Hyperbolic Geometry on the Half-Plane and Poincare Disc

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May 2018

Abstract

The extension from the comfortable Euclidean plane to a non-Euclidean space is both an attractive and a daunting one. In 1829, Lobachevsky provided the first complete "stable" version of a non-Euclidean geometry, and later mathematicians like Poincare developed different models in which these ideas operated. In this paper, we will provide an introduction to the constructs of hyperbolic geometry using two of these models. The first part will be a development of hyperbolic geometry in the plane from an analytic standpoint. We will then use these tools to develop similar ideas in the context of the complex unit disk. Finally, we will present a proof from Unger [3] of the Pythagorean Theorem in the Poincare disc.

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The proofs and ideas in the first four sections are accredited to Anderson [1] unless otherwise stated. The material in the fifth section is accredited to Gamelin [2] and the proofs and ideas in the sixth section are due to Unger [3].

1 Introduction to the Hyperbolic Plane

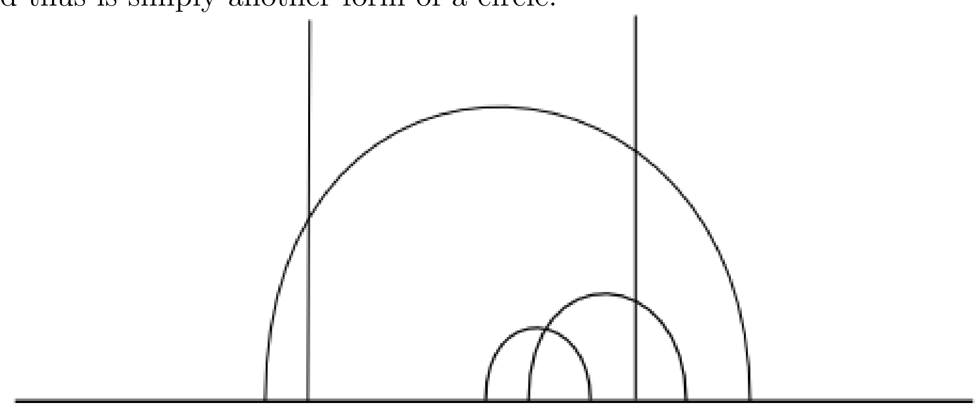
We begin with the planar construction of hyperbolic geometry and the explore what it means to have a curve on the hyperbolic plane:

Definition 1.1. *The* ***hyperbolic plane*** *is defined to be the upper half of the complex plane:*

$$\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$$

Definition 1.2. *A* ***hyperbolic line*** *is the intersection with* $\mathbb{H}$ *of a Euclidean circle centered on the real axis or a Euclidean line perpendicular to the real axis in* $\overline{\mathbb{C}}$ *(the extended complex plane* $\mathbb{C} \cup \{\infty\}$*)*

Recall that in the extended complex plane, a line is just the stereographic projection of a circle on the Riemann sphere that runs through the north pole, and thus is simply another form of a circle.



Definition 1.3. *Two hyperbolic lines are* ***parallel*** *if they are disjoint.*

So far we have not deviated too far from the realms of Euclidean geometry, having only redefined a few terms. However, the following result is where we really start to see major differences:

Theorem 1.4. *Let* l *be a hyperbolic line in* $\mathbb{H}$ *and let* p *be a point in* $\mathbb{H}$ *not on* l*. Then there exist infinitely many distinct hyperbolic lines through* p *that are parallel to* l*.*

This runs completely contrary to the parallel result (pun intended) in the Euclidean plane, where only one such line exists.

Definition 1.5. *A function* $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *is a* ***homeomorphism*** *if* f *is a bijection and if both* f *and* f^{-1} *are continuous.*

2 Mobius Transformations

Recall the familiar definition of a Mobius Transformation:

Definition 2.1. *A function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a* ***Mobius transformation*** *if it is of the form:*

$$f(z) = \frac{az+b}{cz+d}$$

for $a, b, c, d \in \mathbb{C}$

It is clear that Mobius transformations are homeomorphisms since they are a composition of translations, dilations, and inversions which are all homeomorphisms. We now present some useful properties of the Mobius transformations which are critical to their use in developing the hyperbolic metric.

