Harmonic Morphisms on Riemannian Manifolds

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Abstract

The goal of this paper is to define and charaterize harmonic morphisms between Riemannian manifolds, then build up machinery for utilizing them. Much of the paper will follow the works of Bent Fuglede, who pioneered much of the theory of harmonic morphisms in his seminal paper 40 years ago. Later we will take an algebraic approach to the study of harmonic morphisms, building up some categorical machinery for future use. We conclude with an example of utilizing harmonic morphisms by applying them to solving the Dirichlet problem on Riemannian manifolds. Section 1 is dedicated to introductory material, 2 follows the works of Fuglede, and 3 and 4 contain interesting findings of my own.

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1 Introduction

This paper makes use of some concepts from elementary category theory. The basic definitions of categories, morphisms, and functors are assumed to be known by the reader beforehand. However, some definitions will be provided in this section. All category theory used in this paper is explained in greater detail in Steve Awodey's book Category Theory [2].

We will utilize the notion of contravariant functors, which play a critical role in the definition of sheaves.

Definition 1.1. Given categories C, D, a contravariant functor is a functor $F : C \to D$ such that for every morphism $f : A \to B$ in C, the value Ff in D is a morphism $Ff : F(B) \to F(A)$.

Next we define a posetal category.

Definition 1.2. Let $(X, \leq)$ be a partially ordered set, or "poset". This defines a posetal category, which has as objects the elements of X, and there are morphisms $a \to b$ iff $a \leq b$.

When we define the category of Riemannian manifolds with harmonic morphisms, we will show that the isomorphisms are given by diffeomorphisms between manifolds.

1.1 Riemannian Manifolds

A Riemannian manifold is a differentiable manifold M of dimension m equipped with a metric g_M. That is, at each point $p \in M$, a tangent space T_pM to M can be determined, and there is a symmetric, positive definite, bilinear form $g_M(p) : T_pM \times T_pM \to \mathbb{R}$ defined on the tangent space that varies smoothly in p. It should be noted that the metric g_M is not a "metric" in the sense of a metric space, however it can be used to define such a metric.

The metric evaluated at a point $p \in M$ is usually denoted $g_M(p) = g_p$. The tangent space T_pM is a vector space of dimension m that is "tangent to" the manifold at p. An intuitive visual is a plane connected to M that is "centered at p", that is the zero vector in T_pM is thought to be "at p". A norm on T_pM is induced by the metric; for any vector $a \in T_pM$, the norm of a is given by

$$\|a\| = \sqrt{g_p(a, a)}.$$

Angles between vectors can be defined intuitively. For any two vectors $a, b \in T_pM$, the angle between a and b is the number θ such that

$$\cos\theta = \frac{g_p(a,b)}{\|a\|\,\|b\|}.$$

The metric tensor g_{ij} on M is a tool for measuring distances on M using the algebra of vectors, and is indispensible for utilizing calculus on M.

Definition 1.3. Given the Fréchet derivative $J = Df$ of the map $f : \mathbb{R}^m \to M$,the metric tensor on M is defined as the matrix product

$$g = J^T J.$$

One should note then that the determinant $\det g$ is equal to the square of the Jacobian determinant. Thus the components of the metric tensor g are

$$g_{ij} = J_{ik} J_{kj},$$

where the Einstein summation convention is used,

$$a_i b_i = \sum_1^m a_i b_i, \quad a^i = \frac{a_i}{|g|}, \quad g^{ij} = \frac{1}{g_{ij}}.$$

Here, $|g|$ denotes the determinant of g. This convention will be used throughout the article. Another way of computing the metric tensor is via the metric inner product. Let x^j denote the coordinates in $\mathbb{R}^m$ of a point $p = (x^j) \in M$, and U an open subset of M containing p. For every such open set there is a diffeomorphism $\psi : U \to \mathbb{R}^m$ that maps the coordinates of a point $p \in M$ with respect to the manifold to their coordinates in $\mathbb{R}^m$. That is, if p has "local" coordinates y^j in the manifold, then $x^i = \psi^i(y^j)$ are the components of the map ψ. Thus the coordinates y^j can be defined as the components of the inverse map ψ^{-1}. A standard convention is to denote by y^j the map itself, so that $y^j(x^i)$ are the local coordinates of the point p in M and $x^i(y^j)$ are the coordinates of p in $\mathbb{R}^m$. Next we denote by $\partial_j p$ the partial derivatives $(\partial_j y^i)$ of the inverse chart y. With this definition, the components of the metric tensor can be defined equivalently as

$$g_{ij} = g(\partial_i p, \partial_j p).$$

The machinery defined above is a standard notation for expressing coordinates on a manifold.

