The Euler Characteristic

Derek Johnson

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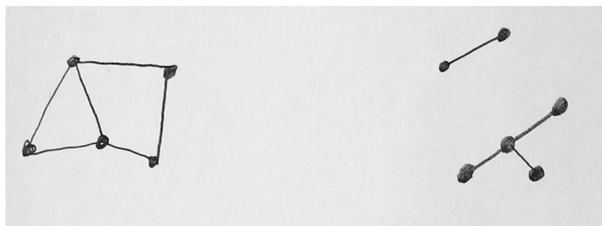
1 Introduction

Euler's Formula for polyhedra has roots in the earliest days of topology. Its extension to a much broader class of objects, called the Euler Characteristic, serves as a powerful tool for discussing more general topological objects. Due to its usefulness as a topological property, the Euler Characteristic lends itself to powerful applications. For example, Euler Characteristics can be used to determine the number of targets detected by a system of sensors with minimal data.

2 Motivation: Graphs and Polyhedra

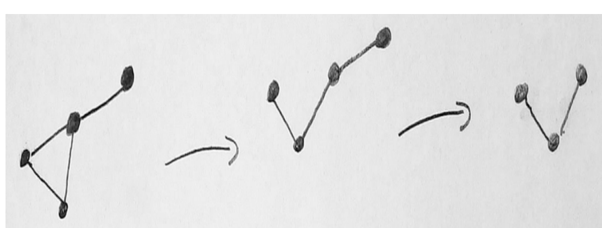
Definition 2.1. A **Planar Graph** is a set $\{v_j\}$ of points in $\mathbf{R}^2$ called vertices together with a set $\{\{v_i, v_j\}\}$ of undordered pairs of distinct vertices called edges such that no edges cross. A **connected graph** is a graph for which for every pair of distinct vertices v_0 and v_n there is a sequence of edges $\{v_0, v_1\}$, $\{v_1, v_2\}$, ... $\{v_{n-1}, v_n\}$ which form a path from v_0 to v_n [5].



An obscure, but occasionally useful result, called the *Jordan Curve Theorem*, states that any simple closed curve in the plane divides the plane into two open, simply connected sets, one bounded and one unbounded [7]. The region bounded by a simple closed curve of edges is called face. If two distinct paths of edges connect two vertices, then those edges form at least one face, and for every face there are at least two paths connecting the vertices bordering the face.

Lemma 2.1. *Let G be a connected graph, V be the number of vertices of G, E be the number of edges of G, F be the number of faces of G, and $\chi(G) = E - V + F$. Then $\chi(G) = 1$.*

Proof. If $F > 0$, remove an edge that borders a face of G. This leaves G connected, since every path between vertices in G that went through the removed edge can go around the other path connecting the vertices of that edge. This also reduces the number of faces of the graph by 1, so the quantity χ is unchanged. Repeating this F times leaves a graph with no faces. Now there is a vertex which has only one edge connecting it to the graph, because otherwise there would be a cycle in the graph. Removing that vertex and edge leaves the graph connected and preserves χ. Eventually, there will be one edge left and, since the graph is connected, two vertices. Thus $\chi(G) = V - E + F = 1$. $\square$

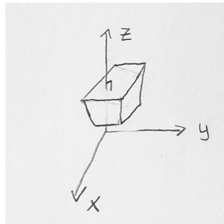


This result can be used to produce a similar result for convex polyhedra (i.e. surfaces which are boundaries of convex open sets in $\mathbb{R}^3$ composed of polygons).

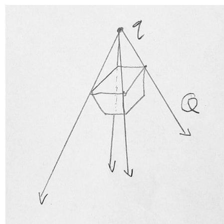
Theorem 2.2. *Let P be a convex polyhedron, V be the number of vertices of P, E be the number of edges of P, and F be the number of faces of P. Then $\chi(P) = V - E + F = 2$.*

Remark. *This result is known as Euler's Formula.*

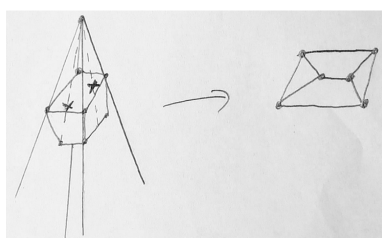
Proof. For the sake of verbal clarity, position P such that there is one face F_0 on top (i.e. parallel to the $x-y$ plane if the z axis is vertical).