Theorem 2.2. *Mobius transformations map circles in the extended complex plane to circles in the extended complex plane.*

Proof. We consider only the case of circles that do not pass through infinity (Euclidean circles) since the other case is similar and easier. It is also clear that dilations and translations map circles to circles, so we only need to show that the same is true for inversions (functions of the form $w = \frac{1}{z}$). Such a circle is of the form $|z-a|^2 = r^2$ so under the inversion this becomes:

$$0 = |1-aw|^2 - r^2|w|^2 = (1-aw)\overline{(1-aw)} - r^2|w|^2$$

$$= (|a|^2 - r^2)|w|^2 - aw - \overline{aw} + 1$$

Setting $w = u + iv$ this becomes:

$$(|a|^2 - r^2)(u^2+v^2) + Au + Bv + 1 = 0$$

Depending on whether or not $r = |a|$, this equation is either of the form of a line or a circle, both of which are circles in the extended complex plane. (Credit to Gamelin [2])

□

Definition 2.3. *For an open set* $\Omega \subset \overline{\mathbb{C}}^k$*, we say a function* $f : \Omega \to \mathbb{C}$ *is* ***invariant*** *under the Mobius transformations if for any Mobius transformation* m *the function satisfies:*

$$f(z_1, \ldots z_k) = f(m(z_1), \ldots m(z_k))$$

Definition 2.4. *Given four distinct points* $z_1, z_2, z_3, z_4 \in \mathbb{C}$*, we define the* ***cross ratio*** *of these points to be:*

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)}$$

Theorem 2.5. *Considering the cross ratio as a function from* $\mathbb{C}^4 \to \overline{\mathbb{C}}$ *then the cross ratio is invariant under the Mobius transformations.*

Theorem 2.6. *The set of all Mobius transformations is precisely the set of all homeomorphisms of* $\overline{\mathbb{C}}$ *that take circles to circles.*

We only sketch the direction of this proof since the full argument is rather long. We have already shown that Mobius transformations are homeomorphisms of $\overline{\mathbb{C}}$ that preserve circles, so it suffices to show that such a homeomorphism must be a Mobius transformaion.

We argue this by considering an arbitrary homeomorphism f and a mobius transformation p that takes the points $f(0), f(1), f(\infty)$ to $0, 1, \infty$ (since Mobius transformations uniquely map three-points to three-points). Since $p \circ f$ takes circles to circles and it takes ∞ to ∞ then it must take the real axis to itself so that $p \circ f(\mathbb{R}) = \mathbb{R}$. Therefore, since $p \circ f$ fixes the real axis then it must either map $\mathbb{H}$ to $\mathbb{H}$ or to the lower half plane. In the former case set $m = p$ and in the latter case set $m = C \circ p$ where C is the complex conjugation $\bar{z}$.

Thus, we have a Mobius transformation m so that $m \circ f(0) = 0, m \circ f(1) = 1, m \circ f(\infty) = \infty$, and $m \circ f(\mathbb{H}) = \mathbb{H}$. The rest of the proof then involves proving that $m \circ f$ is the identity z, which would imply that f is the inverse of m and hence a Mobius transformation.

Theorem 2.7. *The Mobius transformations are conformal and thus preserve angles between curves.*

Define the set of Mobius transformations which are the homeomorphisms of $\mathbb{H}$:

$$\text{Mob}(\mathbb{H}) = \{m : m(\mathbb{H}) = \mathbb{H}\}$$

Theorem 2.8. *Every element of Mob*$(\mathbb{H})$ *takes hyperbolic lines in* $\mathbb{H}$ *to hyperbolic lines in* $\mathbb{H}$

Proof. Indeed, this follows as a consequence of Theorem 2.7 which states that the elements of Mob($\mathbb{H}$) preserve angles between circles in $\overline{\mathbb{C}}$, together with the fact that every hyperbolic line in $\mathbb{H}$ is the intersection of $\mathbb{H}$ with a circle in $\overline{\mathbb{C}}$ centered on $\overline{\mathbb{R}}$, and finally that every element of Mob($\mathbb{H}$) takes circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$

□

Theorem 2.9. *Every element of Mob*$(\mathbb{H})$ *either has the form:*

$$m(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$ *or:*

$$n(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$$

where a, b, c, d *are purely imaginary and* $ad - bc = 1$.