Since the tangent space of a manifold is variable from point to point, we define the general basis of $T_xM, x \in M$ as a set of operators. Here, we name the coordinate chart for clarity.

Definition 1.4. For an open set $U \subseteq M$ with coordinate chart ψ, the basis at a point $p = (y^j) \in M$ of the tangent space T_pM is a set of operators $(\partial_i)_p$ such that, for a function $f \in C^1(M)$,

$$(\partial_i)_p f = \partial_i (f \circ \psi^{-1})(\psi(p)).$$

When no point p is specified, the basis vectors are simply denoted ∂_i and act as regular partial derivative operators. The Tangent bundle on M is the disjoint union of all tangent spaces on M,

$$TM = \coprod_{x \in M} T_xM = \{(x, T_xM) : x \in M\}.$$

A familiar notion is that of vector fields. However, vector fields on Riemannian manifolds have a more sophisticated definition.

Definition 1.5. Given a Riemannian manifold M with tangent bundle TM, a vector field on M is a function that assigns to each point $x \in M$ a vector $v \in T_xM$.

Next we define the Laplace-Beltrami operator, which generalizes the Laplacian operator on Riemannian manifolds.

Definition 1.6. Given a manifold M with metric tensor g, the Laplace-Beltrami operator is defined as

$$\Delta_M = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|}\, g^{ij} \partial_j \right).$$

A function $f : M \to \mathbb{R}$ is called harmonic if $\Delta_M f = 0$. Thus, the Laplace-Beltrami operator generalizes harmonic functions to functions on Riemannian manifolds.

Similarly, we define the gradient of a function on a manifold as follows.

Definition 1.7. For a Riemannian manifold (M, g) and function f on M, the gradient of f is given by

$$\nabla f = g^{ij} \partial_i f \partial_j.$$

Note that the gradient is an element of the tangent space, and so operations via the metric can be used as a norm on the gradient of a function. A special case is when the map is f is the coordinate chart itself. Then we have by definition the identity

$$g(\nabla \psi^k, \nabla \psi^j) = g^{ij}.$$

We also introduce the notion of the Christoffel Symbols of the Second Kind, a set of functions defined on a manifold. The applications of these symbols are marginal in the subject of this paper, so little motivation will be provided. Simply put, the Christoffel Symbols provide a transformation of vectors on the manifold into the tangent space of the manifold. Thus they allow one to project a set of vectors on the manifold M onto the tangent space, allowing for various vector fields on M to be defined. Their use is generally in the subject of parallel transport, which will not be considered in this paper.

Definition 1.8. Given a manifold M with metric g, suppose $\mathbf{r}_p$ is a local chart $U_p \to \mathbb{R}^m$ for a neighborhood $U_p \subseteq M$ of $p \in M$. Then the Christoffel Symbols of the Second Kind are defined as

$$\Gamma_{ij}^k = \partial^k \mathbf{r} \cdot \partial_i \partial_j \mathbf{r} = \frac{1}{2} g^{km} \left(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij} \right).$$

The Christoffel Symbols will be used rarely in this paper, and so no derivation will be provided.

Next we define the notion of a differential of a function.

Definition 1.9. Given a C^1 map $f : N \to M$ between Riemannian manifolds M, N, the differential df_x at a point $x \in M$ is defined as a linear map

$$df_x : T_x M \to T_{f(x)} N$$

that maps the tangent space of M at x to the tangent space of N at $f(x)$.

The differential df of f generalizes the Fréchet derivative of a function to the case of manifold domains, and allows one to preserve the differentiable structure of M when mapping to N via f. Thus it is an important concept in the study of functions between Riemannian manifolds. Similar to the definition we saw in class, a diffeomorphism[1] $f : M \to N$ is called conformal if df is nonzero on M. Later we will develop the notion of semiconformal maps between manifolds, to which the differential will have numerous applications.