Let F_0 be contained in the plane $z = a$. Choose a point q above the interior of F_0 and form a pyramid Q by drawing rays from q through the vertices of F_0 such that P is contained in Q. Then the intersection of Q and the plane $z = a$ is F_0.



Q can be made to contain P by making the faces of Q have smaller slope than those adjacent to F_0. Since P is contained in Q, for any point on P there is a line through q contained in Q. This line exits P in exactly one point, since P is convex. The point must be in F_0, because the line must pass through the plane $z = a$. Each such line passes through a distinct point of F_0 because they all go through q as well. Thus there is a one-to-one map f from the surface of $P - F_0$ to F_0. By labeling the images of the vertices of P under f vertices, the images under f of the edges of P as edges, and the images under f of the faces of P as faces, this construction forms a connected planar graph bordered by the edges of F_0. Since $V - E + F$ is the same for that graph as P with the face F_0 removed, $\chi(P) = 2$. ☐



This result can be used to classify all regular polyhedra, i.e. all platonic solids [5].

Corollary 2.2.1. *There are five regular polyhedra: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron.*

Proof. Let R be a regular polyhedron, and V, E, and F be the vertices, edges, and faces of R, respectively. Let each vertex be the junction of n edges, and each face be bordered by m edges. Then $2E = nV$ and $2E = mF$. By Euler's Formula,

$$2 = V - E + F = V - \frac{nV}{2} + \frac{nV}{m}$$

Therefore

$$4m = V(2m - mn + 2n)$$

Since V and m are positive,

$$2m - mn + 2n > 0$$

$$(2 - m)(n - 2) + 4 > 0$$

$$(m-2)(n-2) < 4$$

$m > 2$ and $n > 2$, so the possibilities are $(m, n) = (3, 3)$, $(3, 4)$, $(4, 3)$, $(5, 3)$, or $(3, 5)$, which correspond with the tetrahedron, the octahedron, the cube, the dodecahedron, and the icosahedron, respectively. $\square$

This result illustrates the power of the Euler Characteristic, in that Euler's formula readily classifies the convex polyhedra. Later, the Euler Characteristic will be used to classify surfaces. A generalization of polyhedra and polygons to higher definitions is called convex polytopes. The following section describes how to extend Euler's Formula to all convex polytopes.

3 Convex Polytopes

Definition 3.1. The **convex hull** of a set A in $\mathbf{R}^n$ is the intersection of all convex sets that contain A; that is, the convex hull of A is the smallest convex set containing A. The convex hull of a finite set of points is called a **convex polytope** [3].

Segments, polygons, and polyhedra are special cases of convex polytopes.

Definition 3.2. For a finite set of points $\{x_1, ..., x_k\}$ in $\mathbf{R}^n$, $x = \sum_{j=1}^{k} \lambda_j x_j$ is an **affine combination** of the $x_i's$ if $\sum_{j=1}^{k} \lambda_j = 1$. The **affine hull** of a set S in $\mathbf{R}^n$ is the set of all affine combinations of finite sets of points in S [3].

The affine hull of a set in $\mathbf{R}^n$ is a translate of a subspace of $\mathbf{R}^n$ in the sense of linear algebra. Thus we may define the dimension of a polytope to be the dimension of its affine hull. A polytope of dimension d will be called a d-polytope.

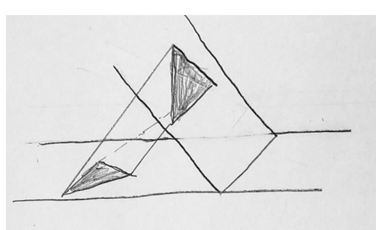
Definition 3.3. A **hyperplane** is a translate of a subspace of $\mathbf{R}^n$. A hyperplane H supports a subset S of $\mathbf{R}^n$ if the distance between H and S is 0 (defined the same as in Folland) and the intersection of one of the half-spaces determined by H and S is empty. If S is a convex subset of $\mathbf{R}^n$ and F is a subset of S such that F is empty, equal to S, or there is a hyperplane H that supports S such that $F = S \cap H$, then F is a **face** of S [3].

