THE GAMMA FUNCTION:
AN ELEGANT REPRESENTATION OF THE N-BALL

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Abstract. The main purpose of this paper is to derive the formula for the volume of the n-ball in terms of the Gamma function, and show that the volume of the unit n-ball decreases to 0 as the dimension gets large. Other properties and consequences are also stated and shown. The reader should be familiar with a basic understanding of Real Analysis in Euclidean spaces, the Gamma function, and Linear Algebra.

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1. Introduction

The volume of simple n-dimensional objects such as the n-ball can become quite the tedious and difficult problem for students learning multivariable calculus. However, with some knowledge about real analysis in Euclidean spaces, this calculation can be made with a simple with a simple change of coordinates and implementation of the Gamma function. However, to understand why such a function can be used, this is where the importance of having a grasp of real analysis in Euclidean spaces lies.

Since the volume of an n-ball can be represented in terms of the Gamma function, this yields a simple representation with some interesting consequences that come easily from the presence of the Gamma function. However, most of these properties are inconsequential and are only interesting because these are properties you might not think about n-dimensional geometry as the dimension gets large.

This paper will be following the 3rd method from D. J. Smith's *How Small Is a Unit Ball?*[2] to derive the volume of an n-ball. This derivation is simply a change of coordinates of the integral that describes the volume of set into hyper-spherical. Other interesting properties, such as the sum of all n-balls of every dimension being convergent for fixed values comes from Smith's article as well. Other properties such as the recursive formula from J. A. Scott's *The Volume of the n-Ball*[4], and that the volume of the unit n-ball vanishes as the dimension gets large from H. R. Parks' *The Volume of the Unit n-Ball*[3] come from their respective articles. This paper will also use material established by G. B. Folland's *Advanced Calculus*[1], as many of its definitions and theorems, and will be used as standard throughout the paper.

2. Definitions and Clarifications

Notation: x may denote a number or a vector, depending on what set of which x is an element of. The "absolute value bars" ($|x|$) will be used interchangeably with the "norm bars" ($||x||$), and all norms will be the Euclidean 2 norm, unless otherwise specified. So context is important when interpreting the symbols in this paper (vectors don't have an ordering, so if $x > 0$, then x is a real number).

Definition 2.1. The ***n*-ball** of radius $r > 0$ centered at x_0 is the set of all points for $x \in \mathbb{R}^n$ such that $|x - x_0| < r$. In set notation, we denote this set as $B_n(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r,\ 0 < r, x_0 \in \mathbb{R}^n\}$.

Note that we will only be concerning ourselves with balls centered about the origin, which we will denote as $B_n(r)$. With this definition, we then have that a 1-ball is an open interval that is symmetric about a point with the length $2r$. That is, a 1-ball centered at the origin is $(-r, r)$. The 2-ball is then the disk of radius r. It is important to note that balls are open sets, so they are not circles or spheres, balls are instead the points contained by circles (being disks), spheres (being balls), or n-spheres.

Definition 2.2. The **volume** of a set S is

$$\mathrm{Vol}(S) = \int_S 1\, dV.$$

This means that the area of disk (volume of a 2-ball) is $\int_0^R \int_0^{2\pi} r\, d\theta dr = 2\pi R^2$ (after changing our coordinate system into polar). The area of the rectangle $[0, a] \times [0, b]$ is $\int_0^a \int_0^b 1\, dydx = ab$. The volume of the unit n-cube would then be $\int_0^1 \cdots \int_0^1\, d^n x = 1$.

Definition 2.3. We define **hyper-spherical coordinates** to be the coordinate system

$$x_1 = t\prod_{j=1}^{n-1} \sin\theta_j, \qquad x_k = t\cos\theta_{k-1}\prod_{j=k}^{n-1} \sin\theta_j \ \text{ for } 2 \le k \le n-1, \qquad x_n = t\cos\theta_{n-1}$$

where $0 \le t \le r$, $0 \le \theta_1 \le 2\pi$, and $0 \le \theta_j \le \pi$ for $2 \le j \le n-1$.

In $\mathbb{R}^3$, this corresponds to the spherical coordinates $(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$ where $\theta = \theta_1$ and $\phi = \theta_2$.