1.2 Sheafs

Sheafs are a useful tool for studying global properties of a topological space. The way they achieve this is by generalizing the local properties of

[1] A diffeomorphism is an invertible, differentiable map between Riemannian manifolds such that its inverse is differentiable. By "differentiable", we mean that the differentiable exists.

the space to global properties via the Axiom of Gluing. We begin by defining a presheaf, as the definition of a sheaf is based heavily on the machinery of presheafs.

Let (X, τ_X) be a topological space. Ordering the topology τ_X by the subset relation, we get a partially ordered set $(\tau_X, \subseteq)$. This defines a posetal category, where the objects are the open sets of X and the arrows are given by the subset relation: there is a morphism $U \to V$ iff $U \subseteq V$.

Definition 1.10. A presheaf is a contravariant functor F on the posetal category $(\tau_X, \subseteq)$.

Thus the presheaf determines a second category C with objects $F(U)$ as the images of the elements of τ_X, and morphisms $F(U) \to F(V)$ iff $F(V) \subseteq F(U)$. For the purposes of this article, we will only consider presheafs that map to categories of sets, for ease of explanation. In this case, the elements of $F(U)$ are called the sections of U, and the morphisms $F(U) \to F(V)$ are called restriction morphisms. As set functions, the morphisms take sections $s \in F(U)$ to restricted sections $s|_V \in F(V)$. Two properties are required to hold on the morphisms in C:

(1) The morphism $F(U) \to F(U)$ is the identity map.
(2) If $s : F(U) \to F(V)$, $t : F(V) \to F(W)$ and $r : F(U) \to F(W)$, then $t \circ s = r$.

Thus the presheaf "axioms" determine that there is exactly one morphism $F(U) \to F(V)$ for any two open sets $U \subseteq V$. Next we define a sheaf on X.

Definition 1.11. A sheaf F is a presheaf that satisfies the following axioms:

(*Locality*) If (U_i) is an open covering of $U \in \tau_X$, and $s, t \in F(U)$ such that

$$s|_{U_i} = t|_{U_i}$$

for every U_i, then $s = t$.

(*Gluing*) If (U_i) is an open covering of $U \in \tau_X$, and for each i there is a section s_i such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

for every U_i, U_j, then there is a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for every U_i.

The sheaf axioms provide existence and uniqueness for a *glued* section defined on any open set. Thus they allow local properties to be extended to the global scale via *gluing*. The sheaf axioms can be stated purely in category theory as well: the gluing axiom requires that the restriction of the sheaf F to the subspace U has a limit L, and the locality axiom requires that $L \cong F(U)$.

As stated above, the sheaf axioms allow one to expand local properties of the open sets $U \in \tau_X$ to global properties on the topological space X. Thus they have a broad application, such as the solution to the "annulus problem" we encountered earlier this quarter. For the purposes of this article, we will consider the Riemannian manifolds that are also equipped with a sheaf that maps each open set to the vector space of harmonic functions defined on it. In the case of general topological spaces, this object is called a Brelot space. Since we will only consider the Brelot spaces that are also Riemannian manifolds, we will refer particularly to these objects simply as Brelot spaces in this article. Later we will see that the application of harmonic morphisms allows one to build up a category of Brelot spaces for deeper study.

2 Harmonic Morphisms

Harmonic morphisms will allow us to map between Brelot spaces and preserve structure. As many of the results of complex analysis hold on the harmonic functions on Riemannian manifolds, this will allow for powerful applications of what is called complex analysis to the study of Riemannian geometry.

For the remainder of this article, we will suppose that M, N are Riemannian manifolds of dimensions m, n respectively.

Definition 2.1. A harmonic morphism $f : M \to N$ is a continuous function such that, for any open set $V \subseteq N$ and harmonic function $v : V \to \mathbb{R}$ on V, $v \circ f$ is harmonic on the open set $f^{-1}(V)$, provided this set is nonempty.