A face of a polytope is itself a polytope, so we can analogously call faces of dimension k k-faces. The following theorem extends Euler's Formula to general polytopes in $\mathbf{R}^n$.

Theorem 3.1. *Denote the number of k-faces of a d-polytope P by $f_k(P)$. Then $\sum_{i=0}^{d}(-1)^i f_i(P) = 1$. Equivalently, since $f_d(P) = 1$, $\sum_{i=0}^{d-1}(-1)^i f_i(P) = 1 - (-1)^d$* [3].

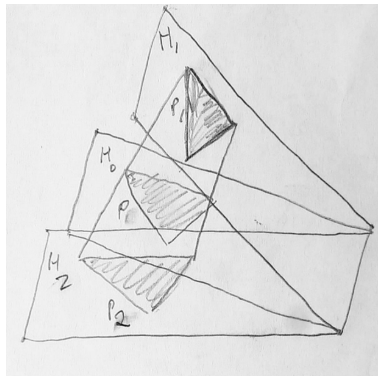
The proof of theorem 2.1 as given in [3] has several steps. The first two will be written as lemmas, and the final step, which ties these together, will be the proof of the theorem. Used in the proof is the following type of set:

Definition 3.4. A **d-prismoid** is a d-polytope which is the convex hull of two at most $d-1$-dimensional polytopes P_1 and P_2 such that for any $d-1$-dimensional hyperplanes H_1 and H_2 containing P_1 and P_2, respectively, $(P_1 \cup P_2) \cap (H_1 \cap H_2) = \emptyset$ [3].



For example, a tetrahedron is a prismoid formed from the convex hull of a triangle and a point.

Lemma 3.2. *Suppose that Theorem 2.1 holds for all polytopes up to dimension $d-1$, and that P is a d-prismoid which is the convex hull of P_1 and P_2. Let H_1 and H_2 be $d-1$-hyperplanes through P_1 and P_2, and H_0 be a hyperlane through the interior of P such that $H_1 \cap H_2 = H_1 \cap H_2 \cap H_0$, let $P_0 = P \cap H_0$. Then Theorem 2.1 holds for P* [3].

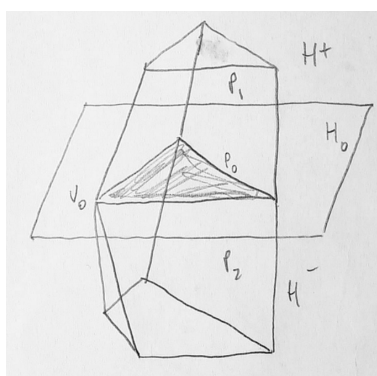


Proof. $f_0(P) = f_0(P_1) + f_0(P_2)$ and $f_k(P) = f_k(P_1) + f_k(P_2) + f_{k-1}(P_0)$ for all $1 \leq k \leq d-1$, since a k-face of P is either a k-face of P_1 or of P_2, or corresponds with a $k-1$-face of P_0. Thus

$$\sum_{k=0}^{d}(-1)^k f_k(P) = \sum_{k=0}^{d-1}(-1)^k(f_k(P_1)+f_k(P_2)) - \sum_{k=0}^{d-2}(-1)^{k+1} f_k(P_0) + (-1)^d f_d(P) = 2-(1-(-1)^{d-1})+(-1)^d = 1$$

so Theorem 2.1 holds for P. □

Lemma 3.3. *Let P be a d-polytope and H_0 be a hyperplane through the interior of P and containing one vertex of P. Let H^+ and H^- be the two closed half-planes with boundary H_0, and let $P_0 = P \cap H_0$, $P_1 = P \cap H^+$, and $P_2 = P \cap H^-$. If Theorem 2.1 holds for P_0, P_1 and P_2, then Theorem 2.1 holds for P* [3].