3 Hyperbolic Metric in the Plane

We are now ready to apply our knowledge of Mobius transformations to analyze the geometry of $\mathbb{H}$. First, let us apply some of our ideas from Euclidean geometry. Given a continuous, nonzero function ρ on $\mathbb{R}^k$, recall that for some piecewise-smooth path $f : [a, b] \to \mathbb{R}^k$ we define the path length as:

$$\text{length}_\rho(f) = \int_f \rho(z)|dz| = \int_a^b \rho(f(t))|f'(t)|dt$$

Definition 3.1. *Given some continuous nonzero function ρ on $\mathbb{H}$ we say that **length is invariant under** $Mob(\mathbb{H})$ if for any piecewise smooth path $f : [a, b] \to \mathbb{H}$ and any $\gamma \in Mob(\mathbb{H})$, we have:*

$$length_\rho(f) = length_\rho(f \circ \gamma)$$

Theorem 3.2. *For every positive constant c, the element of arc length:*

$$\rho(z) = \frac{c}{Im(z)}|dz|$$

on $\mathbb{H}$ is invariant under $Mob(\mathbb{H})$.

Since we would like our arc length on $\mathbb{H}$ to be invariant under Mob($\mathbb{H}$), this leads us to the following definition:

Definition 3.3. *For a piecewise-smooth path $f : [a, b] \to \mathbb{H}$, we define the **hyperbolic length** of f to be:*

$$length_{\mathbb{H}}(f) = \int_f \frac{1}{Im(z)}|dz| = \int_a^b \frac{1}{Im(f(t))}|f'(t)|dt$$

Now that we have some concept of length, we can move forward with trying to define a metric on $\mathbb{H}$:

Definition 3.4. *A **metric** on a set X is a function:*

$$d : X \times X \to \mathbb{R}$$

*satisfying three conditions: (i) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$ (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$ (iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. We say that (X, d) is a **metric space**.*

Now, for any two points x, y let $\Gamma(x, y)$ be the set of all piecewise-smooth paths connecting x and y. Then consider the function:

$$d_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$$

defined by:

$$d_{\mathbb{H}}(x, y) = \inf\{\text{length}_{\mathbb{H}}(f) : f \in \Gamma(x, y)\}$$

Proposition 3.5. *For every element $\gamma \in Mob(\mathbb{H})$ and for every pair of points $x, y \in \mathbb{H}$, we have that:*

$$d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(\gamma(x), \gamma(y))$$

Proof. First notice that for any $f \in \Gamma(x,y)$ then $\gamma \circ f \in \Gamma(\gamma(x), \gamma(y))$. Now, since hyperbolic length is invariant under $\text{Mob}(\mathbb{H})$ we have that:

$$\text{length}_{\mathbb{H}}(\gamma \circ f) = \text{length}_{\mathbb{H}}(f)$$

for every path $f \in \Gamma(x,y)$. So:

$$\begin{aligned} d_{\mathbb{H}}(\gamma(x), \gamma(y)) &= \inf\{\text{length}_{\mathbb{H}}(g) : g \in \Gamma(\gamma(x), \gamma(y))\} \\ &\leq \inf\{\text{length}_{\mathbb{H}}(\gamma \circ f) : f \in \Gamma(x,y\} \\ &\leq \inf\{\text{length}_{\mathbb{H}}(f) : f \in \Gamma(x,y)\} = d_{\mathbb{H}}(x,y) \end{aligned}$$

Since γ is invertible and $\gamma^{-1} \in \text{Mob}(\mathbb{H})$ then we can repeat this argument the other way:

$$d_{\mathbb{H}}(x,y) = \inf\{\text{length}_{\mathbb{H}}(f) : f \in \Gamma(x,y)\}$$

$$\leq \inf\{\text{length}_{\mathbb{H}}(\gamma^{-1} \circ g) : g \in \Gamma(\gamma(x), \gamma(y))\}$$

$$\leq \inf\{\text{length}_{\mathbb{H}}(g) : g \in \Gamma(\gamma(x), \gamma(y))\} = d_{\mathbb{H}(\gamma(x),\gamma(y)}$$

Hence, $d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(\gamma(x), \gamma(y))$

□

Theorem 3.6. $(\mathbb{H}, d_{\mathbb{H}})$ *is a metric space.*

Proof. We must show that $d_{\mathbb{H}}$ satisfies the three defining conditions for a metric on $\mathbb{H}$.