Thus harmonic morphisms pull back harmonic functions on the codomain. It is clear that the composition of harmonic morphisms $g : L \to M$ and $f : M \to N$ is again a harmonic morphism $f \circ g : L \to N$, and that the identity map *id* is harmonic. Thus there is a category BHarm of Brelot spaces with harmonic morphisms. As well, the harmonic morphisms in BHarm induce direct image functors on the sheaves of the Brelot spaces. That is, for a harmonic morphism $f : M \to N$ and sheaves F, G on M, N, respectively, there is a functor $f_* : G \to F$ defined as $f_*G(V) = g \mapsto g \circ f$ for V open in N

such that

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ F(M) & \xleftarrow{f_*} & G(N) \end{array}$$

commutes. Thus there is also a category BSheaf of sheaves on the Brelot spaces of BHarm. This behaviour will be examined in more detail later in the paper. For the remainder of the article we will refer to the sheaf functor that takes Riemannian manifolds in BHarm to their sheaves in BSheaf as F.

Next we turn to characterizing the harmonic morphisms in BHarm, so that we can understand what kind of functions they are and thus utilize them more effectively. The proofs of the following theorems are reduced to large amounts of symbol-pushing, and so will not be provided. They can be found in [5]. For the rest of the paper we will consider only manifolds M, N with $\dim(M) \geq \dim(N)$. We will see that the case $\dim(M) < \dim(N)$ is not very interesting in the context of harmonic morphisms.

2.1 Preliminaries for Characterization

We begin with a theorem by Greene and Wu[6], of which no proof will be given.

Theorem 2.1. *Every non-compact n-dimensional manifold N has an embedding $h : N \to R^{2n+1}$ such that the components h^k are harmonic on N.*

This will allow us to choose "harmonic local coordinates" y^k on a Riemannian manifold, and hence express a function f on N as a collection of harmonic functions $f^k = y^k \circ f$ on $\mathbb{R}^{2n+1}$. Choosing such coordinates will be a useful tool in our characterization of harmonic morphisms. We also note a theorem of Carleman, Aronaszajn, and Cordes [1],[4] that is similar in nature to that of the identity theorem in complex analysis. It is thus called the uniqueness theorem.

Theorem 2.2. *If u is harmonic on a manifold M, and the partial derivatives of all orders vanish at a point of M, then u is constant.*

One can extend this to get a weaker statement, "If a harmonic function on a manifold is zero on an open set, then that function is zero identically". Next we define semiconformal maps between Riemannian manifolds.

Definition 2.2. A semiconformal map is a C^1 function $f : M \to N$ such that, at all points x where the differential df is nonzero, df is surjective onto $T_{f(x)}N$.

Another definition is that, when restricted to the points of M where $df \neq 0$, the map f is a conformal submersion. The meaning of "conformal" on a manifold is the same as usual, using the definition of angle defined in the introduction. The submersion condition reads as "df is surjective", and the conformality condition requires that df is a linear map that *only scales vectors by a constant*. Any semiconformal map f determines a dilation λ on M, with $\lambda(x) = |df|$, and so we have the following equivalent property for f. If $D = (\ker df_x)^{\perp}$ denotes the orthogonal complement of the kernel of df at $x \in M$, then for any two vectors $v, w \in D$, we have the identity

$$g_N^{f(x)}(df(v), df(w)) = \lambda^2(x) g_M^x(v, w),$$

where g_M^x denotes the norm on T_xM assigned by the metric g_M on M.

Next we provide some elementary characterizations of harmonic morphisms and semiconformal maps due to Fuglede [5].

Theorem 2.3. *A C^2 map $f : M \to N$ is a harmonic morphism if and only if there is a function λ, called the dialation of f, on M such that*
(1)

$$\Delta_M(v \circ f) = \lambda^2((\Delta_N v) \circ f)$$

for all C^2 functions $v : N \to \mathbb{R}$.

The characterization can obviously be locallized so as to agree better with our definition of a harmonic morphism.

A similar characterization of semiconformal maps is given here.

Theorem 2.4. *A C^1 map $f : M \to N$ is semiconformal with dialation λ such that if and only if*
(2)

$$g_M(\nabla f^k, \nabla f^l) = \lambda^2(g_N^{kl} \circ f).$$

Here we note that the components of the metric tensor g_N^{kl} are used as functions on the manifold N.
Next we define a third type of map, which will be useful in our characterization. We begin with a definition of the tension field of a function on its target manifold.