Proof.

$$f_0(P) = f_0(P_1)f_0(P_2) - 2f_0(P_0) + 1$$

since a vertex of P_1 or P_2 distinct from all but one of those of P_0. Likewise,

$$f_1(P) = f_1(P_1) + f_1(P_2) - 2f_1(P_0) - f_0(P_0) + 1$$

because an edge of P is either an edge of P_1 or P_2 distinct from those of P_0, or is an edge of P divided by the shared vertex with P_0 into an edge of P_1 and P_2. For $2 \leq k \leq d-2$,

$$f_k(P) = f_k(P_1) + f_k(P_2) - 2f_k(P_0) - f_{k-1}(P_0)$$

since for $2 \leq k \leq d-2$, a k-face of P is a k-face of P_1 or P_2 but not of P_0, or is divided by a $k-1$-face of P_0 into a k-face of P_1 and a k-face of P_2. Lastly,

$$f_{d-1}(P) = f_{d-1}(P_2) - 1 - f_{d-2}(P_0)$$

because a $d-1$-face of P is a $d-1$-face of P_1 or P_2 distinct from P_0, or else is divided by a $d-2$-face of P_0 into a $d-1$-face of P_1 and a $d-1$ face of P_2. Therefore

$$\begin{aligned}\sum_{k=0}^{d}(-1)^k f_k(P) &= f_0(P) - f_1(P) + \sum_{k=2}^{d-2}(-1)^k f_k(P) + (-1)^{d-1} f_{d-1}(P) + (-1)^d f_d(P) \\ &= f_0(P_1) + f_0(P_2) - 2f_0(P_0) + 1 \\ &- (f_1(P_1) + f_1(P_2) - 2f_1(P_0) - f_0(P_0) + 1) \\ &+ \sum_{k=2}^{d-2}(-1)^k [f_k(P_1) + f_k(P_2) - 2f_k(P_0) - f_{k-1}(P_0)] \\ &+ (-1)^{d-1}[f_{d-1}(P_1) - 1 + f_{d-1}(P_2) - 1 - f_{d-2}(P_0)] \\ &+ (-1)^d f_d(P)\end{aligned}$$

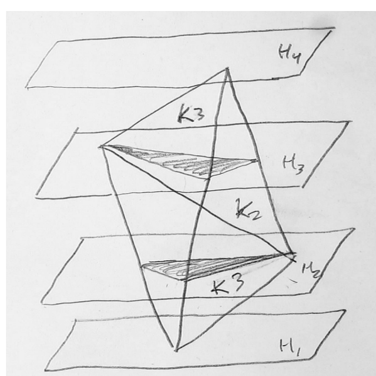
Since Theorem 2.1 holds for P_0, P_1, and P_2,

$$\sum_{k=0}^{d}(-1)^k f_k(P) = 2(1-(-1)^d) - (1-(-1)^{d-1}) + (-1)^d = 1$$

so Theorem 2.1 holds for P. ☐

With these lemmas, we can now prove Theorem 2.1 for all convex polytopes.

Proof. Theorem 2.1 holds for 0-, 1-, 2-, and 3-polytopes. Suppose P is a d-polytope and Theorem 2.1 is established for all polytopes up to dimension $d-1$. Let H be a hyperplane such that every hyperplane parallel to H contains at most one vertex of P (which exists since P has finitely many vertices), and H_1, ... , $H_{f_0(P)}$ be hyperplanes parallel to H and each containing distinct vertices of P, such that H_j separates H_i and H_k for $i < j < k$. Denote by K_i the part of P between H_i and H_{i+1} (including $H_j \cap P$ and $H_{i+1} \cap P$) for $1 \leq i \leq f_0(P) - 1$. Each K_i is a d-prismoid, i.e. the convex hull of $(H_i \cap P) \cup (H_{i+1} \cap P)$, so each K_i satisfies $\sum_{k=0}^{d}(-1)^k f_k(K_i) = 1$. Set $Q_j = \bigcup_{i=1}^{j} K_i$ for $1 \leq j \leq f_0(P) - 2$. Then Q_{j+1}, H_{j+1}, Q_j, and K_{j+1} satisfy the conditions of lemma 2.2 with $P = Q_{j+1}$, $H_0 = H_{j+1}$, $P_1 = Q_j$, and $P_2 = K_{j+1}$. $Q_1 = K_1$, and each K_i satisifies Theorem 2.1, so each Q_i satisfies Theorem 2.1. $Q_{f_0(p)-1} = P$, so Theorem 2.1 holds for all d-polytopes P. By induction, Theorem 2.1 holds for all convex polytopes.



