Signed Sums of kth Powers

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1 Introduction

There are many problems in number theory that involve looking at the representation of the integers or real numbers in the form $n = \sum_{i=1}^{m} \epsilon_i a_i$, where $\{a_i\}$ is a given sequence and the ϵ_i have values restricted to a given set, and m is the number to be found. Some famous problems are the Egyptian Fractions, where $\{a_i\} = 1/i$ and $\epsilon_i \in \{0, 1\}$, and we are examining representations of fractions, and Waring's Problem, which asks what is the minimum number of integers needed to present each natural number as the sum of a kth power. Many other problems of this sort can be formed with suitable choices of a_i and ϵ_i. In this paper, we will examine the result proved by Michael Bleicher [1], in which we choose $a_i = i^k$ for a fixed non-negative integer k, and $\epsilon_i \in \{-1, 1\}$. We will prove that infinitely many representations of any integer n will exist and an algorithm will be determined for finding m, where the algorithm is of polynomial time. Lower bounds of m will be proved and asymptotic estimates of m will be given in each of the following cases (a) k fixed and $n \to \infty$ and (b) n fixed and $k \to \infty$.

2 Preliminary Results

Definition 1. For a non-negative integer k, define $\epsilon_{k,j}$ for $0 \le j \le 2^k$ to be

$$\epsilon_{k,j} = \begin{cases} 1 & k = 0 \\ -\epsilon_{k-1,j} & \text{for } k > 0 \text{ and } 0 \le j < 2^{k-1} \\ \epsilon_{k-1,j-2^{k-1}} & \text{for } k > 0 \text{ and } 2^{k-1} \le j < 2^k \end{cases} \tag{1}$$

Definition 2. For k and l non-negative integers and x real, define

$$D_{k,l}(x) = \sum_{i=0}^{2^k-1} \epsilon_{k,i}(x+i)^l \tag{2}$$

By convention $0^0 = 1$.

We see that $D_{0,0}(x) = \sum_{i=0}^{0} \epsilon_{0,i}(x+i)^0 = 1$ and $D_{k,0}(x) = \sum_{i=0}^{2^k-1} \epsilon_{k,i} = 0$ for $k > 0$ by the definition of $\epsilon_{k,i}$.

Definition 3. Let $f(x)$ be a function defined on the integers. We define $D_k f(x)$ inductively for $k > 0$ by

$$D_0 f(x) = f(x) \tag{3}$$

$$D_k f(x) = D_{k-1} f(x + 2^{k-1}) - D_{k-1} f(x) \tag{4}$$

Lemma 1. *For all non-negative integers k,*

$$D_k f(x) = \sum_{i=0}^{2^k-1} \epsilon_{k,i} f(x+i) \tag{5}$$

Proof. We proceed by induction. For $k = 0$, $D_0 f(x) = \sum_{i=0}^{0} \epsilon_{0,i} f(x+i) = f(x)$, which is true by definition. Now, suppose the lemma holds for $D_{k-1} f(x)$. Then from (1) and (3), we get

$$\begin{aligned}
\sum_{i=0}^{2^k-1} \epsilon_{k,i} f(x+i) &= \sum_{i=0}^{2^{k-1}-1} \epsilon_{k,i} f(x+i) + \sum_{i=2^{k-1}}^{2^k-1} \epsilon_{k,i} f(x+i) \\
&= \sum_{i=0}^{2^{k-1}-1} -\epsilon_{k-1,i} f(x+i) + \sum_{i=2^{k-1}}^{2^k-1} \epsilon_{k-1,i-2^{k-1}} f(x+i) \\
&= -D_{k-1} f(x) + \sum_{i=0}^{2^{k-1}-1} \epsilon_{k,i} f(x + 2^{k-1} + i) \\
&= D_{k-1} f(x + 2^{k-1}) - D_{k-1} f(x) \\
&= D_k f(x)
\end{aligned}$$

□

Remark. If $f(x) = x^l$, we have $D_k f(x) = D_{k,l}(x)$ using (2) and (5).

