y) = \phi(x) \star \phi(y)$.
An isomorphism is a bijective homomorphism.
An automorphism is an isomorphism from a group to itself.

Definition 2.3. The kernel of a homomorphism is the set of all elements mapped to the identity.

Definition 2.4. A free group $F(S)$ is a group generated by a set S where each group element is a word, a finite sequence of the elements of S, the inverses of the elements and S. We can make each word have a unique representation by ensuring there is no x and x^{-1} next to each other and that multiple adjacent x are written in their power form x^n. The group operation is concatenation of two sequences, the identity being the empty sequence and inverses defined as reversing the order of the sequence and replacing each sequence element with its inverse [**?**].

Definition 2.5. A presentation of a group $\langle S, R\rangle$ is a way of constructing a group where S is a set of generators and R a set of relators are elements of the free group of S, words of the generators. Each element of the group is an element of the free group of S, a word, but where each instance of a relator in the word maps to the identity.

So for example, if you have some presentation $\langle a, b|b^2, a^2\rangle$, the word $b^3a^2b = b^2ba^2a = ebea = ba$. We will only focusing on presentations of groups more than groups themselves and so a rigorous treatment of a presentation is not necessary. So we will also assume every group we look at will have a presentation without proof. We focus on presentations which are finite, meaning S and R are finite.

2.2. **Geometry.**

Definition 2.6. A graph of a group $\Gamma(G)$ is a directed graph whose vertices are the elements of a group G with presentation $\langle S, R\rangle$. Say S has n elements $g_1, \ldots, g_n$, then for each vertex $v \in \Gamma(G)$, there are n outgoing edges, labelled by each generator in S so for a generator g_i, v connects to vg_i. We make the slight modification where each generator that satisfies $g_i^2 = e$, we draw one undirected edge instead of two opposite directed edges between the elements.

So starting at any vertex in $\Gamma(G)$ and following along the edges of any relator in R, we return back to the starting vertex.

Definition 2.7. A directed graph is homogeneous if for every edge label g the transform taking $v \to gv$ is an automorphism of the graph.

Its easy to see that every graph of a group is homogenous as it just shifts every vertex to its incident vertex connected by g and there is an obvious inverse that takes $v \to g^{-1}v$. It is also important to note that every automorphism of the graph of a group has this property, to see this, let v_0 be some vertex mapped to gv_0, then for an arbitrary v_1, $v_1 = v_0g'$ as each vertex can be mapped to the identity by its inverse and then back to the desired vertex. Thus $v_1 = v_0g' \to gv_0g' = gv_1$ hence every automorphism is homogenous [1]. This also means that every circular permutation of a word traces out the same area but starts at a different vertex.

Remark 2.8. Thus a suitable directed graph with every vertex having n distinctly labeled inbound edges and n outbound edges is the graph of a group iff it admits an automorphism that takes every vertex to another. This gives us an easy way to show a certain geometrically interpreted graph is a the graph of a group.

Definition 2.9. A simplex is a set of vertices, the dimension n being the cardinality of the set, and we call it an n-face. A simplicial complex $\mathcal{K}$ is a set of simplices such that

(1) *Clousre*: If a simplex $\omega \in K$ and $\tau \subset \omega$ nonempty then $\tau \in \mathcal{K}$.
(2) *Intersection*: If $\omega_0, \omega_1 \in \mathcal{K}$ and $\omega_0 \cap \omega_1 = \tau$ nonempty, then $\tau \in \omega_0, \omega_1$.

An m-complex is a simplicial complex with simplices of dimension at most m.

So a 2-complex is a collection of triangles, lines and points and a 1-complex is a collection of lines and points, which be thought of as a graph. For each graph $\Gamma(G)$, we can extend this to a notion of a 2-complex of the graph, where we fill in areas of closed regions.

Definition 2.10. A 2-complex of a graph $\Gamma(G)$ is denoted as $\Gamma^2(G)$ and is defined as superset of $\Gamma(G)$ where for each vertex $v \in \Gamma(G)$, every relator of the group R_i traces the boundary of one or more connected polygons (the edges being straight lines) starting at v which is then filled in via a triangulation by something called the two ears theorem [2]. We don't do anything in the case where $R_i = x^2$ as that is just a single undirected edge.