Definition 2.4. The Gamma Function is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt.$$

and satisfies the following properties for any $n \in \mathbb{N}$

$$\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(1) = 1, \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi},$$
$$\Gamma(n) = (n-1)!, \qquad \Gamma(n + \tfrac{1}{2}) = (n - \tfrac{1}{2}) \cdots \tfrac{3}{2}\tfrac{1}{2}\sqrt{\pi}.$$

It should be noted that the gamma function converges for all positive x, and converges uniformly for $\delta \le x \le C$ where $C \ge \delta > 0$ (which is to say on compact subsets), and that the gamma function is defined for $-x < 0$ when x is not a natural number, however, we are not concerned with that property. What is important, and most notable about the gamma function is that it's a continuous extension of the factorial function. Moreover, we can define Γ in terms of complex numbers, but that is beyond the scope of this paper. In order to arrive at a gamma function representation for the volume of the n-ball, another function is needed.

Definition 2.5. The Beta Function is defined for $x, y > 0$ by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

and satisfies

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Definition 2.6. We define **Stirling's Formula** as

$$\lim_{x\to\infty} \frac{\Gamma(x)}{x^{x-(1/2)e^{-x}}} = \sqrt{2\pi}$$

and **Stirling's Approximation** as

$$\Gamma(x) \sim \sqrt{2\pi} x^{x-(1/2)} e^{-x}.$$

Note that as defined, Stirling's formula and approximation are two different statements, as the approximation tells us that as $x \to \infty$ that the ratio tends towards 1, the difference however will still tend towards ∞, which is to say that the approximation does not become exact as $x \to \infty$. However, Stirling's approximation can be used for estimates. The formula is the more precise statement, but the approximation will be more useful in this paper, since we might want to make an estimate with Γ before taking its argument to ∞. However, Stirling's formula and approximation are generally regarded as the same statement. An important remark is that Stirling's approximation behaves as follows:

$$\Gamma(\tfrac{x}{2}) \sim \sqrt{2\pi}\left(\frac{x}{2}\right)^{(x-1)/2} e^{-x/2}$$

$$\Gamma(\tfrac{x}{2}+1) = (\tfrac{x}{2})\Gamma(\tfrac{x}{2}) \sim \frac{x}{2}\sqrt{2\pi}\left(\frac{x}{2}\right)^{(x-1)/2} e^{-x/2} = \sqrt{\pi x}\left(\frac{x}{2e}\right)^{x/2}$$

3. Derivation of the Formula

Lemma 3.1.

$$\prod_{k=1}^{n} \int_0^\pi \sin^k \theta\, d\theta = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

Proof. For this proof, we will use induction. So first consider the base case $n = 1$. Then we obtain

$$\int_0^\pi \sin\theta\, d\theta = 2 = \frac{\pi^{1/2}}{\frac{1}{2}\pi^{1/2}} = \frac{\pi^{1/2}}{\frac{1}{2}\Gamma(\frac{1}{2})} = \frac{\pi^{1/2}}{\Gamma(\frac{3}{2})} = \frac{\pi^{1/2}}{\Gamma(\frac{1}{2}+1)}.$$

With the base case proved, we now move onto the inductive step, where we assume that our hypothesis is true for $n-1$, and will show that it holds for n. So for $n \geq 2$ note that

$$\int_0^\pi \sin^n\theta\, d\theta = 2\int_0^{\pi/2} \sin^n\theta\, d\theta$$

since sine is symmetric about $\theta = \frac{\pi}{2}$. Now we will preform a change of coordinates with $\sin\theta = \sqrt{t}$ (informally this is a u substitution where "$\frac{1}{2}t^{-1/2}dt = -\cos\theta d\theta$"). This changes our bounds of integration from $(0, \frac{\pi}{2})$ to $(0, 1)$. Note that since $\sin\theta = \sqrt{t}$, it follows that we're considering a triangle with one side being of length t, and the hypotenuse being of length 1. This means that $\cos\theta = \sqrt{1-t}$, yielding

$$2\int_0^{\pi/2} \sin^n\theta \, d\theta = 2\int_0^1 \frac{t^{n/2}}{2t^{1/2}(1-t)^{1/2}} \, dt = \int_0^1 t^{(n-1)/2}(1-t)^{-1/2} \, dt.$$

However, note that what we have now is in terms of the beta function. So we now obtain

$$\int_0^1 t^{(n-1)/2}(1-t)^{-1/2} \, dt = \int_0^1 t^{(n+1)/2-1}(1-t)^{1/2-1} \, dt = \beta(\tfrac{n+1}{2}, \tfrac{1}{2}) = \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2}+\frac{1}{2})}$$