Lemma 2. *For all non-negative integers k, $D_{k,k}(x)$ is constant and $D_{k,l}(x) = 0$ for $l < k$.*

Proof. We first consider $k = l$ and prove by induction. For $k = 0$, we have $D_{0,0}(x) = 1$ by definition, so it is constant. Now, suppose the lemma holds for $k = m - 1$. Then by Lemma 1 we have

$$D_{m,l}(x) = D_m x^l = D_{m-1}(x + 2^{m-1})^l - D_{m-1}(x)^l \tag{6}$$

Note that by the definition of $D_k f(x)$, we have $\frac{d}{dx} D_k f(x) = D_k f'(x)$. Then by differentiating (6), we get

$$\begin{aligned}
D'_{m,l}(x) &= D'_{m-1}(x + 2^{m-1})^l - D'_{m-1}(x)^l \\
&= l D_{m-1}(x + 2^{m-1})^{l-1} - l D_{m-1}(x)^{l-1} \\
&= l(D_{m-1,l-1}(x + 2^{m-1}) - D_{m-1,l-1}(x))
\end{aligned}$$

By the inductive hypothesis, $D_{m-1,l-1}(x + 2^{m-1}) - D_{m-1,l-1}(x) = 0$ for $m = l$. Thus $D'_{m,m}(x) = 0$, so $D_{m,m}(x)$ must be constant. Because $D_{l,l}(x)$ is constant, $D_{l+1,l}(x) = D_{l,l}(x + 2^l) - D_{l,l}(x) = 0$. Thus it is true that $D_{k,l}(x) = 0$ for $k > l$. □

Definition 4. $D_k = D_{k,k}(x)$

This will allow for the notation to be less cluttered.

Lemma 3. *For every pair of non-negative integers k and n,*

$$\sum_{i=1}^{n} i^k = \frac{n^{k+1}}{k+1} + \frac{nP_{k-1}(n)}{(k+1)!} \tag{7}$$

where $P_{k-1}(n)$ is a integer polynomial in n of order $k-1$ with the convention that $P_{-1}(n) = 0$.

Proof. We proceed by induction on k. For $k = 0$, $\sum_{i=1}^{k} i^0 = n = \frac{n^1}{1} + \frac{nP_{-1}(n)}{1!}$, so it is true for $k = 0$. Now, suppose the lemma is true for all integers j, $0 \leq j < k$, and we want to show it is true for $j = k$.
Define $c_j = j^{k+1} - (j-1)^{k+1}$. By the Binomial Theorem,

$$\begin{aligned} c_j &= j^{k-1} - \sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^i j^{k+1-i} \\ &= -\sum_{i=1}^{k+1} \binom{k+1}{i} (-1)^i j^{k+1-i} \end{aligned}$$

Note that $n^{k+1} = \sum_{j=1}^{n} c_j$ by the definition of c_j. Then by simplifying n^{k+1} and using the inductive hypothesis,

$$\begin{aligned} n^{k+1} &= -\sum_{j=1}^{n} \sum_{i=1}^{k+1} \binom{k+1}{i} (-1)^i j^{k+1-i} \\ &= -\sum_{i=1}^{k+1} \binom{k+1}{i} (-1)^i \sum_{j=1}^{n} j^{k+1-i} \\ &= -(-1)^1 \binom{k+1}{1} \sum_{j=1}^{n} j^{k+1-1} - \sum_{i=2}^{k+1} \binom{k+1}{i} (-1)^i \sum_{j=1}^{n} j^{k+1-i} \\ &= (k+1) \sum_{j=1}^{n} j^k - \sum_{i=1}^{k+1} \binom{k+1}{i} (-1)^i \left(\frac{n^{k+2-i}}{k+2-i} + \frac{nP_{k-i}(n)}{(k+2-i)!} \right) \end{aligned}$$

Because the highest order of $nP_{k-i}(n)$ is $k-2+1 = k-1$ and the largest order of $(k+2-i)!$ is $k+2-2 = k$, $\sum_{i=1}^{k+1} \binom{k+1}{i} (-1)^i \left(\frac{n^{k+2-i}}{k+2-i} + \frac{nP_{k-i}(n)}{(k+2-i)!} \right)$ can be written in the form $\frac{nP_{k-1}(n)}{k!}$. Then we can solve for $\sum_{j=1}^{n} j^k$.

$$\begin{aligned} n^{k+1} &= (k+1) \sum_{j=1}^{n} j^k - \frac{nP_{k-1}(n)}{k!} \\ \sum_{j=1}^{n} j^k &= \frac{n^{k+1}}{k+1} - \frac{nP_{k-1}(n)}{(k+1)!} \end{aligned}$$

which is the desired form. The proof is complete. □

Lemma 4. *For every positive integer n, and every non-negative integer k, there exists an integer N such that*

$$\sum_{i=1}^{N} i^k \equiv 0 \pmod{n} \tag{8}$$

and N can be chosen such that $N \equiv 0 \pmod{n}$.

