ABSTRACT FOURIER ANALYSIS (DRAFT)

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Abstract. In this paper, we will discuss abstract Fourier analysis on arbitrary groups, in particular, we will discuss complete reducibility of representations of finite groups and of compact groups.

Contents

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1. Introduction

Let $f : [-\pi, \pi] \to \mathbb{R}$ be a 2π periodic function, we may associate to it with a sequence *Fourier coefficients* $\hat{f} : \mathbb{Z} \to \mathbb{R}$, defined by:

$$\hat{f}(n) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\theta) e^{-in\theta} d\theta$$

and approximate the original function by the *Fourier series* of f:

$$f(\theta) \sim \sum_{-\infty}^{\infty} \hat{f}(n) e^{in\theta}$$

If $f \in L^2(-\pi, \pi)$, by Theorem 8.43 in [3], we can establish a correspondence between the series and f by showing the series converges to f in norm (of the L^2 space), that is,

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} \left| f(\theta) - \sum_{-N}^{N} \hat{f}(n) e^{in\theta} \right| = 0$$

Moreover, since the integral of $e^{in\theta}$ from $\theta = -\pi$ to π is zero except for $n = 0$, $\langle e^{in\theta}, e^{im\theta} \rangle = 0$ for all $m \neq n$. Hence $\{e^{in\theta}\}_{-\infty}^{\infty}$ forms an *orthogonal basis* for $L^2(-\pi, \pi)$.

The first generalization one may make is to allow f to be a (possibly) non periodic function from $\mathbb{R}$ to $\mathbb{R}$, and consider the "continuous" analogue of Fourier coefficients. This generalization defines us the *Fourier transform* of f, $\mathcal{F}f = \hat{f} : \mathbb{R} \to \mathbb{R}$:

$$\hat{f}(\xi) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\theta) e^{-i\xi\theta} d\theta$$

and the corresponding approximation for the original f:

$$f(\theta) \sim \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi\theta} d\xi \tag{1.1}$$

If we restrict $f \in L^2(\mathbb{R})$, then by Theorem 1.9 in Chapter 5 of [7], eq. (1.1) turns out to be an equality, hence the name: *Fourier inversion* formula. By the same argument as before, the set $\{e^{i\xi\theta}\}$ is an orthogonal basis of $L^2(\mathbb{R})$.

The second and the third possible generalization follows from the observation that the Fourier transform $\mathcal{F}(-)$ is a endomorphism on the group $L^2(\mathbb{R})$ under convolution. If we define an operator $T_t : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(T_t f)(\theta) \triangleq f(\theta + t)$ for every $t \in \mathbb{R}$, and define the group action $\pi : \mathbb{R} \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $\pi(t)f = \mathcal{F}(T_t f)$. A simple calculation by substitution shows, for a given $t \in \mathbb{R}$,

$$\widehat{T_t f}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\theta + t) e^{-i\xi\theta} d\theta = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\theta) e^{-i\xi(\theta - t)} d\theta = \hat{f}(\xi) \cdot e^{it\xi}$$

Hence $(L^2(\mathbb{R}), \pi)$ is a representation of $\mathbb{R}$ in the inner product space $L^2(\mathbb{R})$ (these terms will be defined more precisely in Section 3).

If we replace $\mathbb{R}$ by a finite group G and the inner product space $L^2(\mathbb{R})$ by a generic finite-dimensional vector space V, we obtain what's called *discrete Fourier transform*. Given this generalization, we are also interested in questions arose classic Fourier theory, that is, (1) whether it is possible to find a decomposition of elements in V into linear combinations of a basis and (2) whether an orthogonal basis exist.

The main portion of the paper will be devoted to investigate the same question for the case where G is an non-abelian compact group and the vector space V is a Hilbert space.

In Section 2, we will develop background machineries from measure theory and functional analysis that allow us to pull off the main result in Section 4. In Section 3, we will investigate the second generalization of abstract Fourier analysis, namely discrete Fourier analysis, and in Section 4 we will establish the result of abstract Fourier analysis on compact groups.

2. Preliminaries

Let G be a group, that is, an underlying set $\mathbb{G}$ with operations that give $\mathbb{G}$ the group structure, if we further enforce a topological structure on $\mathbb{G}$, then we can treat it as a topological space given those algebraic operations are continuous. We call G a *topological group*.

Some examples of topological groups include the group $\mathbb{R}^\times$ of real numbers under multiplication: its' topology was induced from the metric $d(x, y) = |x - y|$. Since the topology of $\mathbb{R}^\times$ is the usual one induced by the metric, by $\epsilon - \delta$ argument, it's easy to show the map $(x, y) \mapsto x \times y$ from $\mathbb{R}^2$ to $\mathbb{R}$ is continuous.

Another example of a topological group that is closely related to Fourier analysis is the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, under multiplication. Equivalently, we can represent elements of $\mathbb{T}$ in polar coordinates: $\mathbb{T} = \left\{e^{i\theta} : \theta \in [-\pi, \pi]\right\}$. The multiplicative identity is $e^0 = 1$, and inverse operations is simply $\left(e^{i\theta}\right)^{-1} = e^{-i\theta}$.

Apparently, $\mathbb{T}$ is not a finite group since the interval $[-\pi, \pi]$ is uncountable, it's a topological group where the topological structure is inherited from the usual topological structure of $\mathbb{C}$. Furthermore, $\mathbb{T}$ is a compact group since we know the unit circle is compact in $\mathbb{C}$.

2.1. Measure Theory Backgrounds.

Let X be a topological space, we say X is a *measure space* if there exist a σ-algebra $\mathfrak{M}$ in X, that is, a family of subsets of X containing X and is closed under taking complements and countable union. We call elements of $\mathfrak{M}$ *measurable sets*. Let $f : X \to Y$ be a map from a measurable space X to a topological space Y *measurable* if for every open set V in Y, the pullback $f^{-1}(V)$ is a measurable set.

Let $\mathcal{T}$ be a topology on X, let Ω be a family of all σ-algebras in X that contains $\mathcal{T}$. Then by intersecting all elements in this family, we have a *smallest* σ-algebra $\mathcal{B}$ that contains all open sets of X, we refer to elements of $\mathcal{B}$ as *Borel sets*.

Given $(X, \mathfrak{M})$ a measure space, we define a *measure* to be a map $\mu : \mathfrak{M} \to [0, \infty]$ that is countably additive. This means, for any disjoint countable collection $\{A_j\} \subset \mathfrak{M}$,

$$\mu\left(\bigcup_{j=0}^{\infty} A_j\right) = \sum_{j=0}^{\infty} \mu(A_j)$$

If the σ-algebra above is the collection of Borel sets of X, then we call the measure a *Borel* measure. We say the Borel measure is *regular* if it satisfies, for all measurable set X,

$$\mu(X) = \inf\{\mu(U) : U \supset X, U \text{ open}\} = \sup\{\mu(K) : K \subset X, K \text{ compact}\}$$

If the topological space G is a *locally compact* group (i.e. every point of G has a compact neighborhood), there is, up to a constant multiple, a unique regular Borel measure μ_L that is invariant under left translation. By *left translation invariance*, we mean $\mu_L(X) = \mu_L(gX)$ for all measurable sets X and $g \in G$. We call such measure a *left Haar measure*, it should also satisfy $\mu_L(X) < \infty$ for X compact and $\mu_L(X) > 0$ for all X measurable. (For the existence and uniqueness of μ_L, see [5]). Similarly, we can also define *right Haar measure.* Usually left and right Haar measure do not coincide for general topological groups, when they do, we call the group *unimodular*.