Thus, if we now consider the product of the n terms we now see by our inductive hypothesis and what we've just shown that

$$\begin{aligned}\prod_{k=1}^{n}\int_0^{\pi} \sin^k\theta \, d\theta &= \prod_{k=1}^{n-1}\int_0^{\pi} \sin^k\theta \, d\theta \int_0^{\pi} \sin^n\theta \, d\theta \\ &= \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2}+1)} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2}+1)} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\end{aligned}$$

□

Before continuing, it should be noted that in the following proof we'll be using a change of coordinates into hyper-spherical from Cartesian. More precisely, we'll be using theorem 4.41 from page 183 of Folland (with Folland's notation where $\mathbf{x}$ denotes a vector, which means that $\mathbf{G}$ is a function whose outputs are vectors):

Given open sets U and V in $\mathbb{R}^n$, let $\boldsymbol{G} : U \to V$ be a one-to-one transformation of class C^1 whose derivative $D\boldsymbol{G}(\boldsymbol{u})$ is invertible for all $\boldsymbol{u} \in U$. Suppose the $T \subset U$ and $S \subset V$ are measurable sets such that $\boldsymbol{G}(T) = S$. If f is an integrable function on S, then $f \circ \boldsymbol{G}$ is integrable on T, and

$$\int \cdots \int_S f(\mathbf{x}) \, d^n\mathbf{x} = \int \cdots \int_T f(\mathbf{G}(\mathbf{u})) |\det(D\mathbf{G}(\mathbf{u}))| \, d^n\mathbf{u}$$

When preforming the change of coordinates to polar for a disk, S is the region of the disk in Cartesian, and $\mathbf{G}(r,\theta) = (r\sin\theta, r\cos\theta)$ is the transformation such that $\mathbf{G}(T) = S$. $\det D\mathbf{G} = r$ in this case.

Theorem 3.2. *The volume of the n-ball is given by*

$$V_n = \frac{r^n \pi^{n/2}}{\Gamma(\frac{n}{2}+1)} = \begin{cases} \frac{\pi^m}{m!}, & \text{for } n = 2m \\ \frac{2^{2m+1}\pi^{m/2}m!}{(2m+1)!}, & \text{for } n = 2m+1 \end{cases}$$

Proof. Let f be the map from hyper-spherical to Cartesian, that is, let

$$f(t, \theta_1, ..., \theta_{n-1}) = (t\prod_{j=1}^{n-1}\sin\theta_j, t\cos\theta_1\prod_{j=2}^{n-1}\sin\theta_j, ..., t\cos\theta_{k-1}\prod_{j=k}^{n-1}\sin\theta_j, ..., t\cos\theta_{n-1})$$

and note that when differentiating f, the first column is found by differentiating all the component functions by the first variable, that is $\frac{\partial}{\partial t}x_n = x_n/t$. The subsequent differentiating with respect to the other θ_k variables will cause the $\sin\theta_k$ to become $\cos\theta_k$ and $\cos\theta_k$ to become $-\sin\theta_k$. So we can express $\frac{\partial}{\partial\theta_k}x_n = x_n\cos\theta_k/\sin\theta_k$ for the first kth component functions of f, while the $k+1$st will be $\frac{\partial}{\partial\theta_k}x_n = -x_n\sin\theta_k/\cos\theta_k$, and the remaining $n-(k+1)$st terms will be 0, since those component functions don't depend on θ_k. Explicitly

$$Df = \begin{bmatrix} x_1/t & x_1\cos\theta_1/\sin\theta_1 & x_1\cos\theta_2/\sin\theta_2 & \dots & x_1\cos\theta_{n-1}/\sin\theta_{n-1} \\ x_2/t & -x_2\sin\theta_1/\cos\theta_1 & x_2\cos\theta_2/\sin\theta_2 & \dots & x_2\cos\theta_{n-1}/\sin\theta_{n-1} \\ x_3/t & 0 & -x_3\sin\theta_2/\cos\theta_2 & \dots & x_3\cos\theta_{n-1}/\sin\theta_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ x_n/t & 0 & 0 & \dots & -x_n\sin\theta_{n-1}/\cos\theta_{n-1} \end{bmatrix}$$

For clarity we will express $s_j = \sin\theta_j$, $c_j = \cos\theta_j$, and $k_j = \cot\theta_j$. Thus, with this condensed notation we can expand Df, and now consider $|Df|$ with more clarity than Df,

$$|Df| = \begin{vmatrix} x_1/t & x_1k_1 & x_1k_2 & x_1k_3 & \dots & x_1k_{n-1} \\ x_2/t & -x_2/k_1 & x_2k_2 & x_2k_3 & \dots & x_2k_{n-1} \\ x_3/t & 0 & -x_3/k_2 & x_3k_3 & \dots & x_3k_{n-1} \\ x_4/t & 0 & 0 & -x_4/k_3 & \dots & -x_n/k_{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_n/t & 0 & 0 & \dots & \dots & -x_n/k_{n-1} \end{vmatrix}$$