Proof. We show that taking $N = n(k+1)!$ proves the lemma. $N = n(k+1)! \equiv 0 \pmod{n}$. By the previous lemma, we have

$$\begin{aligned}\sum_{i=1}^{n(k+1)!} i^k &= \frac{(n(k+1)!)^{k+1}}{k+1} - \frac{n(k+1)!P_{k-1}(n)}{(k+1)!} \\ &= \frac{(n(k+1)!)^{k+1}}{k+1} - nP_{k-1}(n) \\ &\equiv 0 \pmod{n}\end{aligned}$$

□

Note that $N = n(k+1)!$ is not the minimal value. For example, $N = 4$ will work for $n = 4$, $k = 3$, but the value the proof yields is 96.

Lemma 5. *For every positive integer n and non-negative integer k, there is a positive integer N, $N \equiv 0 \pmod{n}$ such that for every integer l*

$$\sum_{i=l+1}^{l+N} i^k \equiv 0 \pmod{n} \tag{9}$$

Proof. Choose N as in Lemma 4. Then the sum in (9) covers the identical range (mod n) as the sum in Lemma 4 independent of l, and thus has sum $\equiv 0 \pmod{n}$. □

Let N be the number that depends only on n given by Lemma 5.

Lemma 6. *For every positive integer n, non-negative integer k, and j with $0 \le j < n$, there is a number M_j and some choice of ϵ_i such that*

$$j \equiv \sum_{i=1}^{m_j} \epsilon_i i^k \pmod{n} \tag{10}$$

For $k > 0$, we can choose M_j to satisfy

$$M_j \le \Big(\frac{j+2}{2}\Big) n(k+1)! \tag{11}$$

For $k = 0$, we satisfy (10) and (11) but choosing $M_j = j$, $\epsilon_i = 1$.

Proof. It is obvious that $j \equiv \sum_{i=1}^{j} \epsilon_i \pmod{n}$, where $\epsilon_i = 1$ for all i. For $j = 0$ and $k > 0$, (8) gives a representation with $M_j = N$ and $\epsilon_i = 1$ for all i. It is clear that $N \le (2/2)n(k+1)! = n(k+1)! = N$, so it satisfies (11).

Consider $j > 0$ and $k > 0$. Take $l = qN$ for any positive integer q, and by Lemma 6, we see $\sum_{i=qN+1}^{(q+1)N} i^k \equiv 0 \pmod{n}$. Because $N \equiv 0 \pmod{n}$, $(qN+1)^k \equiv 0 \pmod{n}$. We have $(qN+1)^k = \sum_{i=qN+1}^{(q+1)N} i^k - \sum_{i=qN+2}^{(q+1)N} i^k \equiv -\sum_{i=qN+2}^{(q+1)N} i^k \pmod{n} \equiv 1 \pmod{n}$. Given j, $0 < j < n$, we see

$$j \equiv \begin{cases} \sum_{q=0}^{j/2-1} \Big[(qN+1)^k \sum_{qN+2}^{(q+1)N} i^k\Big] & j \text{ even} \\ \sum_{q=0}^{\lfloor j/2 \rfloor -1} \Big[(qN+1)^k \sum_{qN+2}^{(q+1)N} i^k\Big] + \Big(N[j/2]+1)^k\Big) & j \text{ odd} \end{cases} \tag{12}$$

For j even, $M_j = N(j/2-1) \le \big(\frac{j+2}{2}\big)n(k+1)!$. For j odd, $[j/2] < \frac{j+2}{2}$, so $[j/2]-1 < \frac{j-2}{2}$ and $N \le n(k+1)!$. Then $M_j \le \big(\frac{j+2}{2}\big)n(k+1)!$. Thus (11) holds. □

We will now proceed to prove our main result.

3 Proving Existence

First the existence of a representation of the form $n = \sum_{i=1}^{m} \epsilon_i i^k$ for every n will be proved. Then an algorithm for how to find m will be given and some estimates will be made on the length of the expansion. This result is due to Michael Bleicher [1].

Theorem 1. *For every positive integer n and non-negative integer k, there is a positive integer m and choices of $\epsilon_i = \pm 1$ such that*

$$n = \sum_{i=1}^{m} \epsilon_i i^k$$

Proof. We apply Lemma 6 with $n = D_k$. Then for $0 \leq j < D_k$,

$$j \equiv \sum_{i=1}^{M_j} \epsilon_i i^k \pmod{D_k} \tag{13}$$

Then j and $\sum_{i=1}^{m_j} \epsilon_i i^k$ differ by a multiple of D_k. Let this difference be $\Delta = \pm l D_k$, where $l \geq 0$. Since D_k is constant, $D_k = D_{k,k}(i2^k + M_j + 1) = \sum_{n=0}^{2^k-1} \epsilon_{k,n}(i2^k + M_j + 1 + n)^k = \sum_{n=i2^k+M_j+1}^{(i+1)2^k+M_j} \epsilon_{k,n} n^k$. Suppose that $\Delta > 0$, then