One of the main purpose of inventing the formalism of measure is to develop the theory of integration with respect to a given measure. The idea of computing the integral $\int_X f(x)d\mu(x)$ is to approximate the function f by *simple functions*, that is, a linear combinations of characteristic function on measurable sets. If $f : X \to \mathbb{R}$ is non-negative, define

$$\int_X f(x)d\mu(x) \triangleq \sup_{0\leq s\leq f} \int_X s(x)d\mu(s) = \sup_{0\leq \sum \lambda_i \chi_{E_i} \leq f} \int_X \sum_{i=0}^{n} \lambda_i \chi_{E_i}(x)d\mu(x)$$

Notice, if μ_L is a left Haar measure, then for $\gamma \in G$,

$$\int_G f(\gamma g)d\mu_L(g) = \int_G f(\gamma g)d\mu_L(\gamma g)$$

substituting $h = \gamma g$, we have

$$\int_G f(\gamma g)d\mu_L(g) = \int_G f(g)d\mu_L(g)$$

Similarly, to check μ_R is a right Haar measure, it amounts to check the integral

$$\int_G f(g\gamma)d\mu_L(g) \overset{?}{=} \int_G f(g)d\mu_L(g)$$

Consider the conjugation map $\phi(g) : h \mapsto g^{-1}hg$, it's a automorphism on G (with inverse being $\phi(g)^{-1} : h \mapsto ghg^{-1}$). Since left Haar measures are unique up to a constant multiple, the left Haar measure of $\phi_g(G)$ is a constant multiple of the left Haar measure of G. In another words, there exist a function $\delta : G \to \mathbb{R}_+^\times$ such that

$$\int_G f(\phi_g(h))d\mu_L(h) = \delta(g) \int_G f(h)d\mu_L(h)$$

Given $g_1, g_2 \in G$, since conjugating first by g_1 then by g_2 is equivalent to conjugating by (g_2g_1), we have $\delta(g_2) \cdot \delta(g_1) = \delta(g_2g_1)$, the map δ is a group homomorphism.

Lemma 2.1. *Let μ_L be a left Haar measure of G, let $\delta : G \to \mathbb{R}_+^\times$ be defined as above, then $\delta \cdot \mu_L$ is a right Haar measure of G.*

Proof. We verify the measure $\delta(h)\mu_L(h)$ is right invariant. Let $\widetilde{f} = f \cdot \delta$, then by definition of δ,

$$\int_G \widetilde{f}(g^{-1}hg)d\mu_L(h) = \delta(g) \int_G \widetilde{f}(h)d\mu_L(h)$$

Since μ_L is left invariant, left translate $\widetilde{f}$ by g gives

$$\int_G \widetilde{f}(hg)d\mu_L(h) = \delta(g) \int_G \widetilde{f}(h)d\mu_L(h)$$

Since δ is a homomorphism, we expand the left hand side:

$$\int_G f(hg)\delta(h)\delta(g)d\mu_L(h) = \delta(g)\int_G \widetilde{f}(h)d\mu_L(h)$$

Canceling $\delta(g)$ from both sides, we have

$$\int_G f(hg)\delta(h)d\mu_L(h) = \int_G \widetilde{f}(h)d\mu_L(h) = \int_G f(h)\delta(h)d\mu_L(h)$$

Thus, $\delta \cdot \mu_L$ is a right Haar measure. $\square$

The property of Haar measure on compact groups follows from the Lemma above:

Lemma 2.2. *Let G be a compact group, then*

(1) G is unimodular
(2) $\mu_L(G) < \infty$

Proof. Let $\delta : G \to \mathbb{R}_+^\times$ be define as above. Since G is compact and δ is continuous, the image $\mathrm{Im}(\delta)$ is a compact subgroup of $\mathbb{R}_+^\times$. Since the only compact subgroup of $\mathbb{R}_+^\times$ is trivial, the image is trivial. Then the right Haar measure $\delta \cdot \mu_L$ is equal to μ_L, hence G is unimodular. Since G is compact, by assumption for Haar measure, $\mu(G) < \infty$. $\square$

Since $\mu_L(G) < \infty$ and left Haar measures (also right Haar measure when G is compact) are unique upto a constant multiple, we can pick a measure μ such that the integral is normalized, that is, $\int_G 1d\mu(g) = 1$. Without loss of generality, we will always consider Haar measures that normalizes the integral in later sections. For notational convenience, later in the paper we will use dg in place for $d\mu(g)$.

2.2. **Functional Analysis Machineries.** In order to discuss whether we have "enough" matrix coefficients, we will first need to develop some results on linear operators, in particular on *compact operators*.

Let $\mathfrak{H}$ be a normed vector space with norm $\|-\|$, a linear operator $T : \mathfrak{H} \to \mathfrak{H}$ (i.e. maps compatible with the linear structure of $\mathfrak{H}$) is called *bounded* if there exist a constant C such that

$$\|Tx\| \leq C\,\|x\| \qquad (\forall x \in \mathfrak{H})$$

The smallest such constant is referred to as the *operator norm* of T, denote by $|T|$. It's easy to see T is bounded iff T is continuous (on the metric topology given by the norm).

We call the operator T *self-adjoint* if

$$\langle Tf, g\rangle = \langle f, Tg\rangle \qquad (\forall f, g \in \mathfrak{H})$$

A bounded operator $T : \mathfrak{H} \to \mathfrak{H}$ is *compact* if given any bounded sequence $\{x_1, x_2, \ldots\} \subset \mathfrak{H}$, the image sequence $\{Tx_1, Tx_2, \ldots\}$ has a convergent subsequence.

For $f, g \in \mathfrak{H}$, we call f an *eigenvector* with *eigenvalue* λ of T if $f \neq 0$ and $Tf = \lambda f$. As usual, we call the space $\{f \in \mathfrak{H} : f \neq 0, Tf = \lambda f\} \cup \{0\}$ the λ-eigenspace.

Theorem 2.3 (Spectral theorem for compact operators)**.** *Let T be a compact self-adjoint operator on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{N}$ be the nullspace of T. Then*

(1) the Hilbert space dimension of $\mathfrak{N}^\perp$ is at most countable.
(2) $\mathfrak{N}^\perp$ has an orthonormal basis $\{\phi_i\}_1^\infty$ of eigenvectors of T, and associated set of eigenvalues $\{\lambda_i\}_1^\infty$.

(3) If $\mathfrak{N}^\perp$ is not finite-dimensional, the eigenvalues $\lambda_i \to 0$ as $i \to \infty$.

Proof. See Theorem 4.11-1 in [2]. □

Notice, if $\mathfrak{N}^\perp$ is not finite-dimensional, then each λ_i may only appear finitely many times in the sequence of eigenvalues, hence there are only finitely many eigenvectors associated with λ_i, the λ_i-eigenspace is finite.

Let X, Y be compact topological spaces, Y is has a metric d. Let U be a set of continuous maps $X \to Y$, we call U being *equicontinuous* if for every $x \in X$ and $\epsilon > 0$, there exists a neighborhood $N \ni x$ such that

$$d(f(x), f(x')) < \epsilon \qquad (\forall x' \in N, f \in U)$$

The following theorem relates equicontinuity with compactness:

Theorem 2.4 (Ascoli and Arzela)**.** *Let X be a compact space, $U \subset C(X)$ is a bounded subset of continuous functions, with the sup-norm metric. Suppose U is equicontinuous, then every sequence in U has a uniformly convergent subsequence.*

Proof. See Theorem 3.10-1 in [2]. □

3. Representation Theory for Finite Groups

Definition 3.1 (Group action)**.** Let G be a group, the action of G on set X is a map $\alpha : G \times X \to X$ that is compatible with the group structure. That means, for all $g, h \in G$, $x \in X$,

$$\begin{aligned} \alpha(gh)x &= \alpha(g)(\alpha(h)x) \\ \alpha(1_G)x &= x \end{aligned}$$

Definition 3.2 (Linear Representation)**.** Let G be a group, a *linear representation* ρ of G on a complex vector space V is a group action on V that preserves the vector space structure of V.