Now by pulling out t^{-1} from the first column, k_j from the $j-1$st column, and x_j from the jth row, we now obtain

$$|Df| = t^{-1}\prod_{j=1}^{n} x_j \prod_{j=1}^{n-1} k_j \begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & -k_1^{-2} & 1 & 1 & \dots & 1 \\ 1 & 0 & -k_2^{-2} & 1 & \dots & 1 \\ 1 & 0 & 0 & -k_3^{-2} & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots & -k_{n-1}^{-2} \end{vmatrix}$$

Then by preforming elementary column operations, (subtracting the first column from all the other columns), we then obtain

$$|Df| = t^{-1}\prod_{j=1}^{n} x_j \prod_{j=1}^{n-1} k_j \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & -(1+k_1^{-2}) & 0 & 0 & \dots & 0 \\ 1 & -1 & -(1+k_2^{-2}) & 0 & \dots & 0 \\ 1 & -1 & -1 & -(1+k_3^{-2}) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & -1 & -1 & -1 & \dots & -(1+k_{n-1}^{-2}) \end{vmatrix}$$

However, note that we now have a lower triangular matrix, and so, it's determinate is now just the product of its diagonal terms

$$|Df| = (-1)^{n-1}t^{-1}\prod_{j=1}^{n} x_j \prod_{j=1}^{n-1} k_j \prod_{j=1}^{n-1}\Big(1+\frac{1}{k_j^2}\Big).$$

Note that the reason we are calculating this determinant, is so we can apply our change of coordinates into hyper-spherical, and also note that we will take the absolute value of the determinant of the transformation's derivative, so we can ignore the $(-1)^{n-1}$ term (more precisely we have $\det Df$, and the transformation by the theorem is $|\det Df|$, so we were using $|Df|$ to mean $\det Df$ as it should, but now we're abusing the notation to have $|Df|$ mean $|\det Df|$). Next, recall that $k_j = \cot\theta_k$, and so $1 + k_j^{-2} = c_j^{-2}$, and thus, $k_j c_j^{-2} = (s_j c_j)^{-1}$. Thus, we obtain

$$|Df| = t^{-1} \prod_{j=1}^{n} x_j \prod_{j=1}^{n-1} \frac{1}{\sin\theta_j \cos\theta_j}.$$

However, note that since we're in hyper-spherical it follows that

$$\prod_{j=1}^{n} x_j = t^n \prod_{j=1}^{n-1} \cos\theta_j \prod_{j=1}^{n-1} \sin^j \theta_j.$$

Thus,

$$|Df| = t^{-1} t^n \prod_{j=1}^{n-1} \cos\theta_j \prod_{j=1}^{n-1} \sin^j \theta_j \prod_{j=1}^{n-1} \frac{1}{\sin\theta_j \cos\theta_j} = t^{n-1} \prod_{j=2}^{n-1} \sin^{j-1} \theta_j.$$

Note that the volume of the n-ball of radius $r > 0$, is given by

$$V_n(r) = \int_{B_n(r)} 1 \, dV$$

and by our change of coordinates into hyper-spherical, this yields

$$\begin{aligned} V_n(r) &= \int_0^r \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} |Df| \, d\theta_{n-1} \cdots d\theta_2 d\theta_1 dt \\ &= \int_0^r \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} t^{n-1} \prod_{2}^{n-1} \sin^{j-1} \theta_j \, d\theta_{n-1} \cdots d\theta_2 d\theta_1 dt. \end{aligned}$$

However, note that since each variable appears once in the integrand, it follows that any we can take any permutation of these integrals and obtain the same value. Moreover, we can express our iterated integral as a product of integrals.

$$V_n(r) = \int_0^r t^{n-1} \, dt \int_0^{2\pi} 1 \, d\theta_1 \prod_{j=1}^{n-2} \int_0^{\pi} \sin^j \theta_{j+1} \, d\theta_{j+1}.$$

Moreover, we can express each θ_{j+1} as u, since they're just a dummy variable for integration. We then obtain

$$V_n(r) = \frac{r^n}{n} 2\pi \prod_{j=1}^{n-2} \int_0^{\pi} \sin^j u \, du$$

which by Lemma 3.1 is

$$V_n(r) = \frac{2\pi r^n}{n} \frac{\pi^{(n-2)/2}}{\Gamma(\frac{n-2}{2} + 1)} = \frac{r^n \pi^{n/2}}{(\frac{n}{2})\Gamma(\frac{n}{2})} = \frac{r^n \pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}.$$