$$\begin{aligned} \Delta &= l D_k \\ &= \sum_{i=0}^{l-1} D_k \\ &= \sum_{i=1}^{l-1} \Big(\sum_{n=i2^k+M_j+1}^{(i+1)2^k+M_j} \epsilon_{k,n} n^k \Big) \\ &= \sum_{i=M_j+1}^{l2^k+M_j} \epsilon_i i^k \end{aligned} \tag{14}$$

We add (14) to (13) to get

$$\begin{aligned} \sum_{i=1}^{m} \epsilon_i i^k &= j + \Delta \\ &= \sum_{i=1}^{M_j} \epsilon_i i^k + \sum_{i=M_j+1}^{l2^k+M_j} \epsilon_i i^k \\ &= \sum_{i=1}^{l2^k+M_j} \epsilon_i i^k \end{aligned}$$

If $\Delta < 0$, then we add $-\Delta$ to (13). If $\Delta = 0$, then $j = n$, and $l = 0$. In each case, we get $n = \sum_{i=1}^{l2^k+M_j} \epsilon_i i^k$, so a representation in the desired form is produced with $m = l2^k + M_j$. □

This gives one representation of each integer n in the desired form, but the construction of m seems to be not efficient for large values of n. Some bounds on the least value of such an integer m will be given later in the text. Here are some examples on the expansion.

Example 1. We find an expansion for $n = 160$, $k = 3$: $160 = -1^3 - 2^3 - 3^3 - 4^3 - 5^3 + 6^3 - 7^3 + 8^3 = \sum_{i=1}^{8} \epsilon_i i^3$. One expansion of $n = 160$ can be achieved with $m = 8$.

Example 2. We find an expansion for $n = 15$, $k = 4$: $15 = 1^4 + 2^4 - 3^4 + 4^4 - 5^4 - 6^4 + 7^4 - 8^4 - 9^4 + 10^4 = \sum_{i=1}^{10} \epsilon_i i^4$. One expansion of $n = 15$ can be achieved with $m = 10$.

Example 3. We find multiple expansions for $n = 5$, $k = 2$: $5 = 1^2 + 2^2 = \sum_{i=1}^{2} \epsilon_i i^2$, with $m = 2$. Another representation gives $5 = -1^2 - 2^2 + 3^2 - 4^2 + 5^2 = \sum_{i=1}^{5} \epsilon_i i^2$, with $m = 5$. Alternatively, $5 = 1^2 - 2^2 + 3^2 - 4^2 - 5^2 + 6^2 + 7^2 - 8^2 - 9^2 + 10^2 = \sum_{i=1}^{1} 0\epsilon_i i^2$, with $m = 10$. It becomes obvious that there can be many representations of n for fixed k.

We see that some integers have multiple expansions, so this leads to proving there are infinitely many representations of n with fixed n and k.

Corollary 1. *For very positive integer n and non-negative integer k, there are infinitely many positive integers m and choices of $\epsilon_j = \pm 1$ such that*

$$n = \sum_{i=0}^{m} \epsilon_i i^k$$

Proof. By Lemma 2, we know that D_k is constant, so

$$\begin{aligned} D_k(x) - D_k(x + 2^k) &= \sum_{i=0}^{2^k-1} \epsilon_i (x+i)^k - \sum_{i=0}^{2^k-1} \epsilon_i (x + 2^k + i)^k \\ 0 &= \sum_{i=x}^{2^k-1+x} \epsilon_i i^k + \sum_{i=x+2^k}^{2^{k+1}-1+x} \epsilon_i i^k \\ &= \sum_{i=x}^{2^{k+1}-1+x} \epsilon_i i^k \end{aligned}$$

since ϵ_i can be multiplied by -1 to get the equality. Given a representation $n = \sum_{i=0}^{m} \epsilon_i i^k$, we can take $x = m+1$ and add $\sum_{i=x}^{2^{k+1}-1+x} \epsilon_i i^k$ to n to get $n = \sum_{i=0}^{m+2^{k+1}} \epsilon_i i^k$ which is a new representation. This process can be repeated infinitely many times. Thus there are infinitely many representations of n in the desired form. □

We will proceed to give a better representation of j, where $j = \sum_{i=1}^{M_j} \epsilon_i i^k$ by modifying our procedure. Some definitions will be given to make notation easier.