Now we can focus on paths defined in $\Gamma^2(G)$, and a first important conclusion to make is that $\Gamma^2(G)$ is simply-connected, as we can think of our construction as sewing on arcs of a disk with m arcs, where $R_i = g_1 g_2 \dots g_m$, onto to boundary of the region described by R_i. As all the vertices of the graph are connected is all collections of these disks are obviously simply connected, thus $\Gamma^2(G)$ is as well. Knowing this, we can derive our first important result that translates between group theory and geometry.

Theorem 2.11. *Every word that equals the identity can be simplified completely using relators R_i.*

Proof. Let $a_1 a_2 \dots a_s = 1$ be the word we wish to simplify, then there is a loop γ that takes the path along the edges of the graph $\Gamma(G)$, which can be enlarged from a single point. For any generator g if going along g from the identity still allows you to return to identity, then this generator must show up in some relator R_i because if it didn't then there would no means of deriving g^{-1} to return us to the identity. Thus every edge in the word is part of some relator. Let S be a covering of γ from the regions created by the relators such that all edges of the path are the boundary of some cover. Since the loop is compact, there exists a finite subcovering by the Heine-Borel theorem and so pick the minimal one S'. Then we start off at the identity vertex and move along a_1 via some relator R_j which could possibly be shifted and add the relator's word as $g_{j_1} \dots g_{j_n}$ into our construction c and keep going inductively, using the same relator until a_i differs from relator. Once a_i differs from g_{j_i}, find the suitable R_k that has a_i in it and inject the R_k's word into c at g_{j_i} so that $c = g_{j_1} \dots g_{j_{i-1}} g_{k_1} \dots g_{k_m} g_{j_i} \dots g_{j_n}$. Keep doing this until you obtain the word $a_1 a_2 \dots a_s b_1 \dots b_t = 1$ where the b word is extra relator junk at the end. We know that the a word goes to the identity, so the b word does as well □

3. Simple Tiling Groups

We can now start looking at actual tiling groups starting with tiling Lozenges in a triangular lattice

3.1. **Triangular Tiling Group.** First, we define the region on which we want to tile on. Taking the lattice of $\mathbb{Z}^2$ and shifting every odd row over by 0.5 units to the left and squishing the entire plane vertically by a factor of $\sqrt{.75}$ we connect all adjacent

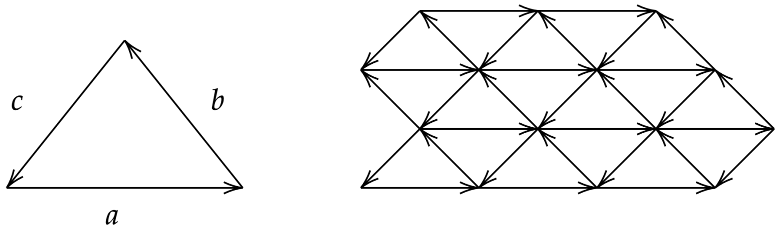

FIGURE 1. Left: Triangle group A's generators showing action of the first relator. Right: A section of the graph of A

vertices by edges to triangulate the plane into equilateral triangles. We label each edge type $S = \{a, b, c\}$ in such a manner: a points to the right along the x-axis, b points up and to the left at a 120° angle, c points down and to the left at a 240° angle. We see that this is in fact homogenous as shifting one vertex, shifts all of them by the same sequence of edges, so by remark 2.7, we have that this is a graph of a group A.

Definition 3.1. $A$is a group with presentation $\langle S|abc, cba\rangle$ where moving along abc produces an upright equilateral triangle and cba produces an upside down equilateral triangle.

These are the fundamental shapes of the graph and everything can be tiled by them. Since $abc = cba = 1$ we know that $ab = c^{-1} = ba$ hence they commute. This group can clearly be seen as isomorphic to $\mathbb{Z}^2$ under addition as we essentially can undo the transformed $\mathbb{Z}^2$ lattice to see that b goes up the y-axis, a goes up the x-axis, and c is down and to the left. An explicit bijection is $a \mapsto (1, 0), b(0, 1), c \mapsto (-1, -1)$

To define a path π, we use 1-complex who's edges trace over the generators S. We can convert this geometric idea into a group setting by looking at the free group of the generators $F(S)$, where we the sequence of edges traced by π is exactly the word in $F(S)$. We denote this word in the free group as $\alpha(\pi)$ to differentiate from the geometric path of π. We can show that π makes a closed shape iff there is a homomorphism $\phi : F(S) \to A$ where $\phi(\alpha(\pi)) = 1$. This is because as we showed in theorem 2.10, that a path π being closed means that the respective word of the region $\alpha(\pi)$ can be written simplified using its relators, so taking $\phi(xy) = \phi(x)\phi(y)$ recursively until you hit the last defined relator by the theorem, in which $\phi(R_i) = 1$ and thus backtracking everything gets mapped to the identity. The other way is simply noting that the sequence of edges returns to itself, hence π must be closed.