Now that we have the volume of the n-ball, we can now find its values for even and odd dimensions. So for $n = 2m$ we see that

$$V_{2m}(r) = \frac{r^{2m} \pi^m}{\Gamma(m+1)} = \frac{r^{2m} \pi^m}{m!}$$

and for $n = 2m + 1$ we see that

$$V_{2m+1}(r) = \frac{r^{2m+1}\pi^{m+1/2}}{\Gamma(m+\frac{1}{2}+1)} = \frac{r^{2m+1}\pi^{m}\pi^{1/2}}{(m+\frac{1}{2})\Gamma(m+\frac{1}{2})} = \frac{r^{2m+1}\pi^{m}}{(m+\frac{1}{2})(m-\frac{1}{2})\cdots(\frac{3}{2})(\frac{1}{2})}$$
$$= \frac{2^{2m+1}r^{2m+1}\pi^{m}}{(2m+1)(2m-1)\cdots(3)(1)} = \frac{2^{2m+1}\pi^{m}m!}{(2m+1)!}$$

□

4. Properties of the n-ball

The first noteworthy consequence that can be seen from the formula for the volume of the n-ball is that volume of the n-ball for any fixed $r > 0$ tends to 0, as $n \to \infty$. Intuitively, Γ is a function that represents factorials on a continuum, and so, it would make sense that a factorial would eventually beat out an exponent. However, this statement can be made a little more precise and stronger with the following theorem:

Theorem 4.1. *The volume of the unit n-ball tends to 0 as n tends towards infinity. Moreover, the volume of the n-ball of radius* $1 \geq r > 0$ *converges uniformly to 0.*

We will actually provide two arguments that show the unit n-ball tending towards 0, as n increases. Both require thinking about n-dimensional geometry, but the first argument uses calculus and Stirling's approximation, while the second does not.

Proof. **ARGUEMENT 1:** First we will show that the unit n-ball can be contained in a polyhedral set. In more sensible dimensions, we're going to be showing that a disk can be contained inside a convex polygon (which will be a set where the lengths of the components sum to value less than the root of the dimension). Let $x = (x_1, ..., x_n)$, and consider the set of points

$$\sum_{j=1}^{n} x_j^2 \leq 1$$

where each $x_j \geq 0$ (note this isn't exactly the n-ball, but something close), it follows that each $x_j \leq 1$. So now we consider $\sum_{j=1} x_j = x \cdot (1, ..., 1)$, which by the Cauchy-Schwarz inequality yields $\sum_{j=1} x_j \leq |x||(1, ..., 1)| = (\sum_{j=1}^{n} x_j^2)^{1/2} \leq \sqrt{n}$. So it follows that the unit n-ball is contained in the polyhedral set

$$\{x : \sum_{j=1}^{n} |x_j| \leq \sqrt{n}\}.$$

Note that we have to take the absolute value of of the x_j's here since these components of the the unit n-ball may lay in the negative component axis. So the next geometrical fact we'll be using is that we can construct this polyhedral set, which is convex, as the union of 2^n congruent copies of the n-simplex. Again, in lower dimensions, this would be equivalent to saying that we can construct a polygon as union of 2^n triangles for an appropriate n. Think of the simple case of hexagon being comprised of 2^3 equilateral triangles (note we won't have enough symmetry to have a property as nice as equilateral in actuality). What matters most at this point is that we've bound the unit n-ball by 2^n congruent n-simplexes. We can determine the volume of the n-simplex by joining a point called that apex to the base of the $n-1$ simplex. The 0-simplex is a point, 1-simplex is a line, 2-simplex is a triangle, and 3 is a pyramid. To find the volume of the line we pick a point on our new axis (in this case the positive real number line), extend a line between the base (the original point) and apex (the new point) and integrate. For the triangle, we pick a point on the positive y-axis, connect a line to the original points from the 1-simplex, and integrate the region they bound. A similar process for the pyramid and further simplexes is done. Note the way we showed how to find the volume was assuming that we were always doing so in the n-dimensional equivalent to the first quadrant/first octant. From Parks' article [3] we are given that the volume of an n-simplex is bh/n, where b is the base, h is the "height" of the apex, and n is the dimension (in $\mathbb{R}^2$ we have $\frac{1}{2}bh$) and that the volume of 2^n congruent n-simplexes is $n^{n/2}/n!$. Thus,

$$V_n(1) \leq \frac{2^n n^{n/2}}{n!}.$$

By Stirling's approximation we obtain,

$$\frac{2^n n^{n/2}}{n!} \sim \frac{2^n n^{n/2}}{\sqrt{2\pi} n^{n-(1/2)} e^{-n}} = \frac{2^{n-(1/2)} e^n}{\sqrt{\pi} n^{(n-1)/2}}$$