For the first condition, let $f : [a,b] \to \mathbb{H}$ be a path in $\Gamma(x,y)$ and recall the definition of $\text{length}_{\mathbb{H}}(f)$:

$$\text{length}_{\mathbb{H}}(f) = \int_f \frac{1}{\text{Im}(z)}|dz| = \int_a^b \frac{1}{\text{Im}(f(t))}|f'(t)|dt$$

By the definition of the hyperbolic plane, the integrand is always nonnegative, and since $d_{\mathbb{H}}(x,y)$ is defined to be the infimum of all the lengths of all paths $f \in \Gamma(x,y)$, then $d_{\mathbb{H}}$ is also nonnegative.

For the second condition, we need to consider the lengths of paths in $\Gamma(x,y)$ and $\Gamma(y,x)$. Let $f : [a,b] \to \mathbb{H}$ be a path in $\Gamma(x,y)$ and consider the composition of f with the function $h : [a,b] \to [a,b]$ given by $h(t) = a+b-t$. Then clearly $f \circ h \in \Gamma(y,x)$ since $(f \circ h)(a) = f(b) = y$ and $(f \circ h)(b) = f(a) = x$. Then consider the following with the reparameterization $s = h(t)$:

$$\text{length}_{\mathbb{H}}(f \circ h) = \int_{f \circ h} \frac{1}{\text{Im}(z)}|dz| = \int_a^b \frac{1}{\text{Im}((f \circ h)(t))}|(f \circ h)'(t)|dt$$

$$= \int_a^b \frac{1}{\text{Im}(f(h)(t))}|f'(h(t))||h'(t)|dt$$

$$= -\int_b^a \frac{1}{\text{Im}(f(s))}|f'(s)|ds$$

$$= \int_a^b \frac{1}{\text{Im}(f(s))}|f'(s)|ds = \text{length}_{\mathbb{H}}(f)$$

Therefore every path in $\Gamma(x,y)$ gives rise to a path in $\Gamma(y,x)$ of equal length, and by the same argument, each path in $\Gamma(y,x)$ gives rise to a path in $\Gamma(x,y)$ of equal length. Therefore, the set of lengths of the paths in each set are equal, and hence have the same infimum, so $d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(y,x)$

Finally, we prove the third condition by contradiction. Suppose that the triangle inequality does not hold for $d_{\mathbb{H}}$ so there exists distinct points $x, y, z \in \mathbb{H}$ so that:

$$d_{\mathbb{H}}(x,z) > d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z)$$

Then set:

$$\epsilon = d_{\mathbb{H}}(x,z) - (d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z)) > 0$$

Since $d_{\mathbb{H}}(x,y) = \inf\{\text{length}_{\mathbb{H}}(f) : f \in \Gamma(x,y)\}$ then there exists a path $f : [a,b] \to \mathbb{H}$ in $\Gamma(x,y)$ with:

$$\text{length}_{\mathbb{H}}(f) - d_{\mathbb{H}}(x,y) < \frac{\epsilon}{2}$$

Similarly, there exists a path $g : [b,c] \to \mathbb{H}$ in $\Gamma(y,z)$ with:

$$\text{length}_{\mathbb{H}}(g) - d_{\mathbb{H}}(y,z) < \frac{\epsilon}{2}$$

Let $h : [a,c] \to \mathbb{H}$ be the piecewise-smooth concatenation of f and g so that $h \in \Gamma(x,z)$. Then we see that:

$$\text{length}_{\mathbb{H}}(h) = \text{length}_{\mathbb{H}}(f) + \text{length}_{\mathbb{H}}(g) < d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) + \epsilon$$

But since $d_{\mathbb{H}}(x,z) \leq \text{length}_{\mathbb{H}}(h)$ by definition, then:

$$d_{\mathbb{H}}(x,z) < d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z) + \epsilon$$

This contradicts our construction of ϵ and hence condition 3 must hold for $d_{\mathbb{H}}$

$\square$

The following result can be constructed from Theorem 2.8, Proposition 3.5, and Theorem 3.6, but we do not present it here.