Definition 5. Fix a positive integer k. Let $D = D_k$.

Definition 6. Let m_j be the least integer which yields the the expansion of j guaranteed by Theorem 1 for the fixed k.

Definition 7. Let $M = \max\{m_j : 0 \le j < D\}$

Since D only depends on k, by Lemma 6 the upper bound of M_j depends only on k. Thus M is determined by k.

Definition 8. Let Q_j be the greatest positive integer such that

$$\sum_{i=m_j+1}^{Q_j N + m_j} i^k < \sum_{Q_j N + m_j + 1}^{(Q_j+1)N + m_j} i^k \tag{15}$$

Let $Q = \max\{Q_j : 0 \le j < D\}$

We see that such a Q_j must exist because the left hand side of (15) is of order Q_j^{k+1} by Lemma 3 and the upper bound of the right hand side of (15) is $((Q_j+1)N+m_j)-(Q_jN+m_j+1)+1)((Q_j+1)N+m_j)^k = N((Q_j+1)N+m_j)^k$, which is of order Q_j^k. Also, $Q > 1$. We now find a lower bound for m_j.

Lemma 7. *For each positive integer j, the length of its shortest expansion m_j satisfies*

$$m_j \geq [((k+1)j)^{1/(k+1)}] \geq [j^{1/(k+1)}] \tag{16}$$

Proof. For $k = 0$, the expansion of j is $\sum_{i=0}^{j} i^0$, so $m_j = j$ which satisfies (16). Now suppose $k > 0$. By Theorem 1,

$$\begin{aligned}
j &= \sum i = 0^{m_j} \epsilon_i i^k \\
&\leq m_j^k + \sum_{i=0}^{m_j-1} i^k \\
&\leq m_j^k + \int_0^{m_j} t^k dt \\
&= m_j^k + \frac{m_j^{k+1}}{k+1}
\end{aligned}$$

$$\begin{aligned}
(k+1)j &\leq m_j^{k+1} + (k+1)m_j^k \\
&< (m_j+1)^{k+1} \\
m_j + 1 &> ((k+1)j)^{1/(k+1)} \\
m_j &\geq ((k+1)j)^{1/(k+1)}
\end{aligned}$$

The last inequality holds because we are working with all integers. Since $(k+1)^{1/(k+1)} > 1$, $((k+1)j)^{1/(k+1)} > j^{1/(k+1)}$, so $m_j \geq [((k+1)j)^{1/(k+1)}] \geq [j^{1/(k+1)}]$, which proves the lemma. □

We will need one more lemma before we can define the algorithm.

Lemma 8. *Let $\{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers that for every $r > 1$, satisfies*

$$\sum_{i=1}^{r} a_i \geq a_{r+1} \tag{17}$$

For fixed n and m, if $\sum_{i=1}^{m} a_i \geq |n|$, then there is a choice of $\epsilon_i = \pm 1$ such that

$$\left| n - \sum_{i=1}^{m} \epsilon_i a_i \right| < a_2 \tag{18}$$

Proof. It is sufficient to prove such an approximation exists for $n > 0$, since the approximation for $-n$ can be found by changing the signs for all of the ϵ_i.

We prove by induction on m. For $m = 1$, we want to show that $|n - \epsilon_1 a_1| < a_2$. The hypothesis gives $n \leq a_1$ and because $\{a_i\}$ is an increasing sequence, $n \leq a_1 < a_2$. Let $\epsilon_1 = 1$. Then it follows $|n - \epsilon_1 a_1| \leq a_1 < a_2$, so (18) is satisfied. If $m = 2$, then we either have $|n - (a_1 + a_2)| < a_2$ or $|n - (a_1 + a_2)| \geq a_2$. If it is case 1, then we are done. For case 2, from the hypothesis $n \leq a_1 + a_2$, we obtain $a_1 + a_2 - n \geq a_2$. Then by subtracting $2a_1$ from both sides, $a_2 - a_1 - n \geq a_2 - 2a_1$. Because a_1 and n are positive, $a_2 > a_2 - a_1 - n$. Since $a_1 < a_2$, $a_2 - 2a_1 > a_1 - 2a_1 = -a_1 > -a_2$. Putting all inequalities together yields $a_2 > a_2 - a_1 - n > -a_2$, so $|n - (-a_1 + a_2)| < a_2$. Thus we have found $\epsilon_1 = -1$, $\epsilon_2 = 1$ such that (18) is satisfied for $m = 2$.