Normally with an exponential in the numerator we would expect this to be unbounded, however, n^n trivially dominates over the exponential (note when $n \geq 5$). Thus, it follows that as $n \to \infty$, $V_n(1) \to 0$.

ARGUMENT 2: We're going to show that the n-ball is contained within the rectangular solid $[-1,1]^4 \times [-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}]^{n-4}$, and proceed from there. Note that for $n = 4$, this is trivially true, but for $n \geq 5$, we make the argument by contradiction. Let the all the points in the unit n-ball be contained in the sets $B = \{j : |x_j| > \frac{1}{\sqrt{5}}\}$ and $S = \{j : |x_j| \leq \frac{1}{\sqrt{5}}\}$ for $x = (x_1, ..., x_n)$. Note that if B had any more than 4 elements, then it would follow that

$$|x| = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2} > \left(5\left(\frac{1}{\sqrt{5}}\right)^2\right)^{1/2} = 1$$

which contradicts that these points describe the unit n-ball. Thus, we have that the unit n-ball is contained within the rectangular solid $[-1,1]^4 \times [-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}]$. Moreover, the unit n-ball is contained within $\binom{n}{4}$ congruent copies of the rectangular solid $[-1,1]^4 \times [-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}]^{n-4}$, which is to say, no matter what solid we're considering, the ordering in where component axis that are in the interval $[-1,1]$ doesn't matter, and we can then only concern ourselves with the $[-1,1]^4 \times [-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}]^{n-4}$ rectangular solid. This is to say, that the rectangle $[0,1] \times [0,2]$ is congruent to $[0,2] \times [0,1]$, so we will just consider the rectangle $[0,1] \times [0,2]$. So the volume of the rectangular solid that contains the unit n-ball has the volume

$$2^4\left(\frac{2}{\sqrt{5}}\right)^{n-4} = \frac{2^n}{5^{n/2}5^{-4/2}} = \frac{25}{(\sqrt{5}/2)^n}.$$

Note that

$$\binom{n}{4} = \frac{n!}{4!(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{24} = \frac{n^4 - 6n^3 + 11n^2 - 6n}{24}$$

and that we can reduce the polynomial to $n^4 - n(6n^2 - 11n + 6)$ where the term on the right is non-positive for $n \geq 4$, so it follows that $24\binom{n}{4} \leq n^4$, and so it follows that the volume of $\binom{n}{4}$ rectangular solids that bound the unit n-ball has the following upper bound that is still dependent on n,

$$\frac{25}{24}\frac{n^4}{(\sqrt{5}/2)^n}.$$

So it follows that since $\sqrt{5} > 2$, that as $n \to \infty$,

$$0 \le V_n(1) \le \binom{n}{4}\frac{25}{(\sqrt{5}/2)^n} \le \frac{25}{24}\frac{n^4}{(\sqrt{5}/2)^n} \to 0.$$

Thus, the volume of the unit n-ball tends to 0 as n tends to ∞.

Thus, by both arguments it follows that the volume of the unit n-ball, $V_n(1)$, tends to 0, as n tends to ∞. For the remaining statement of the theorem, consider $0 < r \le 1$ that $0 \le V_n(r) \le V_n(1)$. Since $V_n(1) \to 0$ as $n \to \infty$, and our upper bound is independent of r, that $V_(r)$ converges uniformly to 0 for $0 < r \le 1$.

□

Next consider the following informal discussion about the convergence of $\sum V_n(r)$. Since the gamma function is the continuation of the factorial function, it should follow that for any fixed $r > 0$, that the sequence of function $V_n(r)$ will eventually be less than $\frac{1}{n!}$. This shows that by comparison that $\sum V_n(r)$ converges pointwise, and for some restricted r, uniformly by the Weierstrass M-test. However, we won't be able to achieve a global uniform bound, and so only have this type of convergence on subsets. This leads to the following theorem:

Theorem 4.2. *The series $\sum_{n=0}^{\infty} V_n(r)$ converges pointwise for all $r > 0$, and uniformly on compact subsets.*