Theorem 3.7. *Given a pair of points $x, y \in \mathbb{H}$, there exists a Mobius transformation $\gamma \in Mob(\mathbb{H})$ such that $\gamma(x) = i\mu$ and $\gamma(y) = i\lambda$ both lie on the positive imaginary axis, and the* ***hyperbolic distance*** *between x and y (i.e. the shortest distance) is given by:*

$$d_{\mathbb{H}}(x,y) = \left| \log \frac{\lambda}{\mu} \right|$$

Definition 3.8. *An* ***isometry*** *of a metric space (X,d) is a homeomorphism f of X that preserves distance. That is, for a pair of points $x, y \in X$:*

$$d(x,y) = d(f(x), f(y))$$

Theorem 3.9. *The set of isometries of $(\mathbb{H}, d_{\mathbb{H}})$ is precisely the set $Mob(\mathbb{H})$.*

4 The Poincare Disc Model and Metric

We now attempt to use the tools we have developed in the hyperbolic plane to construct the Poincare Disc model of hyperbolic geometry, which operates inside the complex unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

We know from complex analysis that there are several Mobius transformations, m, that take $\mathbb{D} \to \mathbb{H}$. For example, let us define such a transformation ζ:

$$\zeta(z) = \frac{\frac{i}{\sqrt{2}}z + \frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}z - \frac{i}{\sqrt{2}}}$$

Now let us define the set of all Mobius transformations which are homeomorphisms of the unit disk:

$$\mathrm{Mob}(\mathbb{D}) = \{m \in \mathrm{Mob} : m(\mathbb{D}) = \mathbb{D}\}$$

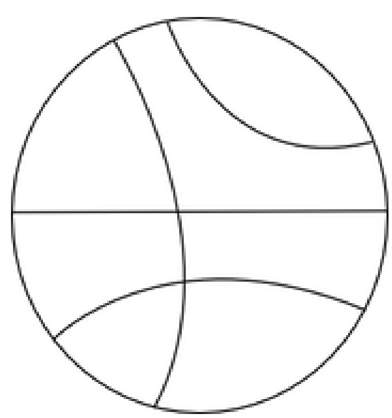
Theorem 4.1. *All elements of Mob($\mathbb{D}$) are of the form:*

$$m(z) = e^{i\theta}\frac{z + a}{1 + \bar{a}z}$$

where $\theta \in \mathbb{R}$ and $a \in \partial\mathbb{D}$.

Definition 4.2. *Given some Mobius transformation $m : \mathbb{D} \to \mathbb{H}$, a **hyperbolic line in** $\mathbb{D}$ is the image under m^{-1} of a hyperbolic line in $\mathbb{H}$*

It is easy to see that because of the complexity of such an inverse image, the hyperbolic lines in $\mathbb{D}$ will generally not look quite as clean as the hyperbolic lines in $\mathbb{H}$



Definition 4.3. *The* ***hyperbolic length*** *of a piecewise-smooth path $f : [a, b] \to \mathbb{D}$ is defined to be:*

$$length_{\mathbb{D}}(f) = length_{\mathbb{H}}(\zeta \circ f)$$

This definition is obviously not ideal since it depends on a specific transformation ζ, but we will amend this now by showing that the actual length is well-defined for any transformation.

Theorem 4.4. *The hyperbolic length of a piecewise-smooth path $f : [a, b] \to \mathbb{D}$ is given by the integral:*

$$length_{\mathbb{D}}(f) = \int_f \frac{2}{1 - |z|^2} |dz|$$

Proof. We may use our formula for hyperbolic length in $\mathbb{H}$:

$$\text{length}_{\mathbb{D}}(f) = \text{length}_{\mathbb{H}}(\zeta \circ f) = \int_{\zeta \circ f} \frac{1}{\text{Im(z)}} |dz|$$

$$= \int_a^b \frac{1}{\text{Im}((\zeta \circ f)(t))} |(\zeta \circ f)'(t)| dt = \int_a^b \frac{1}{\text{Im}(\zeta(f(t))} |\zeta'(f(t))||f'(t)| dt$$

$$= \int_f \frac{1}{\text{Im}(\zeta(z))} |\zeta'(z)| dz$$

Calculating the integrand:

$$\text{Im}(\zeta(z)) = \text{Im}\left(\frac{\frac{i}{\sqrt{2}} z + \frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}} z - \frac{i}{\sqrt{2}}} \right) = \frac{1 - |z|^2}{|-z - i|^2}$$

$$|\zeta'(z)| = \frac{2}{|z + i|^2}$$

So:

$$\frac{1}{\text{Im}(\zeta(z))} |\zeta'(z)| = \frac{2}{1 - |z|^2}$$

And hence:

$$\text{length}_{\mathbb{D}}(f) = \int_f \frac{2}{1-|z|^2}|dz|$$

Now we must show that this is independent of our choice of ζ. Let f be a piecewise continuous path and let p be any Mobius transformation taking $\mathbb{D}$ to $\mathbb{H}$. Since $p \circ \zeta^{-1}$ is a Mobius transformation and takes $\mathbb{H}$ to $\mathbb{H}$ then we have $q = p \circ \zeta^{-1} \in \text{Mob}(\mathbb{H})$