Assume the lemma holds for $l < m$, we want to show (18) holds for $l = m$, where $m > 2$. From the hypothesis,

$$n < \sum_{i=1}^{m} a_i \tag{19}$$

Since $n \geq 0$ and $m > 2$ and (17) is true for $r = m$, subtracting a_m from both sides of (19) gives

$$n - a_m < \sum_{i=1}^{m-1} a_i$$

$$-a_m \leq n - a_m < \sum_{i=1}^{m-1} a_i$$

$$\sum_{i=1}^{m-1} -a_i < -a_m \leq n - a_m < \sum_{i=1}^{m-1} a_i$$

Thus $|n - a_m|$ satisfies (17) with $l = m - 1$, so for a choice $\epsilon_i = \pm 1$,

$$\left| |n - a_m| - \sum_{i=1}^{m=1} \epsilon_i a_i \right| < a_2$$

Then for $n \geq a_m$, we can choose $\epsilon_m = 1$ and for $n < a_m$, we can choose $\epsilon_m = -1$ such that

$$\left| n - \sum_{i=1}^{m} \epsilon_i a_i \right| < a_2$$

This conclude the inductive hypothesis, so the lemma has been proved for all m. □

We can now define the algorithm.

4 The Algorithm

Given $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, we want to find T such that $n = \sum_{i=1}^{T} \epsilon_i i^k$, where $\epsilon_i \in \{-1, 1\}$. The algorithm for finding T will be presented below.

Step 1: Compute D from Definition 5 with the given k.

Step 2: Choose j, $0 \leq j < D$ such that $n \equiv j \pmod{D}$.

Step 3: Find the expansion of j in the desired form (10), which is of length m_j.
From Lemma 6 with $n = D$, there is an upper bound on $m_j \leq M$, so this is a finite process.

Step 4: For each value of j, $j < D$, define a sequence satisfying the hypothesis of Lemma 8 as follows:

Definition 9. Let $a_1^{(j)} = \sum_{i=m_j+1}^{Q_j N + m_j} i^k$. For $m \geq 1$, let

$$a_{m+1}^{(j)} = \sum_{i=(m-1+Q_j)N+m_j+1}^{(m+Q_j)N+m_j} i^k$$

For any $l > m$, $a_l^{(j)} > a_m^{(j)}$ by looking at the bounds of the summation. Using the definition of Q_j from Definition 8, we see that $\sum_{i=1}^{r} a_i^{(j)} > a_{r+1}^{(j)}$. Thus $\{a_m^{(j)}\}$ is a sequence that satisfies the hypothesis of Lemma 8. Note also by the definition of N in Definition 7, we have $a_m^{(j)} \equiv 0 \pmod{D}$ for all m.

Step 5: Given n, let L_n be the least integer such that

$$n \leq \sum_{m=1}^{L_n} a_m^{(j)}$$

Following the inductive procedure in the proof of Lemma 8, we can find a sequence of $\epsilon_i = \pm 1$ such that $|n - \sum_{i=1}^{m} \epsilon_i a_i^{(j)}| < a_2^{(j)}$. We expand the $a_i^{(j)}$'s and redefine a new sequence of ϵ_i's to get

$$\begin{aligned} a_2^{(j)} &> \left| n - \sum_{i=1}^{L_n} \epsilon_i a_i^{(j)} \right| \\ &= \left| n - \Big(\sum_{i=m_j+1}^{Q_j N + m_j} \epsilon_1 i^k + \sum_{i=Q_j N + m_j + 1}^{(1+Q_j)N + m_j} \epsilon_2 i^k + \ldots + \sum_{i=(L_n - 2 + Q_j)N + m_j + 1}^{(L_n - 1 + Q_j)N + m_j} \epsilon_{L_n} i^k \Big) \right| \\ &= \left| n - \sum_{i=m_j+1}^{(L_n - 1 + Q_j)N + m_j} \epsilon_i i^k \right| \end{aligned}$$

Thus we have

$$\left| n - \sum_{i=m_j+1}^{(L_n - 1 + Q_j)N + m_j} \epsilon_i i^k \right| < a_2^{(j)} \tag{20}$$

Step 6: Since all the $a_i^{(j)} \equiv 0 \pmod{D}$, we gave $a_2^{(j)} \equiv 0 \pmod{D}$. Then by the choice of m_j in Definition 5, (20), and Lemma 7, we have

$$n \equiv \sum_{i=m_j+1}^{(L_n - 1 + Q_j)N + m_j} \epsilon_i i^k \pmod{D}$$

By adding $\sum_{i=1}^{m_j} i^k$ to both sides of (20), we get

$$\begin{aligned} \left| n - \sum_{i=1}^{(L_n - 1 + Q_j)N + m_j} \epsilon_i i^k \right| &< a_2^{(j)} + \sum_{i=1}^{m_j} i^k \\ &< \sum_{i=Q_j N + m_j + 1}^{(1+Q_j)N + m_j} i^k + \sum_{i=1}^{m_j} i^k \end{aligned} \tag{21}$$