If the context is clear, we would simply refer to a representation (ρ, V) of group G by the vector space V. We will also write gv in place for $\rho(g)(v)$ when the G-action ρ is clear from the context.

Proposition 3.3. *The set of representations of G on V is in bijection with the set of group homomorphisms from G to $GL(V)$.*

Proof. Let (ρ, V) be a G-representation. By definition of group action, ρ preserves group structure, in particular $\rho(g)$ is invertible for every $g \in G$. Conversely, let $f \in Hom(G, GL(V))$ be given, then for every $g \in G$, $f(g) \in GL(V)$, thus preserves V structure. Furthermore, since f is a homomorphism, it preserves G structure, hence a G-representation. □

Definition 3.4 (Isomorphic class of reps)**.** Let G be a group, (ρ_1, V_1) and (ρ_2, V_2) be two representations of G. An *isomorphism* between those two representations is a linear isomorphism $\phi : V_1 \to V_2$ such that the following diagrams commutes for every $g \in G$:

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "\phi", "\wr"'] & V_1 \arrow[d, "\phi", "\wr"'] \\ V_2 \arrow[r, "\rho_2(g)"] & V_2 \end{tikzcd}$$

In this case, we asy V_1 and V_2 are *equivalent representations*.

An *isomorphic class of representations* of G is thus defined to be a equivalence class of representations that are equivalent to each other.

Generally, if ϕ is not an isomorphism, it's referred to as an *interwining operator* or a *G-linear map*. For notational convenience, let $\mathrm{Hom}^G(V, W)$ denote the space of G-linear maps.

3.1. **Representation Constructions.** Since the definition G-representation relies on a linear structure, a number of constructions of representations are inherited from constructions on vector spaces.

Definition 3.5 (Direct sum of representations)**.** Let (ρ_1, V_1) and (ρ_2, V_2) be two representations of group G, the *direct sum* of two representations is the space $V_1 \oplus V_2$ equipped with action $\rho_1 \oplus \rho_2 = \alpha \circ (\rho_1 \times \rho_2)$. Where $\alpha : GL(V_1) \times GL(V_2) \to GL(V_1 \oplus V_2)$ is a map obtained by coordinate-wise action.

Notice the map $\rho_1 \oplus \rho_2$ may also be identified by the block diagonal matrix:

$$\rho_1 \oplus \rho_2 : g \mapsto \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

Definition 3.6 (Dual representation)**.** Let (ρ, V) be a representation of group G, the *dual representation* is the dual space $V^* = \mathrm{Hom}(V, k)$ with action $\hat{\rho} : G \times V^* \to V^*$:

$$\hat{\rho}(g) : L \mapsto L \circ \rho(g)^{-1}$$

we call $\hat{\rho}$ the *contragradient* of ρ. It's easy to check

$$\begin{aligned} \hat{\rho}(g_1 g_2) &= (L \mapsto L \circ \rho((g_1 g_2^{-1}))) \\ &= (L \mapsto L \circ \rho(g_2^{-1} g_1^{-1})) \\ &= (L \mapsto L \circ \rho(g_2^{-1}) \circ \rho(g_1^{-1})) \\ &= \hat{\rho}(g_1) \circ (L \mapsto L \circ \rho(g_2^{-1})) \\ &= \hat{\rho}(g_1) \circ \hat{\rho}(g_2) \end{aligned}$$

Definition 3.7 (Subrepresentation)**.** Let (ρ, V) be a representation of group G. A *subrepresentation* of V is a G-invariant subspace $W \subset V$, that is, W satisfies

$$\rho(g)(\boldsymbol{w}) \in W \qquad (\forall \boldsymbol{w} \in W, \forall g \in G)$$

and W is a representation of G under the map ρ.

Definition 3.8 (Reducibility)**.** A G-representation (ρ, V) is *irreducible* if it contains no proper invariant subspaces. It's *completely reducible* if it may be decomposed into a direct sum of irreducible representations.

Definition 3.9 (Quotient representation)**.** Let (ρ, V) be a representation of group G, W is a subspace of V. The action map of the *quotient representation* is defined through the action map of the original representation, $\widetilde{\rho} : G \times V/W \to V/W$:

$$\widetilde{\rho}(g)(\boldsymbol{v} + W) \triangleq \rho(g)(\boldsymbol{v}) + W$$

3.2. **Complete Reducibility for Finite Groups.** In this section, we establishes the reducibility theorem of representations of finite groups. The main result is the theorem that every $\mathbb{C}$ representation admits a unique decomposition into irreducible representations.

Definition 3.10 (Unitary representation)**.** Let G be a group, a representation (ρ, V) is *unitary* if there exist a positive definite Hermitian inner product H that is invariant under G-actions, that is,

$$H(v, w) = H(gw, gw) \qquad (\forall v, w \in V, \forall g \in G)$$

A representation is called *unitarisable* if it can be equipped with such an inner product.

Lemma 3.11. *Let V be a unitary representation of finite group G, W is an invariant subspace. Then $W^\perp$ is also an invariant subspace.*

Proof. Let $w' \in W^\perp$, $g \in G$ be given. Let H be the inner product on V that is invariant under G-actions (we call such an inner product G-*equivariant*). Then, we have,

$$\begin{aligned} & H(w, w') = 0 && \text{definition of orthogonal set} \\ \Leftrightarrow\ & H(gw, gw') = 0 && \text{invariance} \\ \Leftrightarrow\ & H(w, gw') = 0 && g^{-1}gw = w \in W \\ \Leftrightarrow\ & gw' \in W^\perp && \text{definition} \end{aligned}$$

Thus, $W^\perp$ is an invariant subspace. ☐

Lemma 3.12. *Let G be a finite group, (ρ, V) be a representation of G ($\dim V < \infty$), W be a subrepresentation of V. Then, there exist a complementary invariant subspace W' of V such that $V = W \oplus W'$.*

Proof. We will prove the theorem by using inner products and Weyl's technique of averaging over the group.

Let H_0 be an arbitrary Hermitian inner product of V, we construct a Hermitian inner product that is invariant under G-actions as follows:

$$H(v, w) \triangleq \sum_{g \in G} H_0(gv, gw)$$

We verify the invariance: let $g \in G$ be given, we would like to show

$$H(gv, gw) = \sum_{g' \in G} H_0(g'gv, g'gw) \stackrel{?}{=} \sum_{h \in G} H(hv, hw) = H(v, w)$$

It suffices to show, for any $g \in G$, there exists a bijection: $Gg \cong G$. From left to the right, let $g' \in G$ be given, we simply put $h = g'g$. From right to left, let $h \in G$ be given, we define $g' = hg^{-1}$.

By 3.11, since V is unitary, $W^\perp$ is an invariant subspace and orthogonal decomposition $V = W \oplus W^\perp$. ☐

Corollary 3.13 (Complete reducibility)**.** *Any representation of a finite group (hence must be finite dimensional) admits an orthogonal decomposition into irreducible sub-representations.*

Proof. Finite decomposition using 3.12. ☐

Now, we would like to show that such a decomposition is unique.

Lemma 3.14. *Let V, W be representations of a finite group G, $\phi \in \mathrm{Hom}^G(V, W)$, then*

(1) $\operatorname{Im}\phi$ *is an invariant subspace of* W
(2) $\ker\phi$ *is an invariant subspace of* V.

Proof.

(1) Let $w \in \operatorname{Im}\phi$ be given, then there exist $v \in V$ such that $\phi(w) = w$. Since ϕ commutes with group actions,
$$\phi(gv) = g\phi(v) = gw$$
Thus, $gw \in \operatorname{Im}\phi$.