Since $\zeta \circ f$ is a piecewise smooth path in $\mathbb{H}$ and the invariance of hyperbolic length on $\mathbb{H}$ is invariant under the Mobius transformations by Theorem 3.2, then:

$$\text{length}_{\mathbb{H}}(\zeta \circ f) = \text{length}_{\mathbb{H}}(q \circ \zeta \circ f) = \text{length}_{\mathbb{H}}(p \circ f)$$

Hence, $\text{length}_{\mathbb{D}}(f)$ is well defined.

□

Define $\Theta(x, y)$ to be the set of all piecewise-smooth paths $f : [a.b] \to \mathbb{D}$ connecting x and y where $f(a) = x$ and $f(b) = y$. Also define:

$$d_{\mathbb{D}}(x, y) = \inf\{\text{length}_{\mathbb{D}}(f) : f \in \Theta(x, y)\}$$

Theorem 4.5. $(\mathbb{D}, d_{\mathbb{D}})$ *is a metric space, and any Mobius transformation m taking $\mathbb{H}$ to $\mathbb{D}$ is a distance preserving homeomorphism between $(\mathbb{H}, d_{\mathbb{H}})$ and $(\mathbb{D}, d_{\mathbb{D}})$.*

Proof. Let m be any Mobius transformation taking $\mathbb{H}$ to $\mathbb{D}$. Also let $\Gamma(z, w)$ be the set of all piecewise-smooth paths $f : [a, b] \to \mathbb{H}$ with $f(a) = z$ and $f(b) = w$. For each pair of points $z, w \in \mathbb{H}$ we have:

$$d_{\mathbb{H}}(z, w) = \inf\{\text{length}_{\mathbb{H}}(f) : f \in \Gamma(z, w)\} = \inf\{\text{length}_{\mathbb{D}}(m \circ f) : f \in \Gamma(z, w)\}$$

$$\leq \inf\{\text{length}_{\mathbb{D}}(g) : g \in \Theta(m(z), m(w)\}$$

$$\leq d_{\mathbb{D}}(m(z), m(w))$$

Similarly, if $x, y \in \mathbb{D}$ and $x = m(z)$ and $y = m(w)$ for points $z, w \in \mathbb{H}$ then we see that:

$$d_{\mathbb{D}}(m(z), m(w)) = d_{\mathbb{D}}(x, y) = \inf\{\text{length}_{\mathbb{D}}(f) : f \in \Theta(x, y)\}$$

$$= \inf\{\text{length}_{\mathbb{H}}(m^{-1} \circ f) : f \in \Theta(x, y)\}$$

$$\leq \inf\{\text{length}_{\mathbb{H}}(g) : g \in \Gamma(z, w)\}$$

$$\leq d_{\mathbb{H}}(z, w)$$

Since $d_{\mathbb{H}}(z, w) = d_{\mathbb{D}}(m(z), m(w))$ for all $z, w \in \mathbb{H}$ and all m taking $\mathbb{H}$ to $\mathbb{D}$, and as $d_{\mathbb{H}}$ is a metric on $\mathbb{H}$, then we have that $d_{\mathbb{D}}$ is a metric on $\mathbb{D}$. Morevoer, this shows that m is a distance-preserving homeomorphism between $(\mathbb{H}, d_{\mathbb{H}})$ and $(\mathbb{D}, d_{\mathbb{D}})$

$\square$

Corollary 4.6. *The set of isometries of* $(\mathbb{D}, d_{\mathbb{D}})$ *is precisely the set* $Mob(\mathbb{D})$.

Proof. This follows from the fact that Mob($\mathbb{H}$) is exactly the set of isometries of $(\mathbb{H}, d_{\mathbb{H}})$ by Theorem 3.9, and that any Mobius transformation m taking $\mathbb{H}$ to $\mathbb{D}$ is a distance-preserving homeomorphism and hence an isometry.