3.2. **Lozenges.** A lozenge is a gluing of two adjacent equilateral triangles and removing the connecting edge. There are three lozenges depending on which side of the three sides the upside down triangle is glued to the upright triangle.

These three lozenges are $L_1 = aba^{-1}b^{-1}$, $L_2 = bcb^{-1}c^{-1}$ and $L_3 = cac^{-1}a^{-1}$ where circular permutations of these descriptions are all equivalent.

Definition 3.2. We call the set of these three lozenges a tile set, denoted by Σ and the group L with presentation $\langle S|L_1, L_2, L_3\rangle$ is the lozenge group.

As each $L_1, L_2, L_3 = 1$, we can move the inverses over to the other side to see that $ab = ba, bc = cb, ac = ca$ so all the generators commute. Thus it must be isomorphic

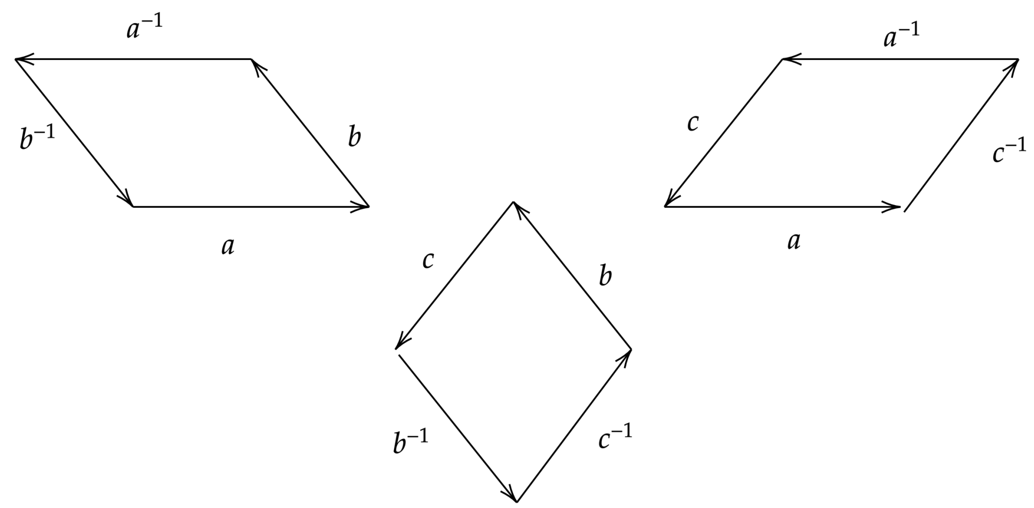

FIGURE 2. From left to right: L_1, L_2, L_3

to $\mathbb{Z}^3$ under addition as it can be seen to have the presentation $\langle a = (1,0,0), b = (0,1,0), c = (0,0,1) | aba^{-1}b^{-1}, bcb^{-1}c^{-1}, cac^{-1}a^{-1}\rangle$.

Now that have lozenges and our triangular region we want to tile in, we can find necessary conditions for there to be a valid tiling of these lozenges.

Theorem 3.3. *If R is a region tiled by some group of tiles Σ and tile group M, then the image of the boundary of the region $I(\pi)$ of the map of $\phi : F(S) \to M$ applied to $\alpha(\pi)$), is the identity.*

Proof. Suppose R can be tiled by the tile set Σ, then if R is composed of just one tile, the boundary must be a circular permutation of one of the relators and hence maps to the identity. Suppose we $I(\pi)$ maps to the identity for a tiling up to size k, then for a tiling of size $k+1$ we know that there is some tile with no edge touching another tile by finiteness. So pick the vertex of the edge such that if you start a path clockwise around the tile, you will first hit the other endpoint of that edge. Now define the boundary π_0 as this clockwise path around the tile. Then define the boundary π_1 around the rest of the region R minus the tile, starting at the same vertex but going counterclockwise. Then as this region is tiled by k tiles, we know that $I(\pi_0) = I(\pi_1) = 1$, so as $\alpha(pi_0)\alpha(pi_1) = \alpha(pi)$ where π is the boundary of R, we have by the properties of homomorphisms that $I(\pi) = 1$ □

This is however not a sufficient condition for a tiling which we will see why later on.