(2) Let $v \in \ker\phi$ be given, then $\phi(v) = 0 \in W$. Since ϕ commutes with group actions, and group actions preserves vector space structure,
$$\phi(gv) = g\phi(v) = g0 = 0$$
Thus $gv \in \ker\phi$.

□

Lemma 3.15 (Schur's Lemma). *Let* V, W *be irreducible representations of finite group* G, $\phi \in \operatorname{Hom}^G(V, W)$. *Then*

(1) Either ϕ *is an isomorphism or* $\phi \equiv \mathbf{0}$.
(2) If $V = W$, *then* $\phi = \lambda \cdot I$ *for some* $\lambda \in \mathbb{C}$, *where* I *is the identity map.*

Proof.

(1) Follows from 3.14.

(2) Since $\mathbb{C}$ is algebraically closed, ϕ has at least one eigenvalue λ, let $\boldsymbol{v} \neq 0$ be the associated eigenvector, then $\phi\boldsymbol{v} = \lambda\boldsymbol{v} \Rightarrow (\phi - \lambda \cdot I)\boldsymbol{v} = \mathbf{0}$, hence $(\phi - \lambda \cdot I)$ is not injective, by (1), it must be 0, thus $\phi = \lambda \cdot I$.

□

Theorem 3.16. *(Uniqueness of decomposition) Any representation* (ρ, V) *of a finite group* G *admits a unique orthogonal decomposition into irreducible subrepresentations.*
$$V = \bigoplus V_j^{\oplus e_j}$$
or equivalently, we may express the decomposition through the action
$$\rho = \sum_j m_j \rho_j$$

Proof. Existence is given by 3.13. Suppose $V = \bigoplus W_k^{\oplus f_k}$ is another decomposition, then consider the identity map $\mathbf{1} : V \to V$ that maps $\bigoplus V_j^{\oplus e_j}$ to $\bigoplus W_k^{\oplus f_k}$, then by Schur's Lemma, it must be the case that $\mathbf{1}$ maps $V_j^{\oplus e_j}$ into $W_k^{\oplus f_k}$ where V_j and W_j are isomorphic and $e_j = f_k$. This proves the uniqueness. □

4. Compact Groups

We would like to answer the similar question about complete reducibility on general compact groups. Is it possible to develop similar results on finite reducibility for general compact group? Is it possible for us to adapt proofs in the finite case to compact groups?

We will provide a affirmative answer to the first question: we will be able to show the reducibility of representation of a compact group into a orthonormal set of irreducible representations, each of them is finite dimensional. However, unfortunately we cannot adapt most of the proofs in the finite case to the compact case (in fact, the only theorem we developed previously may be carried over to the compact case is Schur's Lemma and complete reducibility of *finite* representations).

In short, the orthogonality will be established by Schur's orthogonality theorem, and the complete reducibility will be established by Peter-Weyl's Theorem.

4.1. **Complete Reducibility for Finite Representations.** Let G be a compact group, similar to the finite case, we can define the notion of a representation of G: at the most general setting, a *representation* of G is simply a group homomorphism $f : G \to H$, where the group H is not even necessarily associated with a linear structure. However, in this paper, we will be only interested in *linear representations*, thus $H = \mathrm{End}(V)$ for some vector space V. However, in contrast with the finite case, here V may not have a finite basis. When V is infinite dimensional and is associated with a topology, we say (π, V) is *irreducible* if it has no proper nonzero invariant *closed* subspaces.

In the following sections, we will be interested in general of V being a Hilbert space, that is, an inner product space which is complete under the norm induced by the inner product. In particular, we will be interested in the Hilbert space $L^2(G)$ of square integrable functions on G. The inner product is given by conjugate integration (this is why we developed the theory of integration on compact groups with Haar measures earlier):

$$\langle f_1, f_2 \rangle_{L^2} \triangleq \int_G f_1(g)\overline{f_2(g)}dg \tag{4.1}$$

In addition, $L^2(G)$ is indeed a complete metric space with the metric induced by the inner product defined above, for a detailed examination, see Theorem 3.11 in [6].

In fact, similar to the case of finite group, in the general case of G being a compact group, we can show the existence of an G-equivariant inner product. This also means, every finite dimensional representation of a compact group is *unitarisable*.

Theorem 4.1. *Let G be a compact group and (π, V) be a finite dimensional representation of G, then V admits a G-equivariant inner product.*

Proof. We will again apply Weyl's trick of averaging over the group G, but this time by integrating over G: let $H_0(,)$ be an arbitrary inter product on G, define

$$H(v, w) \triangleq \int_G H_0(\pi(g)v, \pi(g)w)dg$$

Then, $H(,)$ is G-equivariant by construction. □

If we replace "sum over group finite group G" by "integrate over compact group G" in Lemma 3.11 and Lemma 3.12, etc. The result of complete reducibility follows for *finite* representations of compact group G.

Theorem 4.2 (Complete reducibility for finite representations)**.** *Any finite representation of a compact group G admits an orthogonal decomposition into irreducible sub-representations.*

Now, let's step to the complete reducibility of arbitrary representations.

4.2. **Matrix Coefficients.** Let G be a compact group, (π, V) be a representation of G. Then if V is finite dimensional, by choosing a basis for V, we can write out the matrix representation of the linear map $\pi(g)$ for every $g \in G$. Or equivalently, we have a collection of functions $\phi_{ij} : G \to \mathbb{C}$ that tells us what is the value of the (i,j)th matrix entry of $\pi(g)$. Even more importantly, if we are able to construct such a collection of functions, then we also obtain a representation of G! The question is, how do we construct this collection of functions?

Let's brute force through one possible construction. Under the given basis $B = \{e_i\}_{i=1}^n$, suppose we have a vector $v = (v_1, \dots, v_n)^T = \sum_1^n v_i e_i$ and an element $g \in G$. Suppose the matrix $\pi(g)$ is represented by:

$$\begin{pmatrix} \phi_{11}(g) & \cdots & \phi_{1n}(g) \\ \vdots & \ddots & \vdots \\ \phi_{n1}(g) & \cdots & \phi_{nn}(g) \end{pmatrix}$$

Computing $\phi(g)(v)$ gives, $\pi(g)(v) = (\sum_1^n \phi_{1i}(g) \cdot v_i, \cdots, \sum_1^n \phi_{ni}(g) \cdot v_i)^T$, hence $\pi_{ij}(g) = L_i(\pi(g)e_j)$, where L_i is a linear functional on V that picks out the i-th component of the vector, $L_i(\sum a_j e_j) = a_i$.

This motivates us to define *matrix coefficients*, an abstract characterization of those family of functions independent of choices of a basis or even a concrete vector space V.

Definition 4.3 (Matrix Coefficients)**.** Let G be a group, (π, V) be a finite dimensional representation of G, then *matrix coefficients* on G are maps of the form

$$\phi : g \mapsto L(\pi(g)v)$$

for $L \in V^*$, $v \in V$.

Let $\mathcal{M}_\pi$ denote the set of matrix coefficients of representation (π, V), then it forms a vector space. Furthermore, we can show an intimate relation between matrix coefficients and the dual representation.