Replacing m_j and Q_j by M and Q respectively increases the right-hand side of (21), so we get

$$\left| n - \sum_{i=1}^{(L_n - 1 + Q_j)N + m_j} \epsilon_i i^k \right| < \sum_{i=QN+M+1}^{(1+Q)N+M} i^k + \sum_{i=1}^{m_j} i^k \tag{22}$$

The right-hand side of (22) is independent of both n and j, so there is a constant C that depends only on k such that

$$\left| n - \sum_{i=1}^{(L_n - 1 + Q_j)N + m_j} \epsilon_i i^k \right| < C$$

Therefore, for some l, $0 < l < C/D$, it follows that

$$n - \sum_{i=1}^{(L_n-1+Q_j)N+m_j} \epsilon_i i^k = \pm lD \tag{23}$$

Then from Definition 3, a possible redefinition of ϵ_i and the fact that $D = D_{k,k}(x)$ is independent of x, where the $\pm$ agrees with (23), we have

$$\begin{aligned} n &= \sum_{i=1}^{(L_n-1+Q_j)N+m_j} \epsilon_i i^k \pm \sum_{i=1}^{l} D_{k,k}((L_n-1+Q_j)N+m_j+1+(i-1)2^k) \\ &= \sum_{i=1}^{(L_n-1+Q_j)N+m_j} \epsilon_i i^k \pm \sum_{i=(L_n-1+Q_j)N+m_j+1}^{(L_n-1+Q_j)N+m_j+l2^k} \epsilon_i i^k \\ &= \sum_{i=1}^{(L_n-1+Q_j)N+m_j+l2^k} \epsilon_i i^k \end{aligned}$$

Thus we have the desired expansion of

$$n = \sum_{i=1}^{T} \epsilon_i i^k$$

where $T = T(n) = (L_n - 1 + Q_j)N + m_j + l2^k$. The algorithm is of polynomial time if it is upper bounded by a polynomial expression in its input size, which is true in our case because $T(n)$ is given in a polynomial in n. This completes the algorithm.

It remains to calculate an upper bound for the length of the expansion $T(n)$. Since m_j's are bounded above by M and Q_j's are bounded above by Q, and Q, N, M and l only depend on k, L_n is the only term in $T(n)$ that depends on n. In the following proofs, we suppress the subscript in j to make the notation simpler, where $a_m^{(j)}$ will be replaced by a_m and we will write Q and M instead of Q_j and m_j.

Lemma 9. *For fixed k and sufficiently large n, the length of the sum $T(n)$ determined by the algorithm satisfies the following inequality:*

$$T(N) \leq [((k+1)n)^{1/(k+1)}] + l2^k + 1 \tag{24}$$

Proof. We examine L_n found in Step 4 of the algorithm. By its definition and the definition of a_{m+1}, it follows that

$$\begin{aligned} n &> \sum_{m=1}^{L_n-1} a_m \\ &= a_1 + \sum_{m=1}^{L_n-2} a_{m+1} \\ &> \sum_{i=M+1}^{QN+M} i^k + \sum_{m=1}^{L_n+2} \Bigg(\sum_{i=(m-1+Q)N+M+1}^{(m+Q)N+M} i^k \Bigg) \\ &= \sum_{i=M+1}^{QN+M} i^k + \sum_{i=QN+M+1}^{(L_n-1+Q)N+M} i^k \end{aligned}$$

From the definition of $T(n) = (L_n - 1 + Q_j)N + m_j + l2^k$, we see that

$$n > \sum_{i=M+1}^{T-l2^k} i^k$$

We use a lower integral approximation on the sum to obtain

$$\begin{aligned} n &> \int_{M+1}^{T-l2^k} i^k di \\ &= \frac{(T - l2^k)^{k+1}}{k+1} - \frac{M^{k+1}}{k+1} \\ (k+1)n &> (T - l2^k)^{k+1} - M^{k+1} \end{aligned}$$

Since T depends only on n, and T grows arbitrarily large as $n \to \infty$, for sufficiently large n

$$\begin{aligned} (k+1)n &> (T - l2^k)^{k+1} - M^{k+1} \\ &> (T - l2^k - 1)^{k+1} \\ ((k+1)n)^{1/(k+1)} &> T - l2^k - 1 \\ T &< [((k+1)n)^{1/(k+1)}] + l2^k + 1 \end{aligned}$$