3.3. **Translating to $\mathbb{Z}^3$.** As the lozenge group is isomorphic to $\mathbb{Z}^3$,, there is a very nice treatment of transforming the lozenges into this space. We consider the tesselation of cubes such that a corner of each cube is pointing upwards and an edge going up to this vertex is aligned to the x axis. We can look at the graph $\Gamma(L)$ as the graph of the vertices of these cubes and edges connecting their vertices, directing each labelled edge so it moves upwards along the z axis and labelling them in the same sense as before. So moving along *abc* instead of returning to the original vertex, its ends at the original point shifted up vertically by 3. Thus projecting the image of any graph of the cubes from above onto the xy plane produces a graph of the triangles while

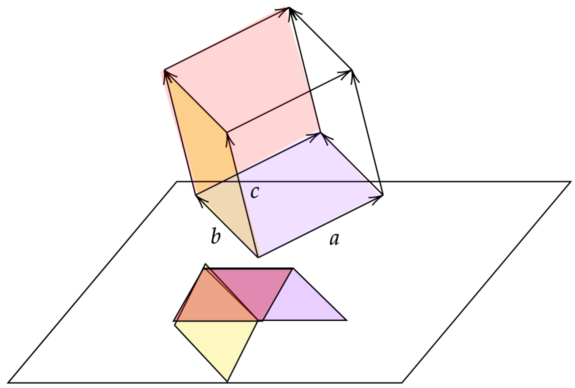

FIGURE 3. From left to right: L_1, L_2, L_3

the projection of each face of the cube produces a lozenge, one variety for each pair of similarly oriented faces.

So for a general boundary π of a region, we can lift the edges into the graph of the $\Gamma(L)$, by starting at a vertex plane and moving it up vertically some amount to match a cube's vertex, then matching the edge of the boundary with the edge of the graph one by one until you've exhausted all edges of π, however we can't conclude that the region closes up again. This is because although you will definitely return to the same spot projected down onto the xy plane, the starting z coordinate might not match up with the ending one. This is the case when $I(\pi) \in L \neq 1$, the image of the boundary in the lozenge group, thought of in $\Gamma(L)$. This is equivalent to the change in height from the starting to the end vertex and is isomorphic to $\mathbb{Z}$ under addition. We can interpret this as the map from $L \to A$ as the projection from the graph onto the plane and kernel of the map being the vertices above and below the starting one, as they are all mapped to that point. As L is isomorphic to $\mathbb{Z}^3$ and A isomorphic to $\mathbb{Z}^2$, the homomorphism acting as a projection $\mathbb{Z}^3 \to \mathbb{Z}^2$ has a kernel of $\mathbb{Z}$.

As each a, b, c go up 1 on their labeled edges and their inverses go down one, the height difference can be seen by summing all the powers of a, b, c in a word.

Remark 3.4. So every tileable region must have the same multiplicity of generators and their inverses

. The converse doesn't hold as we can consider the simple example of $abcb^{-1}a^{-1}c^{-1}$ which traces out two upright triangles and thus can't be tiled by a lozenge. To find sufficient conditions for tiling, we look at defining the height and distance between vertices more explicitly.

4. Tiling conditions for lozenges

4.1. **The distance function.** We consider a 2-complex region R made up of triangles of $\Gamma(A)$ and for any two vertices v and w we can define a distance function between them $d(v, w)$

Definition 4.1. Let v, w be vertices in R, then $d(v, w)$ is the shortest-path distance from v to w along only positively oriented edges, with each edge having 1 unit of distance.

Restricting to positively directed edges means that our path ,which can be described as a word in $F(S)$, has no inverses in it, so the distance obviously isn't symmetric but is still defined for $d(w, v)$ if its defined for $d(v, w)$. We can show this by circling around each triangle the path is connected to and getting back to the identity.

Theorem 4.2. *If R connected, then $d(v, w)$ is well defined for all vertices v, w in R. $d(v, w)$ is also not necessarily symmetric.*

Proof. When we consider R as connected, $d(v, w)$ is defined for all v, w in R as you can always get a path by first, from v, going along one generator, say a until you hit a boundary vertex x. Then for w you do the same but with an inverse generator like a^{-1} until you hit a boundary y. So from x we follow the boundary until you get to an inverse generator, then follow the other two appropriate generators on the triangle in which the boundary is defined on and continue until you get to y. This is possible as the boundary of R is obviously connected. Hence we have found a path from $v \to x \to y \to w$ so $d(v, w)$ is well-defined.