Proposition 4.4. *f is a matrix coefficient of (π, V) iff $\check{f}(g) \triangleq f(g^{-1})$ is a matrix coefficient of the dual representation $(\hat{\pi}, V^*)$.*

Proof. By the identification $V \simeq V^{**}$, from left to right:

$$\check{f}(g) = f(g^{-1}) = L(\pi(g^{-1})v) = (L \circ \pi(g)^{-1})v = v^{**}(\hat{\pi}(g))$$

From right to left:

$$f(g) = \check{f}(g^{-1}) = v^{**}(\hat{\pi}(g^{-1})) = (L \circ \pi(g))v = L(\pi(g)v)$$

$\square$

Since the vector space associated with matrix coefficients are finite dimensional, by straightforward linear algebra, we may check the set of linear functionals $B^* = \{L_i\}_{i=1}^n$ induced from the basis $B = \{e_i\}_{i=1}^n$ for V is a basis for the dual space V^*. Hence $\dim(\mathcal{M}_\pi) \le \dim(V)^2$. This allows us to prove the following theorem.

Let $\lambda(g)f$ and $\rho(g)f$ denote left and right translations by g respectively,

$$\lambda(g)(f) = x \mapsto f(g^{-1}x), \qquad \rho(g)(f) = x \mapsto f(xg) \tag{4.2}$$

Theorem 4.5. *Let f be a function on G. Then the followings are equivalent:*

(1) $\lambda(g)f$ spans a finite dimensional vector space.
(2) $\rho(g)f$ spans a finite dimensional vector space.

(3) f is a matrix coefficient of a finite dimensional representation.

Proof. Firstly, suppose (3), let (π, V) be a finite dimensional representation of G and $f(h) = L(\pi(h)v)$, then $(\lambda(g)(f))(h) = f(g^{-1}x) = L(\pi(g^{-1}x)v)$ and $(\rho(g)(f))(h) = f(xg) = L(\pi(xg)v)$, they are also matrix coefficients of V. Because $\dim(V) < \infty$, $\dim(\mathcal{M}_\pi) \leq \dim(V)^2$, the vector spaces spanned by left and right translations also have finite dimension, hence (3) $\Rightarrow$ (1), (2).

Suppose $\rho(g)(f)$ spans a vector space $V \subset \mathrm{Hom}(G, \mathbb{C})$, $\dim(V) < \infty$, then (ρ, V) is a representation of G: the action is $\rho(g)(v) = x \mapsto v(xg)$. Define a linear functional $L \in V^* \subset \mathrm{Hom}(\mathrm{Hom}(G, \mathbb{C}), \mathbb{C})$ by $L(\phi) = \phi(1)$, where 1 is the unit of G. Then $L(\rho(g)f) = (\rho(g)(f))(1) = f(g1) = f(g)$, f is indeed a matrix coefficient of V, hence (2) $\Rightarrow$ (3).

Finally, suppose $\lambda(g)(f)$ spans a vector space $V \subset \mathrm{Hom}(G, \mathbb{C})$, $\dim(V) < \infty$. Let $\check{f}(h) = f(h^{-1})$ as usual, and define $\widetilde{V}$ by

$$\begin{aligned}\widetilde{V} &= \left\{\lambda(g)f \circ (-)^{-1} \in \mathrm{Hom}(G, \mathbb{C}) : g \in G\right\} \\ &= \left\{h \mapsto f(g^{-1}h^{-1}) : g \in G\right\} = \left\{\rho(g)(\check{f}) : g \in G\right\}\end{aligned}$$

Then $\dim(\widetilde{V}) < \infty$, by the previous argument, $\check{f}$ is a matrix coefficient, by Proposition 4.4, f is a matrix coefficient. Hence, (1) $\Rightarrow$ (3). $\square$

4.3. **Schur's Orthogonality.** Before we dive into the discussion of orthogonality of matrix coefficients, we firstly recall a remarkable theorem in functional analysis that uniquely determines the forms of continuous linear functionals on a Hilbert space by its inner product.

Theorem 4.6 (F. Riesz representation theorem in a Hilbert space)**.** *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. Then, given any continuous linear functional $l \in H^*$, there exist one and only one vector $y_l \in H$ such that*

$$l(x) = \langle x, y_l \rangle \qquad (\forall x \in H)$$

Proof. See Theorem 4.6-1 in [2]. $\square$

Hence, if (π, H) is a finite dimensional representation of compact group G, then all matrix coefficients of H are of the from $g \mapsto L(\pi(g)v) = \langle \pi(g)v, y_L \rangle$ for some $y_L \in H$.

Lemma 4.7. *Let G be group, (π_1, V_1) and (π_2, V_2) be (complex) representations of G. Let $\langle , \rangle$ be any inner product on V_1. If $v_i, w_i \in V_i$, then the map $T : V_1 \to V_2$ defined by*

$$T(w) = \int_G \langle \pi_1(g)w, v_1 \rangle \cdot \pi_2(g^{-1})v_2 dg$$

is an interwining operator between V_1 and V_2.

Proof. We want to show, given $h \in G$,

$$\pi_2(h)(T(v)) \stackrel{?}{=} T(\pi_1(h)(v))$$

Let's expand the definition on the right hand side:

$$\begin{aligned}T(\pi_1(h)(v)) &= \int_G \langle \pi_1(g)(\pi_1(h)(v)), v_1 \rangle \cdot \pi_2(g^{-1})v_2 dg \\ &= \int_G \langle \pi_1(gh)(v), v_1 \rangle \cdot \pi_2(g^{-1})v_2 dg\end{aligned}$$

By change of variable $g \mapsto gh^{-1}$ gives:

$$\begin{aligned}
T(\pi_1(h)(v)) &= \int_G \langle \pi_1(g)(v), v_1\rangle \cdot \pi_2(hg^{-1})v_2 d(gh^{-1}) \\
&= \int_G \langle \pi_1(g)(v), v_1\rangle \cdot \pi_2(h)(\pi_2(g^{-1})v_2) d(gh^{-1}) \\
&= \int_G \langle \pi_1(g)(v), v_1\rangle \cdot \pi_2(h)(\pi_2(g^{-1})v_2) dg && \text{(right invariance)} \\
&= \int_G \pi_2(h)(\langle \pi_1(g)(v), v_1\rangle \cdot (\pi_2(g^{-1})v_2)) dg && \text{(linearity)} \\
&= \pi_2(h)(\int_G \langle \pi_1(g)v, v_1\rangle \cdot \pi_2(g^{-1})v_2 dg) && \text{(linearity)} \\
&= \pi_2(h)(T(v))
\end{aligned}$$

□

Theorem 4.8 (Schur orthogonality, between representations)**.** *Let G be a compact group, (π_1, V_1), (π_2, V_2) be two irreducible representations of G. Then either $\mathcal{M}_{\pi_1} \perp \mathcal{M}_{\pi_2}$ in $L^2(G)$ or the representations are isomorphic.*

Proof. We will prove the result by negating one of the conclusion and use it to prove the other. Suppose $\mathcal{M}_{\pi_1} \not\perp \mathcal{M}_{\pi_2}$, then there exist matrix coefficients $f_1 \in \mathcal{M}_{\pi_1}$ and $f_2 \in \mathcal{M}_{\pi_2}$ such that $\langle f_1, f_2\rangle \neq 0$. By Riesz representation theorem, we may characterize f_1 and f_2 by:

$$f_1(g) = \langle \pi_1(g)w_1, v_1\rangle \quad f_2(g) = \langle \pi_2(g)w_2, v_2\rangle$$

for some $w_i, v_i \in V_i$. To avoid notational confusion, let $H(\cdot,\cdot)$ denote the inner product of the Hilbert space, and let $\langle \cdot,\cdot\rangle$ denote the inner product of both V_1 and V_2. Then by our assumption,

$$H(f_1, f_2) = \int_G \langle \pi_1(g)w_1, v_1\rangle \cdot \overline{\langle \pi_2(g)w_2, v_2\rangle} dg \neq 0$$