Definition 2.3. Given a C^2 map $f : M \to N$, the tension field along f is a vector field on N whose components are given by
(3)

$$\tau^k(f) = \Delta_M f^k + g_M(\nabla f^i, \nabla f^j)(\Gamma_{ij}^k \circ f).$$

A harmonic mapping $f : M \to N$ is any map with

$$\tau^k(f) = 0$$

for every index k.
First we note the trivial fact that the identity function on a manifold is a harmonic map. Thus we have the identity
(4)

$$\Delta_M y^k + g^{ij}\Gamma^k_{ij} = 0.$$

This will be useful in proving the following lemma.

Lemma 2.5. *The tension field of a C^2 semiconformal mapping $f : M \to N$ with dilation λ is given by*
(5)

$$\tau^k(f) = \Delta_M f^k - \lambda^2((\Delta_N y^k) \circ f).$$

Thus in the case of local harmonic coordinates (y^k), the tension map is equal to the Laplacian of f on M.

Proof. Substituting equation (2) into the definition of the tension field, we have

$$\tau^k(f) = \Delta_M f + \lambda^2(g^{ij}_N \circ f)(\Gamma^k_{ij} \circ f),$$

And making use of the tension field of the identity function gives

$$g^{ij}_N \Gamma^k_{ij} = -\Delta_N y^k,$$

giving the desired result

$$\tau^k(f) = \Delta_M f - \lambda^2((\Delta_N y^k) \circ f).$$

□

2.2 Characterization of Harmonic Morphisms

Next we give a theorem characterizing harmonic morphisms in terms of semiconformal and harmonic maps. The proof can be found in [5], however I have filled in some steps that Fuglede considered "trivial".

Theorem 2.6. *The harmonic morphisms $f : M \to N$ are precisely the semiconformal harmonic maps, and the dilation of f is given as in Theorem 2.3.*

Proof. First we note by the chain and product rules the following two identities of the Laplace-Beltrami operator:

(6)
$$\Delta_M(f^k f^l) = f^k \Delta_M f^l + f^l \Delta_M f^k + 2g_M(\nabla f^k, \nabla f^l),$$

(7)
$$\Delta_N(y^k y^l) = y^k \Delta_N y^l + y^l \Delta_N y^k + 2g_N^k l.$$

Next we turn to the proof, starting with the only if direction. Suppose then that f is a harmonic morphism with dilation λ. Composition on the right with f in equation (7) and multiplication by λ^2 gives

$$\begin{aligned}\lambda^2(\Delta_N y^k y^l) \circ f &= \lambda^2 \left((y^k \circ f)((\Delta_N y^l) \circ f) + (y^l \circ f)((\Delta_N y^k) \circ f) + 2g_N^{kl} \circ f\right) \\ &= \lambda^2 \left(f^k((\Delta_N y^l) \circ f) + f^l((\Delta_N y^k) \circ f) + 2g_N^{kl} \circ f\right) \\ &= f^k \Delta_M f^l + f^l \Delta_M f^k + 2\lambda^2 g_N^{kl} \circ f,\end{aligned}$$

where (1) is invoked to exchange $\lambda^2((\Delta_N y^l) \circ f) = \Delta_M f^l$. Next we subtract $\Delta_M f^k f^l$, which we note by (1) is equal to the left-hand side. Thus we obtain

$$0 = 2\lambda^2(g_N^{kl} \circ f) - 2g_M(\nabla f^k, \nabla f^l),$$

giving equation (2). Thus f is semiconformal. Next we use the semiconformality of f to prove it is a harmonic map. Taking local coordinates y^k in N, we see that (1) reads

$$\Delta_M(y^k \circ f) = \Delta_M f^k = \lambda^2((\Delta_N y^k) \circ f),$$

and subtracting off the right hand side gives

$$\Delta_M f^k - \lambda^2((\Delta_N y^k) \circ f) = 0,$$

and the left-hand side is the tension field given by (5). Thus f is a harmonic map as well.