Proof. Instead we will consider the convergence of

$$\sum_{n=0}^{\infty} n^{(n+1)/2} V_n(r).$$

From the remark made about Stirling's approximation, it follows that

$$V_n(r) = \frac{r^n \pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \sim \frac{r^n \pi^{n/2}}{\sqrt{\pi n}(\frac{n}{2e})^{n/2}} = \frac{r^n (2\pi e)^{n/2}}{n^{(n+1)/2}}\sqrt{\pi}.$$

So it follows that

$$\left[n^{(n+1)/2} V_n(r)\right]^{1/n} \sim \left[\frac{r^n (2\pi e)^{n/2}}{\sqrt{\pi}}\right]^{1/n} = \frac{r(2\pi e)^{1/2}}{\pi^{1/2n}}.$$

Note that as n tends towards ∞, that we find that our expression tends to $r\sqrt{2\pi e}$. So it follows by the root test (since all the terms involved are nonnegative) that

$$\sum_{n=0}^{\infty} n^{(n+1)/2} V_n(r)$$

converges for all $r < (2\pi e)^{-1/2}$ and diverges for all $r > (2\pi e)^{-1/2}$. The equality then tells us that we're considering

$$n^{(n+1)/2} V_n(\frac{1}{\sqrt{2\pi e}}) \sim \frac{1}{\sqrt{\pi}}$$

which as $n \to \infty$ does not tend to 0, which tells us that this series diverges for $r = (2\pi e)^{-1/2}$. Note that for all $n \in \mathbb{N}$ it follows that

$$0 \leq V_n(r) \leq n^{(n+1)/2} V_n(r)$$

So it follows by comparison that $\sum V_n(r)$ converges for all $0 < r < (2\pi e)^{-1/2}$. However, there more that can be said. Let $0 < R < (2\pi e)^{-1/2}$ be fixed. Then it follows that for each fixed r, there exists an N_r dependent on r, such that, when $n \geq N_r$ we have that $0 \leq V_n(r) \leq n^{(n+1)/2} V_n(R)$. Note that we can assert this since we're choosing each N_r for each value of r, meaning the convergence for all $r > 0$ for $\sum V_n(r)$ is not uniform, but pointwise. However, for uniform convergence, assume that $0 < r \leq C$ where C is finite. Then it follows that $0 \leq V_n(r) \leq V_n(C)$ where $V_n(C)$ converges for any finite C as argued above. Thus, but the Weierstrass M test, $\sum V_n(r)$ converges uniformly on compact subsets of $\mathbb{R}^n$.

□

Again, it should be noted that the $\sum V_n(r)$ on converges pointwise, and uniformly on compact subsets. This is to say, that we cannot assert that globally $\sum V_n(r)$ converges for any $r > 1$. We do have as a corollary to theorem 4.2 that the volume of the unit n-ball tends to 0, as n tends towards ∞. Now onto something interesting. A neat little coincidence occurs when considering the series, but only with even dimensions.

Theorem 4.3. $\sum_{n=1}^{\infty} V_{2n}(r) = e^{\pi r^2}$.

Proof. The proof is rather cute. Note that

$$V_{2n}(r) = \frac{r^{2n}\pi^n}{\Gamma(n+1)} = \frac{r^{2n}\pi^n}{n!}.$$

Now if we substitute this into our series we obtain

$$\sum_{n=1}^{\infty} V_{2n}(r) = \sum_{n=1}^{\infty} \frac{\pi^n r^{2n}}{n!} = \sum_{n=1}^{\infty} \frac{(\pi r^2)^n}{n!}.$$

If the last equality looks familiar, that's because its the power series expansion of $e^{\pi r^2}$ ($e^x = \sum \frac{x^n}{n!}$).

□

Finally we go back to the unit n-ball. The volume of the unit n-ball satisfies a recursive property. This recursive property doesn't even require an explicit formula for $V_n(1)$ (which will be abbreviated as V_n). However, from Lemma 3.1, we can actually verify that the volume of the unit n-ball is given by $\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ if somehow that when trying to derive the volume for the unit n-ball you decided to use a non-algebraic function, and guess what it might be. However, we will require that we define for $n \in \mathbb{N}$, $I_n = \int_0^{\pi/2} \sin^n x \, dx$, and the following lemmas:

Lemma 4.4. *For $n \in \mathbb{Z}$ such that $n \geq 0$, $nI_n = (n-1)I_{n-2} = \pi$*