$\square$

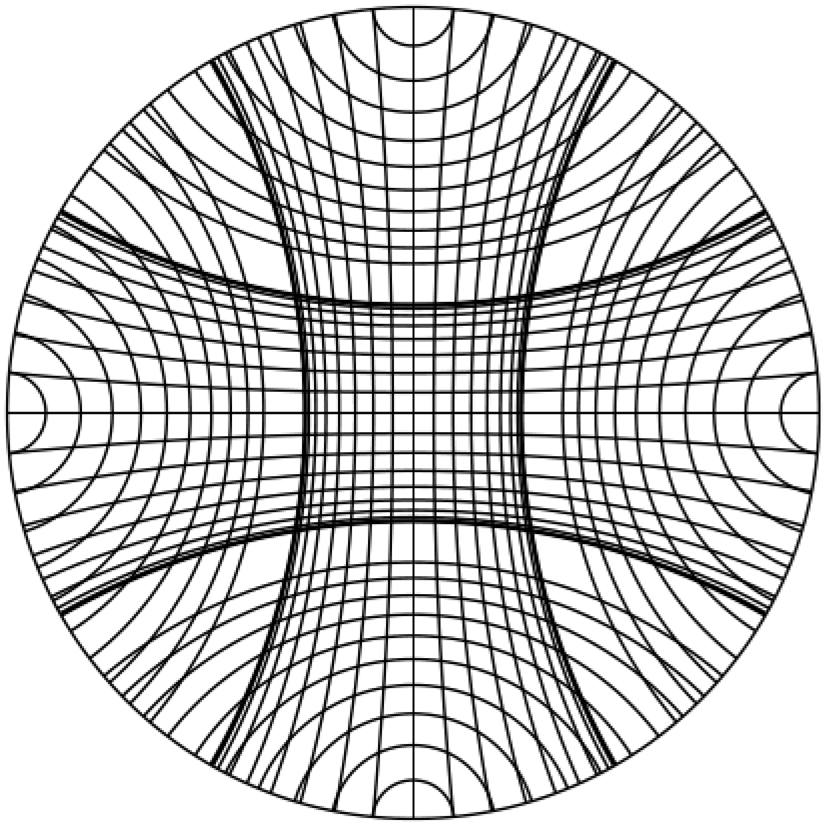
5 Hyperbolic Geodesics

We begin with theorem about distances in the Poincare disc.

Theorem 5.1. *For any two distinct points* z_0, z_1 *in* $\mathbb{D}$ *there is a unique shortest curve in* $\mathbb{D}$ *from* z_0 *to* z_1 *in the hyperbolic metric, namely, the arc of the circle passing through* z_0 *and* z_1 *that is orthogonal to the unit circle.*

Definition 5.2. *The described paths in Theorem 5.1 are called* ***hyperbolic geodesics***

We can consider hyperbolic geodesics as the Poincare disc analogs of straight lines in the Euclidean plane.



Proposition 5.3. *The hyperbolic distance from* 0 *to* z *is given by the formula:*

$$d(0,z) = \log \frac{1+|z|}{1-|z|}$$

Proof. The proof is a simple calculation:

$$d(0,z) = \int_0^{|z|} \frac{2}{1-t^2} dt = \int_0^{|z|} \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt = \log \frac{1+|z|}{1-|z|}$$

$\square$

As we can see, the distance from 0 to z tends towards infinity as z approaches the boundary of the disk. Hence, we can see in the depictions of shortest paths that the lines tend to curve in towards the center of the disk where the distance between points is shorter in the hyperbolic metric, as opposed to moving directly in between the points.

6 The Hyperbolic Pythagorean Theorem

We now present the proof from Ungar [3] of a model of the Pythagorean Theorem in the Poincare disc model. Clearly the Euclidean Pythagorean Theorem does not apply since we have different concepts of triangles and length. However, as it turns out, there does exist a natural formulation of the hyperbolic Pythagorean theorem.

Definition 6.1. *A* ***geodesic triangle*** *or* ***hyperbolic triangle*** *is a region bounded by three distinct hyperbolic geodesics*

Definition 6.2. *We define the operation of* ***Mobius addition***, *$\oplus$, as:*

$$z_0 \oplus z = e^{i\theta} \frac{z_0 + z}{1 + \bar{z} z_0}$$

where $\theta \in \mathbb{R}$ and $z_0 \in \mathbb{D}$.