The if direction can be found in [5], or derived by the reader with adequate symbol-pushing, making use of (2) and (5). □

Remark. Since every harmonic morphism must be semiconformal, the case that $\dim M < \dim N$ implies that df cannot be surjective at any point $x \in M$. Thus by contrapositive of the semiconformal condition, it follows that $df = 0$ identically. That is, every harmonic morphism $f : M \to N$ with $\dim M < \dim N$ is constant.

Another, less general characterization of harmonic morphisms will be useful later, when we talk about solving the Dirichlet problem on Riemannian manifolds. As stated by Fuglede in [5], the result is as follows.

Theorem 2.7. *Suppose that* $\dim M = \dim N = 2$*. Then the harmonic morphisms* $M \to N$ *are precisely the* C^2 *semiconformal mappings of* $M \to N$*.*

Another powerful result by Fuglede is the openness of harmonic morphisms, stated as follows.

Theorem 2.8. *Every non-constant harmonic morphism* $f : M \to N$ *is an open mapping. That is, for any open subset* U *of* M*,* $f(U)$ *is open in* N*.*

The proofs are provided in [5].

3 The Categories BHarm and BSheaf

Here is studied the categories BHarm and BSheaf. Only elementary properties of the categories are given, providing machinery for further study. We begin by characterizing the isomorphisms in BHarm.

Theorem 3.1. *The harmonic diffeomorphisms are precisely the isomorphisms in BHarm.*

Proof. ($\Rightarrow$) Of course, the harmonic diffeomorphisms are bijections, so each has a unique inverse function. It is shown in [3] that the inverse of every harmonic morphism is again a harmonic morphism, and Fuglede notes that injective harmonic morphisms exist only between Riemannian manifolds of the same dimension [5]. Thus every harmonic diffeomorphism is an isomorphism in the category BHarm.

($\Leftarrow$) Suppose f is a harmonic morphism with harmonic inverse f^{-1}. Then the inverse must be differentiable by (2.5), and f must be a bijection for f^{-1} to be well defined, so f is a diffeomorphism.
Thus the isomorphisms in BHarm are precisely the harmonic diffeomorphisms.

□

Of course, the isomorphisms in BSheaf are precisely the "bijections" between sheaves, however there is a deeper implication to the diffeomorphisms in BHarm when we consider the objects of each sheaf. Remember that the direct image functor f_* induced by a harmonic morphism $f : M \to N$ maps the sheaf objects $F(V)$ to linear transformations $F(V) \to F(U)$ for U, V open in M, N, respectively (such that $f(U) = V$).

Theorem 3.2. *Suppose that* $f : M \to N$ *is a harmonic diffeomorphism. Then the direct image functor* f_* *maps the objects of* $F(N)$ *to vector space isomorphisms.*

Proof. Let U, V be open sets of M, N, respectively, and $g, h \in F(V)$ such that $g \circ h = h \circ f$ on $\mathrm{im} f$. Via (2.8), f is an open map, so $g - h$ is zero on an open set. Thus, via (2.2), we have that $g = h$. Thus $f_* F(V)$ is injective. The same argument holds for the direct image functor f_*^{-1} induced by the inverse f^{-1}. Thus by Cantor-Schröder-Bernstein theorem, $f_* F(V) \cong f_*^{-1} F(U)$, and so the injectivity of f_* implies it is bijective. Thus f_* is a vector space isomorphism. □

4 Harmonic Morphisms and the Dirichlet Problem

The Dirichlet problem is an important object in the study of harmonic functions. The problem is stated as follows.

Given a Riemann integrable function ϕ on the boundary of an open connected set $\Omega \subseteq \mathbb{C}$, find a harmonic function u on Ω that extends continuously to the boundary so that $u = \phi$ on $\partial\Omega$.