By complex conjugate, invariance and linearity,

$$\begin{aligned}
&\int_G \langle \pi_1(g)w_1, v_1\rangle \cdot \overline{\langle \pi_2(g)w_2, v_2\rangle} dg \\
&= \int_G \langle \pi_1(g)w_1, v_1\rangle \cdot \langle v_2, \pi_2(g)w_2\rangle dg \\
&= \int_G \langle \pi_1(g)w_1, v_1\rangle \cdot \langle \pi_2(g^{-1})v_2, \pi_2(g^{-1})\pi_2(g)w_2\rangle dg \\
&= \int_G \langle \pi_1(g)w_1, v_1\rangle \cdot \langle \pi_2(g^{-1})v_2, w_2\rangle dg \\
&= \int_G \langle \langle \pi_1(g)w_1, v_1\rangle \cdot \pi_2(g^{-1})v_2, w_2\rangle dg \\
&= \left\langle \int_G \langle \pi_1(g)w_1, v_1\rangle \cdot \pi_2(g^{-1})v_2 dg, w_2 \right\rangle \neq 0
\end{aligned}$$

Let $T : V_1 \to V_2$ be define as in Lemma 4.7 and plugging the definition, we have,

$$\langle T(w_1), w_2\rangle \neq 0$$

Hence $T : V_1 \to V_2$ is not zero, by Schur's Lemma (in compact case), T is an isomorphism. $\square$

Theorem 4.8 shows that if we have two non-isomorphic irreducible representations of a compact group G, then any pair of matrix coefficients of each representation is orthogonal. We now consider the orthogonality of matrix coefficients of the same irreducible representation, we will give an explicit formula for computing the inner product of matrix coefficients of the same irreducible representation.

Theorem 4.9 (Schur's Orthogonality, in one representation)**.** *Let G be a compact group, (π, V) be an irreducible representation of G. V is equipped with inner product $\langle \cdot, \cdot \rangle$. Then fixing $v_1, v_2, w_1, w_2 \in V$, there exist a constant $d > 0$ such that*

$$\int_G \langle \pi(g)w_1, v_1 \rangle \cdot \overline{\langle \pi(g)w_2, v_2 \rangle} dg = d^{-1} \langle w_1, w_2 \rangle \cdot \langle v_2, v_1 \rangle$$

Proof. Firstly, let v_1, v_2 be fixed. Let $T : V \to V$ be defined similar to Lemma 4.7 by:

$$T(w) = \int_G \langle \pi(g)w, v_1 \rangle \pi(g^{-1}) v_2 dg$$

Then, by Schur's Lemma, $T = \lambda I$ for some constant λ (depending on v_1 and v_2), and

$$\begin{aligned}
&\int_G \langle \pi(g)w_1, v_1 \rangle \cdot \overline{\langle \pi(g)w_2, v_2 \rangle} dg \\
=&\int_G \langle \pi(g)w_1, v_1 \rangle \cdot \langle v_2, \pi(g)w_2 \rangle dg \\
=&\int_G \langle \pi(g)w_1, v_1 \rangle \cdot \langle \pi(g^{-1})v_2, w_2 \rangle dg \\
=&\left\langle \int_G \langle \pi(g)w_1, v_1 \rangle \cdot \pi(g^{-1})v_2 dg, w_2 \right\rangle \\
=&\langle T(w_1, w_2) \rangle = \lambda \langle w_1, w_2 \rangle
\end{aligned}$$

Now, let w_1, w_2 be fixed. Let $\widetilde{T} : V \to V$ be defined by

$$\widetilde{T}(v) = \int_G \langle \pi(g)v, w_2 \rangle \cdot \pi(g^{-1}) w_1 dg$$

Then, $\widetilde{T} = \widetilde{\lambda} I$, and

$$\begin{aligned}
&\int_G \langle \pi(g)w_1, v_1 \rangle \cdot \overline{\langle \pi(g)w_2, v_2 \rangle} dg \\
=&\int_G \langle \pi(g)w_1, v_1 \rangle \cdot \langle v_2, \pi(g)w_2 \rangle dg \\
=&\int_G \langle \pi(g)w_1, v_1 \rangle \cdot \langle \pi(g^{-1})v_2, w_2 \rangle dg
\end{aligned}$$

Substitute g by g^{-1}, since Haar measures on compact groups are unique upto a constant multiple, let c be the constant, then $d(g^{-1}) = cdg$, hence

$$\begin{aligned}
\dots =& c\int_G \langle \pi(g^{-1})w_1, v_1\rangle \cdot \langle \pi(g)v_2, w_2\rangle dg \\
=& c\int_G \langle\langle \pi(g)v_2 w_2\rangle \pi(g^{-1})w_1, v_1\rangle dg \\
=& c\langle \int_G \langle \pi(g)v_2 w_2\rangle \pi(g^{-1})w_1 dg, v_1\rangle \\
=& c\langle \widetilde{T}(v_2), v_1\rangle = c\widetilde{\lambda}\langle v_2, v_1\rangle
\end{aligned}$$

Combining the two results above, there exist a constant d such that

$$\int_G \langle \pi(g)w_1, v_1\rangle \cdot \overline{\langle \pi(g)w_2, v_2\rangle} dg = d^{-1}\langle w_1, w_2\rangle \cdot \langle v_2, v_1\rangle$$

If we let $w_1 = w_2$, $v_1 = v_2$, then the left hand side is positive the definition of inner product on the Hilbert space, then d must be positive. $\square$

4.4. **Peter-Weyl's Theorem.** The goal of the this main section is to prove an arbitrary representation of a compact group can be reduced into a direct sum of *finite dimensional* irreducible representations. In particular, if we have the reducibility, by Theorem 4.8 and Theorem 4.9, we can show the components are pairwise orthogonal.

The main technique used in the proofs follows from the idea of using matrix coefficients to construct suitable finite dimensional subrepresentations of the given representation. Thus, what lies in the heart of Peter-Weyl's Theorem is the observation that there exist an "adequate" supply of matrix coefficients on the given group G. Hence, we sometimes refer to this result (Theorem 4.14) as the Peter-Weyl's Theorem.

Throughout the discussion of Peter-Weyl's Theorem, we will be mainly interested in normed vector space of continuous functions on G, or space of L^p functions on G, where the p-norm is defined as, for $1 \leq p < \infty$

$$\|f\|_p \triangleq \left\{ \int_G |f(g)|^p \right\}^{1/p}$$

when $p = \infty$, we define the sup-norm by

$$\|f\|_\infty \triangleq \operatorname{esssup}_G f = \inf\{a \in \mathbb{R} : \mu\{g \in G : f(x) > a\} = 0\} = \sup_G |f(x)| \quad \text{if } f \in C(G)$$

We define $L^p(G)$ to be the set containing all functions f on G such that $\|f\|_p < \infty$.

Let $C(G)$ denote the collection of continuous functions on compact group G, then $C(G)$ forms a ring with multiplication being *convolution*:

$$(f_1 \star f_2)(g) = \int_G f_1(gh^{-1})f_2(h)dh = \int_G f_1(h)f_2(h^{-1}g)dh$$

Given $\phi \in C(G)$, let $T_\phi : C(G) \to C(G)$ be the linear operator by left convolution: $T_\phi : f \mapsto \phi \star f$.