Notice the connection between this definition and our homeomorphisms of the unit disk in Theorem 4.1. This operation can be viewed as a "Mobius left translation"

Definition 6.3. *The* ***Poincare hyperbolic distance function*** *is defined as:*

$$d(a, b) = \left| \frac{a - b}{1 - \bar{a} b} \right| = |a \ominus b|$$

where we use the notation $a \ominus b = a \oplus (-b)$.

Proposition 6.4. *The Poincare hyperbolic distance function satisfies the Mobius triangle inequality:*

$$d(a, c) \leq d(a, b) \oplus d(b, c)$$

Proof. Define $\gamma_a = (1 - |a|^2)^{-\frac{1}{2}}$ for any $a \in \mathbb{D}$. Then $\gamma_a = \gamma_{|a|}$ is a monotonically increasing function of $|a|$ that satisfies:

$$\gamma_{\left||a| \oplus |b|\right|} = \gamma_{|a| \oplus |b|} = \gamma_{|a|}\gamma_{|b|}(1 + |a||b|) \geq \gamma_a \gamma_b |1 + \bar{a}b| = \gamma_{a \oplus b} = \gamma_{|a \oplus b|}$$

Since $\left||a| \oplus |b|\right| = |a| \oplus |b|$ and since $\gamma_z = \gamma_{|z|}$ is a monotonically increasing function of $|z|$, then this inequality implies:

$$|a| \oplus |b| \geq |a \oplus b|$$

for all $a, b \in \mathbb{D}$. Noting that:

$$(-x \oplus a) \oplus (x \ominus b) = \frac{1 - x\bar{a}}{1 - \bar{x}a}(a \ominus b)$$

for all $x, a, b \in \mathbb{D}$, then we have:

$$d(a, b) = |a \ominus b| = \frac{1 - x\bar{a}}{1 - \bar{x}a}(a \ominus b) = |(-x \oplus a) \oplus (x \ominus b)|$$

$$\leq | - x \oplus a| \oplus |x \ominus b| = d(a, x) \oplus d(x, b)$$

which proves the inequality.

□

Theorem 6.5. ***(The Hyperbolic Pythagorean Theorem)*** *Let $\triangle abc$ be a hyperbolic triangle whose vertices are the points $a, b, c \in \mathbb{D}$ and whose sides are $A = -b \oplus c$, $B = -c \oplus a$, and $C = -a \oplus b$. If the two sides A and B are orthogonal, then $|A|^2 \oplus |B|^2 = |C|^2$*

Proof. Let $\triangle abc$ be any hyperbolic triangle whose vertices are the points $a, b, c \in \mathbb{D}$. A **hyperbolic right triangle** is a hyperbolic triangle one of whose angles is $\frac{\pi}{2}$. Furthermore, let $\triangle abc$ be a hyperbolic right triangle whose sides A and B are orthogonal.

Its right angle can be moved to the center of $\mathbb{D}$ via appropriate Mobius transformations in the form of Mobius addition, such that its two orthogonal sides now lie on the real and imaginary axes of $\mathbb{D}$. Since Mobius transformations of the disk preserve both the hyperbolic length of geodesic segments and the measure of hyperbolic angles, then the resulting triangle $\triangle a'b'c'$ obtained by moving $\triangle abc$ is congruent to this former triangle in the sense that the two triangles possess equal hyperbolic lengths for corresponding sides and equal measures for corresponding angles.

The vertices of the relocated hyperbolic right triangle $\triangle a'b'c'$ are $a' = x$, $b' = iy$, and $c' = 0$ for some $x, y \in (-1, 1)$. The hyperbolic length of the geodesic segment joining two points a and b of the disc is $d(a, b) = |b \ominus a|$. Accordingly, the hyperbolic lengths of the sides A, B, C of the triangle $\triangle a'b'c'$ are $|A|, |B|, |C|$ given by:

$$|A|^2 = |b' \ominus c'|^2 = y^2$$
$$|B|^2 = |a' \ominus c'|^2 = x^2$$
$$|C|^2 = |a' \ominus b'|^2 = |x \ominus iy|^2 = \left| \frac{x - iy}{1 - ixy} \right|^2 = x^2 \oplus y^2$$

Therefore:

$$|A|^2 \oplus |B|^2 = |C|^2$$

□

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