An obvious example for why this isn't symmetric is $v = 1$ $w = ab$, then $d(v, w) = 2$, the via the path ab, while $d(w, v) = 1$ via the path c. □

For any closed path on the graph, we know the length must be some multiple of 3 as word that returns to the identity can be written using substitutions of relators and each relator is of size 3 by theorem 2.11. So the distance from v to w along any path is actually equal to $d(v, w)$ mod 3 as we can take the path made by $d(v, w) = d_1$ call it P_1 and then the path made by $d(w, v) = d_2$ call it P_2. Then any other path from v to w, P' has distance d_3 so $d_3 + d_2 = 3x$ must be a multiple of three so $d_1 + d_2 = 3y$ means $d_3 \equiv d_1 \mod 3$.

4.2. **The height function.** Now look at the same procedure we did previously for lifting the boundary of the region R into $\Gamma(L)$. Instead of looking at an arbitrary region, we look at one that is tiled by lozenges so each lozenge can be lifted to a face of $\Gamma^2(L)$, the 2-complex of faces of the cubes.

Definition 4.3. For each vertex v in R, let $h(v)$ be the height of the vertex in the graph $\Gamma(L)$ such that each incident vertex w has $h(w) = h(v) \pm 1$ depending on if its a generator or an inverse of a generator.

This function is unique up to an additive constant, depending on how far we wish to lift the vertices up from the plane. It's also well defined, as the region is tiled by lozenges so there is no edge in $\Gamma(L)$ that goes to some higher or lower height than what was defined. If there was then the sum of the exponents wouldn't equal zero when starting the word at that vertex and would contradict the tiling property stated in remark 3.4. We also lift the middle edge, that would cut the lozenge into two triangles, up into $\Gamma(L)$, the new edge would lift to a diagonal that cuts through a face of the cube and would contribute a height difference of -2 as each positively oriented edge increases by 1 and we know that the word of two generators is equal to the inverse of the third, $ab = ba = c^{-1}, bc = cb = a^{-1}, ac = ca = b^{-1}$. Hence the height difference (which deletes the additive constant) between any two vertices is either the same as the distance, if the minimal path doesn't go through any of these diagonals, or the difference is greater than the distance if the path does go through a diagonal. Hence $h(w) - h(v) \geq dv, w)$ is a necessary condition for tiling.

Though its difficult to find exact sufficient conditions for tiling a region, however, we do have an algorithm we can utilize to figure out whether or not a region can be

tiled by lozenges.
Let R be a triangular region with boundary π, then we can assign a height $h(v)$ for every vertex $v \in \pi$. Obviously if this isn't well defined and multiple heights get assigned to a vertex, then there is no tiling. We move onto constructing the height function for all vertices inside R. First we note that $h(v)$ has a range of $[n, m]$ and so we can start working on vertices with height n. We also color each triangle either white or black, such that each adjacent triangle has an alternating color. Using strong induction, suppose we have worked through all vertices of height less than or equal to some k. Then for vertex v with height equal to k, we consider each edge e going from v to w such that looking right along that directed edge, we see a black triangle. Then if the $h(w)$ is already defined and greater than $k+1$, the tiling is impossible so stop the algorithm. Else, leave it with its defined value and if its not defined at all, set $h(w) = k+1$.

If every vertex can be given a height function successfully, we know there is a tiling on R and every adjacent edge has either height difference one or two. Then we can simply delete all edges with a two height difference and we are left with all the edges that form a lozenge tiling [1].

5. Conclusion

Tiling groups are a neat way to breaking apart geometric problems into ones about algebra in certain instances. These tiling groups can be applied to many other types of objects such as dominoes and hexagons being tiled in square and triangular grids respectively. It is even possible to extend tiling to 3 dimensions or above, which is especially useful in natural 3d structures. As long as you can construct a graph of the objects you wish to tile have it realize the homogenous property of graphs, it can be put into the form of a presentation of a group. Although in many instances, a general solution to the tiling problem doesn't exist, we can find useful facts and especially necessary conditions for tiling through group theory that can help us strengthen our understanding of the problems.

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