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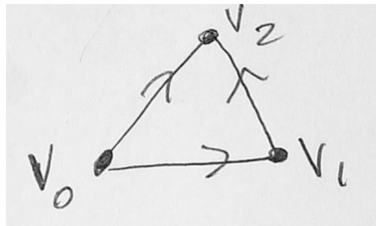
4 Δ-Complexes and Simplicial Homology

Definition 4.1. An **n-simplex** is the convex hull of $n+1$ points in $\mathbf{R}^m$ that do not all lie in a hyperplane of dimension less than n [4].

A 2-simplex is a triangle, a 3-simplex is a tetrahedron, etc. The points determining an n-simplex, $\{v_i\}$, are called its vertices, and the simplex is denoted $[v_1, ..., v_n]$. It is important in what follows to consider this to be an order set of vertices. For example, $[v_0, v_1, v_2]$ may be drawn as



A face of a simplex is a subsimplex whose vertices are any nonempty subset of the original simplex, ordered the same way.

Definition 4.2. A **Δ-complex** is an object obtained by gluing disjoint simplices along common faces [4].

The orientations of the edges of a **Δ**-complex are determined by the orderings of the vertices in its component simplices. Since the construction of a **Δ**-complex X never involves gluing two points in the interior of a face of X, a δ-complex can be thought of as a collection of open simplices $\{e^n_\alpha\}$, where n is the dimension of e^n_α [4]. In order to describe simplicial homology, I will first give a brief summary of basic group theory.

4.1 Intermezzo: Group Theory

Definition 4.3. A **group** [2] is a set G and an operation $* : G \times G \to G$ satisfying these properties:

1. Identity: There is an element e of G such that, for any $g \in G$, $e * g = g * e = g$. e is called the **identity** of G.
2. Associativity: For all f, g, and $h \in G$, $f * (g * h) = (f * g) * h$.
3. Inverses: For every $g \in G$ there is an element $g^{-1} \in G$, called the **inverse** of g, such that $g * g^{-1} = g^{-1} * g = e$ [2].

A group is called **abelian** [2] if $a * b = b * a$ for all $a, b \in G$.

A **subgroup** [2] H of a group G is a subset which satisfies the group axioms with operation $*$ restricted to H.

A **homomorphism** [2] is a function $\phi : G \to H$, where G and H are groups, such that $\varphi(a *_G b) = \varphi(a) *_H \varphi(b)$ for all a and b in G. The **kernel** [2] of φ is the subset $\ker\varphi = \varphi^{-1}(e_H)$ of G. A homomorphism is called an **isomorphism** [2] if it is bijective as a set function. If an isomorphism exists between two groups, the groups are called isomorphic, and can be treated as the same group.

If H is a subgroup of a group G, the sets of left and right cosets of H are

$$\{gH = \{gh : h \in H\} : g \in G\}$$

and

$$\{Hg = \{hg : h \in H\} : g \in G\}$$

respectively. H is **normal** [2] if $gH = Hg$ for all g in G. This is equivalent to the statement $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$. The kernel of any homomorphism is a normal subgroup of the domain. If H is a normal subgroup of G, the **quotient group** [2] G/H is the set of cosets of H with operation

$$(aH) * (bH) = abH$$

This is well defined, since if $a' = ah$ and $b' = bh'$, then

$$a'b'H = ahbh'H = abb^{-1}hbH = abh''H = abH$$

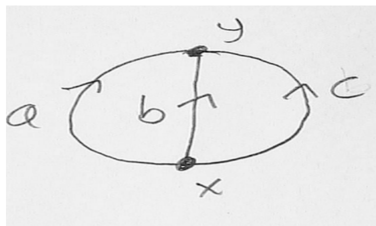
for some $h'' \in H$. That G/H satisfies the group axioms follows from the fact that G does.
A **free abelian group** [2] on a set S is the set of formal sums

$$\sum_{s \in S} \alpha_s s$$

with $\alpha_s \in \mathbf{Z}$ and operation given by summing corresponding components of α_s. The set S is said to generate the free abelian group on S.