For the remainder of this section we will consider such domains Ω to be subsets of the complex plane with smooth boundary, and any manifolds will be assumed to be of dimension 2. Via the Riemann mapping theorem, we have that there is a conformal map $\Omega \to \mathbb{D}$ (note that conformal maps on the complex plane are harmonic morphisms), where $\mathbb{D}$ denotes the Poincaré disk. Thus the problem can be restated as "Given a $\phi \in \mathcal{R}[-\pi, \pi]$, find a harmonic function u on $\mathbb{D}$ such that $u(e^{i\theta}) = \phi(\theta)$ for $\theta \in [-\pi, \pi]$". In this case, the solution to the Dirichlet problem is given by the integral

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \phi(t)\, dt.$$

The map $T : \mathcal{R}[-\pi, \pi] \to F(\mathbb{D})$ that takes Riemann integrable functions to their Dirichlet solutions on $\mathbb{D}$ is a linear transformation, provided u is assumed to be bounded. The Dirichlet solution u to ϕ on $\mathbb{D}$ is denoted in general as $u(z) = T_\phi(z)$. Since the bounded solutions to the Dirichlet problem are unique, it follows that T is injective onto a subset of the space $F(\mathbb{D})$. Similarly, if $u \in F(\mathbb{D})$ is a bounded harmonic function on $\mathbb{D}$ that extends continuously to $\mathbb{D}$, we have that u is the unique Dirichlet solution to $u|_{\partial\mathbb{D}}$. Thus T is an injective linear map from the set of Riemann integrable functions on $[-\pi, \pi]$ to the subspace of bounded harmonic functions on $\mathbb{D}$ that extend continuously to $\partial\mathbb{D}$.

4.1 Solutions to the Dirichlet Problem on Riemannian Manifolds

Here is described a method of solving the Dirichlet problem on Riemannian manifolds of dimension 2. For ease of explanation, we will assume slightly stronger conditions than are necessary for solving the Dirichlet problem on a Riemannian manifold. Thus weaker conditions may be assumed, however as such, they greatly increase the difficulty of the following proof. Additionally, the proof is designed to avoid the concept of measuring distance on the manifold, as the topic delves into subjects not discussed in this paper, namely the calculus of variations.

Consider a Riemannian manifold M of dimension 2. Let $\gamma \in C^\infty[-\pi, \pi]$ define a smooth simple closed curve in M, so that imγ encloses (that is, is the boundary of) a connected open subset V of M. A 1916 result by Lichtenstein[7] states that every Riemannian 2-fold admits conformal local coordinates in $\mathbb{R}^2$. Of course, this allows one to also choose conformal local coordinates in $\mathbb{C}$. Thus we will choose a conformal coordinate chart $f : \Omega \to \mathbb{C}$ for some open set Ω such that $\overline{V} \subseteq \Omega$. Since every conformal map is trivially semiconformal as well, and f is necessarily a smooth diffeomorphism, we have by (2.7) that f is a harmonic morphism, and by (2.8) we have that the image $f(V)$ is an open connected subset of the complex plane. Therefore, via the Riemann mapping theorem, there is a conformal holomorphic bijection $f(V) \to \mathbb{D}$. Such a map is again a harmonic morphism (via 2.7), and compositions of harmonic morphisms are harmonic morphisms. Thus we will suppose *wlog* that f maps V to the Poincaré disk. Next let $\psi : \partial V \to \mathbb{R}$ be a continuous function on ∂V. Choose a function ϕ on $\partial\mathbb{D}$ such that $\phi \circ f = \psi$ on ∂V. Then we can solve the Dirichlet problem for ϕ on $\mathbb{D}$, and the solution is given by

$$u(z) = \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} (\phi \circ f \circ \gamma)(t)\, dt.$$

Note that because f is a harmonic morphism, $u \circ f$ is harmonic on V trivially. Next choose an arbitrary sequence $x_n \to x$, $x_n \in \mathbb{D}$, $x \in \partial\mathbb{D}$. f is a diffeomorphism, so it is bijective, implying that for every such sequence there is a unique associated sequence $y_n \to y$, $y_n \in V$, $y \in \partial V$ such that $f(y_n) = x_n$ for every $n \in \mathbb{N}$, and so by continuity $f(y_n) \to x$ (note that convergence of this sequence relies on some metric having been determined on M, however the specifics of this metric are not important). Therefore, since u is a solution to the Dirichlet problem on $\mathbb{D}$, we have that $(u \circ f)(y_n) \to (\phi \circ f)(x) = \psi(x)$ as $n \to \infty$. The sequence x_n (and therefore the associated y_n) is arbitrary, so $u \circ f$ extends continuously to ψ on ∂V by sequential continuity. Thus $u \circ f$ is a solution to the Dirichlet problem on V for ψ. $\square$

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