Since T must be an integer, we obtain $T \leq [((k+1)n)^{1/(k+1)}] + l2^k + 1$, which proves the lemma. □

Theorem 2. *If for fixed k, $L(n)$ is the length of the shortest expansion of n as a sum in the desired form, then $L(n)$ is asymptotic to $[(k+1)n]^{1/(k+1)}$ as $n \to \infty$.*

Proof. We want to show that $\lim_{n\to\infty} \frac{L(n)}{[(k+1)n]^{1/(k+1)}} = 1$. The upper bound of $L(n)$ is $[((k+1)n)^{1/(k+1)}] + l2^k + 1$ by Lemma 9 and the lower bound of $L(n)$ is $[((k+1)n)^{1/(k+1)}]$ by Lemma 7. Using the Squeeze Theorem, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \frac{[((k+1)n)^{1/(k+1)}]}{[(k+1)n]^{1/(k+1)}} \leq \lim_{n\to\infty} \frac{L(n)}{[(k+1)n]^{1/(k+1)}} \leq \lim_{n\to\infty} \frac{[((k+1)n)^{1/(k+1)}] + l2^k + 1}{[(k+1)n]^{1/(k+1)}} \\ &1 \leq \lim_{n\to\infty} \frac{L(n)}{[(k+1)n]^{1/(k+1)}} \leq 1 \\ &\lim_{n\to\infty} \frac{L(n)}{[(k+1)n]^{1/(k+1)}} = 1 \end{aligned}$$

Thus $L(n)$ is asymptotic to $[(k+1)n]^{1/(k+1)}$ as $n \to \infty$. □

We now change our perspective to what happens if n is fixed and k tends to infinity.

Theorem 3. *For a fixed value of n, let $l(k)$ be the shortest expansion of n as a sum in the desired form. Then $l(k) \geq k + 2$ as $k \to \infty$.*

Proof. Let us denote $l(k)$ by l. Since $n = \sum_{i=1}^{l} i^k = l^k + \sum_{i=1}^{l-1} i^k$, for k large enough such that $2^k > n$, we must have

$$l^k - \sum_{i=1}^{l-1} i^k < n$$

By replacing the sum with an upper integral approximation, we get

$$\begin{aligned} n &> l^k - \Big(1 + \int_2^l x^k dx\Big) \\ &= l^k - 1 - \frac{l^{k+1}}{k+1} + \frac{2^{k+1}}{k+1} \\ l^k\Big(1 - \frac{l}{k+1}\Big) &< n - \frac{2^{k+1}}{k+1} + 1 \\ l^k\Big(\frac{l}{k+1} - 1\Big) &> \frac{2^{k+1}}{k+1} - n - 1 \end{aligned} \tag{25}$$

For k large enough such that $\frac{2^k}{k+1} > n+1$, (25) and the fact that $l^k > 0$ yields

$$\begin{aligned} & l^k\Big(\frac{l}{k+1} - 1\Big) > 0 \\ & \frac{l}{k+1} - 1 > 0 \\ & \frac{l}{k+1} > 1 \\ & l > k+1 \\ & l \geq k+2 \end{aligned}$$

The last inequality yields because we are dealing with integers. Thus we have an asymptotic estimate of l as $k \to \infty$ with n fixed. As a direct consequence, for fixed n, $\liminf_{k\to\infty} \frac{l(k)}{k} \geq 1$ because $l(k)$ is lower bounded by $k+2$. □

This concludes our main findings. We will turn our attention to further conjectures of the same sort by changing the choices of ϵ_i or the choices of a_i.

5 Concluding Remarks

There have been several generalizations of this problem. Bleicher [1] poses one question about generalization, which asks whether we can generalize the problem to $\{a_i\}$ being an increasing sequence of integers such that $a_i > c^i$ for a constant $c > 0$ and every positive integer i, and whether or not there is an upper bound on the possible choices of c. These are answered by Feng-Juan Chen and Yong-Gao Chen [2], with the first problem in the affirmative and the second problem in the negative. Yu [3] generalizes this result to a polynomial $a_i = f(i)$ with the condition that there does not exist an integer $d > 1$ such that it divides the values $f(x)$ for all x and proves that for a given l, every integer n can be written as $n = \sum_{i=l}^{m} \epsilon_i f(i)$. There are infinitely more questions of this sort that are waiting to be answered.

References

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