4.2 Homology: Motivation

Simplicial comlexes are assigned simplicial homology groups to formalize the number of holes of different dimensions these objects have. To motivate the definition of homology groups (see [4]), consider the following oriented graph, G_1:



Let C_1 be the free abelian group on the edges a, b, and c, identified with chains of edges for which a positive coefficient represents a traversal from x to y. A cycle is a chain of edges such that x and y are entered as many times as they are exited. Expressed algebraically, the number of times a chain

$$ka + lb + mc$$

enters y is

$$k + l + m$$

and the number of times a chain leaves y is

$$-k - l - m$$

since each time a chain leaves y an edge is traversed backwards. Likewise, the chain enters x

$$-k - l - m$$

times and leaves x

$$k + l + m$$

times. Thus a chain is a cycle iff

$$k + l + m = 0$$

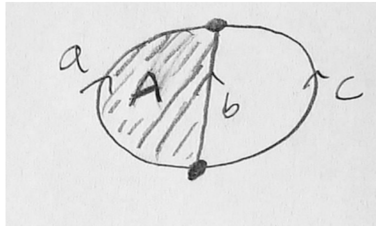
Let C_0 be the free abelian group on $\{x, y\}$ and define a homomorphism, called the boundary homomorphism

$$\partial : C_1 \to C_0$$

which sends a, b, and c to $y - x$, i.e. each edge is sent to an oriented sum of its endpoints. This gives

$$\partial(ka + lb + mc) = (k + l + m)y - (k + l + m)x$$

so the set of cycles in C_1 is the kernel of ∂. If $k+l+m = 0$, then $l = -k+m$, and $ka+lb+mc = k(a-b)-m(b-c)$ so $(a - b)$ and $(b - c)$ generate the subgroup of cycles of C_1. That the subgroup of cycles in C_1 i generated by two cycles expresses the fact that there are two holes in the graph.
Consider now a new diagram G_2 obtained from G_1 by filling in the region bounded by $a - b$:



$(a-b)$ no longer encloses a hole, since $(a-b)$ can be continuously deformed to a point. This may be expressed by forming the quotient group

$$C_1/ < (a-b) >$$

If C_2 is the group generated by A and $\partial_2(A) = a - b$ then the quotient of C_1 by the subgroup generated by $a - b$ is

$$\frac{\ker\partial_1}{\mathrm{Im}\partial_2}$$

which is defined as the homology group $\mathrm{H}_1(X_2)$.

4.3 Simplicial Homology

Let X be a Δ-complex and $\Delta_n(X)$ be the free abelian group generated by the open n-simplices e_α^n of X. The boundary homomorphism [4]

$$\partial_n : \Delta_n(X) \to \Delta_{n-1}(x)$$

is defined to be

$$[v_0, ..., v_n] \to [v_1, ..., v_n] + \sum_{i=1}^{n-1} (-1)^i [v_0, ..., v_{i-1}, v_{i+1}, ..., v_n] + (-1)^n [v_0, ..., v_{n-1}]$$

An explicit calculation shows that, for all n,

$$\partial_n \partial_{n+1} = 0$$

Thus

$$\mathrm{Im}\partial_{n+1} \subset \ker\partial_n$$

so the n^{th} **homology group** [4], H_n, may be defined as

$$\ker\partial_n / \mathrm{Im}\partial_{n+1}$$

For a complex X we can define the Euler Characteristic $\chi(X)$ to be

$$\sum_n (-1)^n \mathrm{rank}\mathrm{H}_n(X)$$

where $\mathrm{rank}\mathrm{H}_n(X)$ denotes the number of generators of $\mathrm{H}_n(X)$. The n^{th} **Betti Number** of X is defined to be $\mathrm{rank}\mathrm{H}_n(X)$, so one can equivalently write the Euler Characteristic of a complex as an alternating sum of the Betti Numbers of X. This is equivalent to the sum of the n-faces of X. Homology groups are topological invariants, so this definition of Euler Characteristic shows that it is topologically invariant [4].