Proposition 4.10. *Let $f \in C(G)$ be given, then*

$$\|f\|_1 \leq \|f\|_2 \leq \|f\|_\infty$$

Proof. By Cauchy-Schwarz inequality, let $\langle\cdot,\cdot\rangle$ be the inner product defined as Equation (4.1), let 1 denote the constant function 1,

$$\|f\|_1 = \langle |f|, 1\rangle \le (\langle |f|, |f|\rangle)^{1/2} \cdot (\langle 1, 1\rangle)^{1/2} = \left(\int_G |f(g)|^2 \, dg\right)^{1/2} = \|f\|_2$$

The second inequality is trivial,

$$\|f\|_2 = \left(\int_G |f(g)|^2 \, dg\right)^{1/2} \le \left(\int_G \|f\|_\infty^2 \, dg\right)^{1/2} = \|f\|_\infty$$

$\square$

Proposition 4.11. *Let G be a compact group, $\phi \in C(G)$. Then T_ϕ is a bounded operator on $L^1(G)$. Further, if $f \in L^1(G)$, then $T_\phi f \in L^\infty(G)$ and*

$$\|T_\phi f\|_\infty \le \|\phi\|_\infty \|f\|_1$$

Proof. Let $f \in L^1(G)$ be given. We estimate the sup-norm of $T_\phi f$ by

$$\|T_\phi f\|_\infty = \sup_{g\in G} \left| \int_G \phi(gh^{-1}) f(h) dh \right| \le \|\phi\|_\infty \int_G |f(h)| \, dh$$

Put $C \triangleq \|\phi\|_\infty$, $C < \infty$ since ϕ is a continuous function on a compact set hence obtain a finite extrema. Since $f \in L^1(G)$, $\|f\|_1 < \infty$, thus $T_\phi f \in L^\infty(G)$.

Furthermore by Proposition 4.10, $\|T_\phi f\|_1 \le \|T_\phi f\|_2 \le \|T_\phi f\|_\infty$, We have, $\|T_\phi f\|_1 \le C \|f\|_1$, thus by definition the operator T_ϕ is bounded in $L^1(G)$. In fact, T_ϕ is bounded in all three norms: $1, 2, \infty$. $\square$

Proposition 4.12. *Let G be a compact group, $\phi \in C(G)$, then*

(1) T_ϕ is a bounded operator on $L^2(G)$ and $|T_\phi| \le \|\phi\|_\infty$.
(2) T_ϕ is compact.
(3) If $\phi(g^{-1}) = \overline{\phi(g)}$, it is self-adjoint.

Proof. (1) Let $f \in L^2(G)$ be given, then $f \in L^1(G)$ by Proposition 4.10 and by the argument in Proposition 4.11,

$$\|T_\phi f\|_2 \le \|T_\phi f\|_\infty \le \|\phi\|_\infty \|f\|_1 \le \|\phi\|_\infty \|f\|_2$$

Hence T_ϕ is bounded on $L^2(G)$ and $|T_\phi| \le \|\phi\|_\infty$.

(2) Let B be a set of bounded functions in $L^2(G)$, without loss of generality, we consider the unit ball in $L^2(G)$. Since $L^2(G) \subset L^1(G)$, it suffices to consider the unit ball in $L^1(G)$, that is:

$$B \triangleq \left\{ f \in L^1(G) : \|f\|_1 \le 1 \right\}$$

We want to show the image set $T_\phi(B)$ is sequentially compact, that is, every infinite sequence in $T_\phi(B)$ has a convergent subsequence. We will establish the result by using Ascoli and Arzela Theorem 2.4. First off, by Proposition 4.11, we know $T_\phi(B)$ is bounded, hence it suffice to show $T_\phi(B)$ is equicontinuous.

Let $\epsilon > 0$ be given, since $\phi \in C(G)$ and G is compact, ϕ is uniformly continuous, there exist a neighborhood N of the identity $1_G \in G$ such that

$$|\phi(kg) - \phi(g)| < \epsilon \qquad (\forall k \in N)$$

Then given $f \in B$, $k \in N$, $g \in G$,

$$\begin{aligned}|T_\phi f(kg) - T_\phi f(g)| &= \left|\int_G (\phi(kgh^{-1}) - \phi(gh^{-1}))f(h)dh\right| \\ &\leq \int_G \left|\phi(kgh^{-1}) - \phi(gh^{-1})\right| \cdot |f(h)|\, dh \\ &\leq \epsilon \|f\|_1 \leq \epsilon\end{aligned}$$

By definition, $T_\phi(B)$ is equicontinuous, hence apply Theorem 2.4, $T_\phi(B)$ is sequentially compact, T_ϕ is compact.

(3) Suppose we have $\phi(g^{-1}) = \overline{\phi(g)}$, then

$$\begin{aligned}\langle T_\phi f_1, f_2\rangle &= \int_G T_\phi f_1(g) \cdot \overline{f_2(g)} dg \\ &= \int_G (\int_G \phi(gh^{-1}) f_1(h) dh) \cdot \overline{f_2(g)} dg \\ &= \int_G (\int_G \overline{\phi(hg^{-1})} f_1(h) dh) \cdot \overline{f_2(g)} dg \\ &= \int_G f_1(h) \cdot \overline{T_\phi f_2(h)} dh \\ &= \langle f_1, T_\phi f_2\rangle\end{aligned}$$

Hence T_ϕ is self-adjoint.

□

Proposition 4.13. *Let G be a compact group, $\phi \in C(G)$, $\lambda \in \mathbb{C}$, then the λ-eigenspace*

$$V(\lambda) = \left\{f \in L^2(G) : T_\phi f = \lambda f\right\}$$

is invariant under $\rho(g)$ for all $g \in G$, where $\rho(g)$ is defined as usual as in Equation (4.2)*.*

Proof. Let $g \in G$ be given,

$$\begin{aligned}(T_\phi(\rho(g)f))(x) &= \int_G \phi(xh^{-1}) f(hg) dh \\ &= \int_G \phi(xgh^{-1}) f(h) dh \qquad \text{substitute } h \to hg^{-1} \\ &= (\rho(g)(T_\phi f))(x)\end{aligned}$$

□

Theorem 4.14 (Peter-Weyl, Part 1)**.** *The matrix coefficients of G are dense in $L^2(G)$.*

Proof. Let $f \in L^2(G)$ be given, we need to show for any $\epsilon > 0$, there exist a matrix coefficient f' (thus associated with a finite dimensional representation), such that $\|f' - f\| < \epsilon$.

Let $\epsilon > 0$ be given, since G is compact, f is uniformly continuous on G, hence left translation of f by g: $\lambda(g)f = h \mapsto f(g^{-1}h)$ is uniformly continuous on G. Then there exist a neighborhood $U \ni 1_G$ such that $\|\lambda(g)f - \lambda(1_G)f\|_\infty = \|\lambda(g)f - f\|_\infty < \epsilon/2$ for all $g \in U$.

Define $\phi : U \to \mathbb{C}$ be a continuous map satisfying $\int_G \phi(g)dg = \int_U \phi(g)dg = 1$, and $\phi(g) = \phi(g^{-1})$. Then the operator T_ϕ is self-adjoint and compact by Proposition 4.12. We collect the following facts:

(1) $\|T_\phi f - f\| < \epsilon/2$:
$$\begin{aligned}|T_\phi f(h) - f(h)| &= \left|\int_G \phi(g)f(g^{-1}h) - \phi(g)f(h)dg\right| \\ &= \left|\int_U \phi(g)f(g^{-1}h) - \phi(g)f(h)dg\right| \\ &\leq \int_U \phi(g)\cdot\left|f(g^{-1}h) - f(h)\right|dg \\ &\leq \int_U \phi(g)\cdot\|\lambda(g)f - f\|_\infty\, dg \leq \frac{\epsilon}{2}\end{aligned}$$
(2) T_ϕ is a compact operator on $L^2(G)$ by Proposition 4.12.
(3) Let λ denote an eigenvalue of T_ϕ, by spectral theorem Theorem 2.3, the eigenspaces $V(\lambda) \subset L^2(G)$ are all finite dimensional except perhaps $V(0)$. Furthermore, the spectral theorem tells us spaces $V(\lambda)$ are mutually orthogonal and they span $L^2(G)$.
(4) By Proposition 4.13, maps in $V(\lambda)$ are all T_ϕ invariant.