Proof. We will proceed with integration-by-parts (where informally "$u = \sin^{n-1} x$" and "$dv = \sin\ dx$")

$$I_n = \int_0^{\pi/2} \sin^n x\ dx = -(n-1)\sin^{n-1} x \cos x\Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}\cos^2 x\ dx$$
$$=(n-1)\int_0^{\pi/2} \sin^{n-2} x\ dx - (n-1)\int_0^{\pi/2} \sin^n x\ dx.$$

Note that since we can move the rightmost term to the other side of the equality and obtain

$$n\int_0^{\pi/2} \sin^n x\ dx =(n-1)\int_0^{\pi/2} \sin^{n-2} x\ dx$$
$$nI_n =(n-1)I_{n-2}$$

□

Lemma 4.5. *For $n \in \mathbb{N}$, $2nI_nI_{n-2} = \pi$*

Proof. Now for $n \geq 1$ we will show by induction $2nI_nI_{n-1} = \pi$. The base case $n = 1$ yields

$$2I_1I_0 = 2\int_0^{\pi/2} \sin x\ dx \int_0^{\pi/2} dx = \pi\Big[-\cos x\Big]_0^{\pi/2} = \pi.$$

So now we move onto the inductive step, where we assume that our hypothesis holds true for $n-1$, and will prove it for n: $2nI_nI_{n-1} = 2(n-1)I_{n-1}I_{n-2} = \pi$. Where the first equality comes from Lemma 4.4.

□

Theorem 4.6. *The volume of the unit n-ball satisfies the following recursive property: $nV_n = 2\pi V_{n-1}$.*

Proof. Note that for $n > 2$ we can obtain that volume of V_n by summing the scaled versions of the V_{n-1} along a single axis orthogonal to the $n-1$ ball. If we consider V_3 then this becomes more apparent. What we have is the unit disk lying in the xy-plane, and now we want to find the volume of the unit ball. To do so, the simplest way would be to sum disks of radius r over every plane parallel to the xy-plane on the continuum $[-1,1]$ on the z-axis. This would yield $V_3 = \int_{-1}^{1} r^2 V_2\ dz$ where more explicitly we see how r scales the volume of the unit disk to form a continuum of disks from $[-1,1]$ on the z-axis, and the reason why r is squared is more easily seen if we combine the r and V_2 and we see that the integrand is equivalent to $V_2(r)$. So if we generalize this idea we obtain $V_n = 2\int_0^1 r^{n-1}V_{n-1}\ dz$. r forms the following right triangle with the x_{n-1} axis, $\sqrt{r^2 + x_{n-1}^2} = 1$. If we now parametrize our integral with $r = \sin\phi$ and $z = \cos\phi$ (thus yielding $dz = -\sin\phi\ d\phi$, and our bounds change from $[0,1]$ to $[\frac{\pi}{2}, 0]$, and yes the interval is written incorrectly), we obtain

$$V_n = -2\int_{\pi/2}^{0} \sin^{n-1}\phi \sin\phi V_{n-1}\ d\phi = 2V_{n-1}\int_0^{\pi/2} \sin^n\phi\ d\phi = 2V_{n-1}I_n.$$

Note that since this process works for any V_n, it follows that this works for V_{n-1}, so by the same argument, we obtain

$$V_n = 2V_{n-1}I_n = 2V_{n-2}I_nI_{n-1}$$
$$nV_n = n2V_{n-2}I_nI_{n-1} = 2\pi V_{n-2}.$$

Where the last equality comes from Lemma 4.5.

□

Thus, by theorem 4.6 it follows that if you didn't know what the volume of the unit n-ball was, and made the guess is was $\frac{\pi^{(n)/2}}{\Gamma(\frac{n}{2}+1)}$, then if $n\frac{\pi^{(n)/2}}{\Gamma(\frac{n}{2}+1)} = 2\pi\frac{\pi^{(n-2)/2}}{\Gamma(\frac{n-2}{2}+1)}$ held, then you would have a formula for the unit n-ball.

For more derivations of the volume of the n-ball, and further readings on properties of the n-ball, the reader is encouraged to see Smith's *How Small Is a Unit Ball?* [2].

References

[1] G. B. Folland, *Advanced Calculus* (3rd ed.), Prentice Hall, Upper Saddle River, 2002.
[2] D. J. Smith, *How Small Is a Unit Ball?*, Mathematics Magazine **62**, 1989, 101-107.
[3] H. R. Parks, *The Volume of the Unit n-Ball*, Mathematics Magazine **86**, 2013, 270-274.
[4] J. A. Scott, *The Volume of the n-Ball*, The Mathematical Gazette **82**, 1998, 104-106.