5 Surfaces

Definition 5.1. A **compact surface** [6] is a connected, compact set for which for every point there is a neighborhood in the surface homeomorphic to the open disk in $\mathbb{R}^2$.

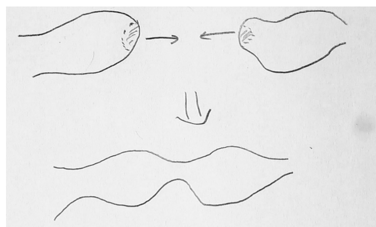
Definition 5.2. A **triangulation** [6] of a compact surface X is a collection of subsets $\{T_1, ..., T_n\}$ of X homeomorphic to triangles in $\mathbf{R}^2$ that satisfy

1. $\bigcup_i T_i = X$

2. Either $T_i = T_j$, or $T_i \cap T_j$ is empty, a vertex, or an edge.

If X is a triangulated surface, the Euler Characteristic $\chi(X)$ is defined as before as $V - E + F$, where V is the number of vertices, E the number of edges, and F the number of faces of a triangulation of X. The Euler Characteristic of a surface is independent of the triangulation of the surface used to determine it [6]. For example, the sphere S^2 has Euler characteristic 2, since it is homeomorphic to convex polyhedra. As another example, the torus T^2 has Euler Characteristic 0.

Definition 5.3. If S_1 and S_2 are compact surfaces, the surface $S_1\#S_2$, called the **connected sum** [6] of S_1 and S_2, is constructed by by removing a small open disk from S_1 and S_2 and gluing the boundaries of the holes.



Theorem 5.1. *If S_1 and S_2 are two surfaces* [6], then

$$\chi(S_1\#S_2) = \chi(S_1) + \chi(S_2) - 2$$

Proof. Triangulate both surfaces, and remove the interior of a triangle from each. This reduces the Euler Characteristic of both surfaces by 1. Joining the boundaries of these two triangles forms the connected sum of the surfaces, and preserves the sum of the Euler Characteristics since three vertices and three edges are removed. Consequently, the Euler Characteristic of the connected sum is 2 less than the sum of the Euler Characteristics of the surfaces. □

The Euler Characteristic provides a way of classifying all compact surfaces [6]:

Theorem 5.2. *If S and T are two compact surfaces with the same Euler Characteristic, and S and T are both orientable or both non-orientable, then S and T are homeomorphic. Further, if S and T are the connected sum of n tori, then $\chi(S) = \chi(T) = 2 - 2n$.*

6 Target Enumeration via Euler Characteristic Integrals

The idea is to use the Euler Characteristics of the "fields of vision" of each sensor in an array for which each sensor counts the number of objects it sees in order to determine the total number of objects detected. For example, consider the case where each sensor is a point of $\mathbf{R}^2$ which detects object within a radius R. If h is the number of objects counted by the sensor at each point, then the total number of objects is

$$\frac{1}{\pi R^2} \int_{\mathbf{R}^2} h dA$$

since each object is detected by a region of sensors of area πR^2 [1].

Definition 6.1. Let $A = \{A_k\}$ be a collection of families of subsets of $\mathbb{R}^n$, each closed under intersections and complements, such that A is closed under products and projections, and A_1 is the family of all finite unions of points and open intervals. A set in A_n is called **definable** [1]. A function is called definable if its graph is definable.

An important result, called the *triangulation theorem* [1], states that definable sets are homeomorphic to simplicial complices and thus have well-defined Euler Characteristics.