Let f_λ be the projection of f on the basis $V(\lambda)$:
$$f_\lambda = \sum_{w\in V(\lambda)} \langle f, w\rangle w$$
By orthogonality between $V(\lambda)$s, we have
$$\sum_\lambda \|f_\lambda\|_2^2 = \|f\|_2^2 < \infty$$
For some positive real number q, define f', f'' by:
$$f'' \triangleq \sum_{|\lambda|>q} f_\lambda, \qquad f' \triangleq T_\phi f''$$
Then both f' and f'' are contained in a finite dimensional vector space: $\bigoplus_{|\lambda|>q} V(\lambda)$. By (4), and apply Theorem 4.5, it follows f is a matrix coefficient of a finite dimensional representation.

Lastly, we will show f' indeed approximates f within ϵ: choose q so that
$$\left\|\sum_{0<|\lambda|<q} f_\lambda\right\|_1 \leq \left\|\sum_{0<|\lambda|<q} f_\lambda\right\|_2 = \sqrt{\sum_{0<|\lambda|<q}\|f_\lambda\|_2^2} < \frac{\epsilon}{2\|\phi\|_\infty}$$
Then,
$$\begin{aligned}\|T_\phi(f - f'')\|_\infty &= \left\|T_\phi\left(f_0 + \sum_{0<|\lambda|<q} f_\lambda\right)\right\|_\infty = \left\|T_\phi\left(\sum_{0<|\lambda|<q} f_\lambda\right)\right\|_\infty \\ &\leq \|\phi\|_\infty \cdot \left\|\sum_{0<|\lambda|<q} f_\lambda\right\|_1 < \frac{\epsilon}{2}\end{aligned}$$

Therefore, use T_ϕ invariance,

$$\|f - f'\|_\infty = \|f + T_\phi(f - f - f'')\|_\infty = \|f - T_\phi f\|_\infty + \|T_\phi f - T_\phi f''\|_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Corollary 4.15. *The matrix coefficients of compact group G are dense in $L^2(G)$.*

Proof. Since $C(G)$ is dense in $L^2(G)$, it follows from Theorem 4.14. □

Theorem 4.16 (Peter-Weyl, Part 2)**.** *Let H be a Hilbert space and G be a compact group. Let $\pi : G \to End(H)$ be a unitary representation. Then H is a direct sum of finite dimensional irreducible representations.*

Proof. We will prove firstly, if H is nonzero then it has an irreducible finite dimensional invariant subspace. Then, we will extract a finite dimensional representation of G on H.

Let $v \in H$ be a nonzero vector, since the group action π is continuous and G is compact, it's uniformly continuous in G. Hence there exist a neighborhood N of $1_G \in G$ such that

$$\|\pi(g)v - \pi(1_G)v\| = \|\pi(g)v - v\| \leq \frac{\|v\|}{2} \qquad (\forall g \in N)$$

Define $\phi : N \to \mathbb{C}$ be a continuous map satisfying $\int_G \phi(g)dg = 1$. Then

$$\left\langle \int_G \phi(g) \cdot \pi(g)vdg, v \right\rangle = \langle v, v\rangle - \left\langle \int_N \phi(g) \cdot (v - \pi(g)v)dg, v \right\rangle$$

and by Cauchy Schwarz,

$$\begin{aligned}
&\left\| \left\langle \int_N \phi(g) \cdot (v - \pi(g)v)dg, v \right\rangle \right\| \\
&\leq \left\| \int_N \phi(g) \cdot (v - \pi(g)v)dg \right\| \cdot \|v\| \\
&\leq \int_N \|\phi(g)\| \, \|v - \pi(g)v\| \, dg \cdot \|v\| \\
&\leq \int_N \|\phi(g)\| \cdot \frac{\|v\|}{2} dg \cdot \|v\| = \frac{\|v\|^2}{2}
\end{aligned}$$

Hence, the integral $\int_G \phi(g) \cdot \pi(g)vdg \neq 0$.

By Theorem 4.14, since $\phi \in C(G)$, for arbitrary $\epsilon > 0$, there exist a matrix coefficient f associated to a finite dimensional representation (ρ, W) such that $\|f - \phi\|_\infty < \epsilon$. Thus, in this case,

$$\left\| \int_G (f - \phi)(g) \cdot \pi(g)vdg \right\| \leq \epsilon \|v\|$$

Take small enough ϵ, then $\left\|\int_G f(g) \cdot \pi(g)vdg\right\| = \left\|\int_G \phi(g) \cdot \pi(g)vdg\right\| + \epsilon \|v\| \neq 0$.

By Proposition 4.4, $\check{f}$ is a matrix coefficient of the dual representation $(\hat{\rho}, W^*)$ (which has the same dimension). By definition of matrix coefficient, $\check{f}(g) = L(\hat{\rho}(g)w)$ for some $L \in (W^* \to \mathbb{C}) \simeq W$, $w \in W^*$. Define $T : W^* \to H$ by

$$T(x) \triangleq \int_G L(\hat{\rho}(g^{-1})x) \cdot \pi(g)vdg$$

using the same argument used to prove Lemma 4.7, we know $T \in \mathrm{Hom}^G(W^*, H)$ and it's nonzero since when evaluated at w:

$$T(w) = \int_G L(\hat{\rho}(g^{-1})w) \cdot \pi(g)v dg = \int_G \check{f}(g^{-1}) \cdot \pi(g)v dg = \int_G f(g) \cdot \pi(g)v dg \neq 0$$

Since $\dim(W^*) < \infty$, the image $T(W^*)$ is a nonzero finite dimensional invariant subspace of H!

Now, let Σ be the set of all sets of finite dimensional irreducible invariant subspaces, order its elements by inclusion. By Zorn's Lemma, the partial order Σ has a maximal element $\widetilde{S}$. If $\widetilde{S}$ spans H, we are done with $\widetilde{S}$ being a complete decomposition of H. Otherwise, the complement of the span of S is a nonzero, by previous arguments, it contains an invariant irreducible subspace U, then $\widetilde{S}$ is not maximal since $\widetilde{S} \cup \{U\}$ is larger than $\widetilde{S}$, we have a contradiction! $\square$

We have shown that a Hilbert space representation of a compact group may be completely reduced into a direct sum of *finite dimensional* irreducible representations. If we instantiate the result with $G = \mathbb{T}$, $H = L^2(\mathbb{R})$ and an action $\widetilde{\pi} : \mathbb{T} \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$\widetilde{\pi}(e^{i\theta})f \triangleq e^{i\theta} \cdot \hat{f}$$

where $\hat{f}$ is the complete decomposition of f into orthogonal components given by Peter-Weyl's Theorem (which is exactly the Fourier transform of f). If we chain $\widetilde{\pi}$ together with the quotient map $\mathbb{R} \to \mathbb{R} / \mathbb{Z} \simeq \mathbb{T}$ that sends $t \in \mathbb{R}$ to $e^{it} \in \mathbb{T}$, then we recover the map $\pi : \mathbb{R} \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$ from the introduction.

5. Conclusion

The theory Fourier of analysis was long developed throughout the history: it was first developed in 16th century to study the physical phenomenon of vibration of strings and to solve partial differential equations associated with it. It later became one of the core problems in analysis and relates to varies areas of mathematics. For example, it has relates intimately to the representation theory on compact groups as we have explored in this paper.

In fact, if we put more restriction on the case we studied and let G be a compact *abelian* group (of which $\mathbb{T}$ still is one archetypal example), there is an even more remarkable symmetry between representation theory and Fourier analysis: characters of G form an orthonormal basis of $L^2(G)$ and Fourier Transform and Fourier inversion formulas gives an isomorphism between G and the dual group $\hat{G}$ of its characters.

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