Definition 6.2. Suppose X is a simplicial complex and $CF(X)$ is the abelian group of functions having finite range and definable level-sets $\{\phi : X \to \mathbf{Z}\}$ with basis $\mathbf{1}_\sigma \to 1$ for each simplex σ of X. The **Euler Integral** [1] is the homomorphism

$$\int_X d\chi : CF(X) \to \mathbf{Z}$$

given by

$$\int_X \sum_\alpha c_\alpha \mathbf{1}_{\sigma_\alpha} d\chi = \sum_\alpha c_\alpha \chi(\alpha)$$

Consider the problem of enumerating the number of targets in X given

1. The height function, h, giving the number of targets detected by the sensor at each location in X.
2. For each set U_α consisting of all sensors that detect object α, $\chi(U_\alpha) = N \neq 0$

Theorem 6.1. *Let a be the number of targets of the problem stated above* [1]. Then the problem above is solved by

$$\frac{1}{N}\int_X h d\chi = \frac{1}{N}\int_X \sum_\alpha (\mathbb{1}_{U_\alpha}) d\chi = \frac{1}{N}\sum_\alpha \int_X \mathbb{1}_{U_\alpha} d\chi = \frac{1}{N}\sum_\alpha \chi(U_\alpha) = a$$

This theorem looks similar to the first example of this section. Note that exchanging the summation and the integral is fine since there are assumed to be a finite number of targets. This result is difficult to apply in practice, however, since real sensors can only be implemented in discrete arrays. To circumvent this issue, one can use a system of discrete sensors as an approximation to the structure of the target space [1].

Lemma 6.2. *Let $h \in CF(X)$, and write $\{h = s\} = \{x : h(x) = s\}$ and similarly for $\{h > s\}$, $\{h \geq s\}$, $\{h < s\}$, and $\{h \leq s\}$. Then* [1]

$$\begin{aligned}\int_X h d\chi &= \sum_{s=-\infty}^{\infty} s\chi\{h = s\} \\ &= \sum_{s=0}^{\infty} \chi\{h > s\} - \chi\{h < s\}\end{aligned}$$

Proof. The first equality follows from the definition of the Euler Integral. For the second part, write

$$\begin{aligned}\sum_{s=-\infty}^{\infty} s\chi\{h = s\} &= \sum_{s=0}^{\infty} s(\mathbb{1}_{\{h\geq s\}} - \mathbb{1}_{\{h>s\}}) + \sum_{s=0}^{-\infty} s(\mathbb{1}_{\{h\leq s\}} - \mathbb{1}_{\{h<s\}}) \\ &= \sum_{s=0}^{\infty} s(\mathbb{1}_{\{h>s-1\}} - \mathbb{1}_{\{h>s\}}) + \sum_{s=0}^{-\infty} s(\mathbb{1}_{\{h<s+1\}} - \mathbb{1}_{\{h<s\}}) \\ &= \sum_{s=0}^{\infty} \mathbb{1}_{\{h>s\}} - \sum_{s=0}^{-\infty} \mathbb{1}_{\{h<s\}}\end{aligned}$$

which follows from the fact that h is integer valued and the sums telescope. $\square$

This gives a new way of calculating the Euler Integral which is somewhat more robust to perturbations of the sensor field [1].

Definition 6.3. A function is called **upper semicontinuous** [1] if

$$f(x) \leq f(x_0) + \epsilon$$

for all x sufficiently close to x_0 in the domain.

Proposition 6.1. *Let $h : \mathbb{R}^2 \to \mathbb{N}$ be upper semicontinuous and have finite range and definable $\{h = s\}$, and G be a graph in $\mathbb{R}^2$ with vertices N such that the restriction of h to N is known but not necessarily the coordinates of the points in N. If G is dense enough to give the right connectivity of the sets $\{h < s\}$ and $\{h > s\}$ for each s in the range of h, then the number of targets is given by Lemma 6.2* [1].

7 Conclusion

The Euler Characteristic is an important topological property that is useful in classifying many objects. By studying its description at several levels of sophistication, one attains a deeper understanding of the connections between geometry, topology, and algebra. Because of its usefulness as a topological property, the Euler Characteristic can be used to determine information about certain data without exact geometric information, such as in the case of target enumeration via Euler Integrals.

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