Fractional Calculus

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Abstract

This paper aims to give the reader a comfortable introduction to Fractional Calculus. Fractional Derivatives and Integrals are defined in multiple ways and then connected to each other in order to give a firm understanding in the subject. The reader is expected to be versed in undergraduate complex analysis, meaning that they should also be familiar with real analysis. In the concluding remarks, the readers familiar with measure theory will find a brief discussion of how to extend the topics discussed in the paper to more general analysis.

Contents

1 Historical Background

In September of 1695, Leibniz wrote a letter to l'Hôpital regarding derivatives of a "general order"[1]. L'Hôpital wrote back asking "what if the order is 1/2?"(Anastassiou, 5). This is regarded as the start of Fractional Calculus. In 1832, Liouville noticed that the well-know fact

$$D^{(m)}(e^{az}) \ = \ a^m e^{az} \quad \forall m \in \mathbb{N}$$

(where $D^{(m)}f(z)$ is the m^{th} derivative of f with respect to z) could be extended for complex numbers. That is $m \in \mathbb{N}$ could be replaced with $\alpha \in \mathbb{C}$, and we can define

$$D^{\alpha}(e^{az}) \ = \ a^{\alpha} e^{az} \quad \forall \alpha \in \mathbb{C}.$$

Where D^{α} is today called the *fractional derivative*. There are various ways of defining the fractional derivative. I will focus primarily on the real version of the Riemann-Liouville Fractional Derivative, discussed in Chapter 2 of Podlubny [9]. Podlubny gives few formal definitions and theorems, so I have written my own based on what was in the text. I will also briefly discuss The Riemann-Liouville Fractional Derivative in the complex case, given by Osler (646-647) [8], as well as the Caputo Fractional Derivatice, defined by Podlubny in chapter 2.4. As this paper aims to introduce the reader to Fractional Calculus, following these three definitions there will be properties, theorems, and examples regarding the material discussed. Should the reader like, a list of the various definitions, equations, and theorems are provided at the end of this paper, immediately before the references. I will begin with some preliminaries that will be helpful in deriving some of the results in the paper.

2 Preliminaries

This section includes only definitions which appear in the paper that will not be defined at the time they are mentioned.
<u>**Simply Connected**</u>:
Some examples of domains that are *not* simply connected domains are annuli, punctured disks, and punctured planes (Gamelin, 252). The reader familiar with topology may be aware that being simply connected is analogous to having genus 0. The following definition is *Compex Analyis*(Gamelin, 252-253)[4]:

Definition 2.1. *Let $\gamma(t)$ for $a \le t \le b$ be a closed path in a domain D. Let z_1 be the constant path at some point in D. We say that γ is <u>deformable to a point</u> if $\forall 0 \le s \le 1$, there exist closed paths $\gamma_s(t)$ for $a \le t \le b$ such that $\gamma_s(t)$ depend continuously on s and t, $\gamma_0(t) = \gamma(t)$, and $\gamma_1(t) \equiv z_1$. We say that a domain D is <u>simply connected</u> if every closed path in D is deformable to a point.*

That is to say, γ is deformable to a point if there exists a sequence of curves $\gamma_s(t)$ that depend continuously on s and t, with the initial path (that is, $s = 0$) γ_0 being equal to γ, and with the final path ($s = 1$) being equal to the "constant

path", e.g. point at $z_1 \in D$.
The Complex-Valued Gamma Function:
The following is from the first chapter of *Fractional Differential Equations* (Podlubny).

Definition 2.2. *The Gamma Function, denoted* $\Gamma(z)$*, is given by*

$$\Gamma(z) \;=\; \int_0^\infty \frac{t^{z-1}}{e^t} dt$$

The Gamma function converges on the right half plane $\mathfrak{Re}(z) > 0$ as shown on page 2 of Podlubny

Proposition 2.3. $\Gamma(z+1) \;=\; z\Gamma(z)$

Proof. Let $u = t^z$ and let $dv = e^{-t}$. Using integration by parts,

$$\Gamma(z+1) \;=\; \int_0^\infty e^{-t}t^z dt \;=\; -e^{-t}t^z|_0^\infty + \int_0^\infty e^{-t}z^{z-1}dt \;=\; \Gamma(z)$$

■

The following definition is quite frequent among authors (Podlubny, 62). In fact, many authors in fractional calculus think of the integral of a function f to just be the -1^{st} derivative. While this definition is one I am not personally fond of, it is used by the authors being discussed. The following definition can been seen as an alternate statement of the fundamental theorem of calculus.

Definition 2.4. *Let* $f(\tau)$ *be a continuous and integrable function. Then define the integral of* f *by*

$$f^{(-1)}(t) \;=\; \int_a^t f(\tau)d\tau$$

3 The Real Case

Definition 3.1 (Riemann-Liouville Fractional Derivative)**.** *Let* $f(t)$ *be an* $m+1$ *times differentiable function. We say* ${}_aD_t^p f(t)$ *is the* p^{th} *fractional derivative with respect to t (with lower bound/terminal a), with* $(m \leq p < m+1)$*. It is given by*

$${}_aD_t^p f(t) \;=\; \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau \qquad (m \leq p < m+1)$$

We will now see how this can be extended to a derivative of arbitrary order (rather than just p between m and $m+1$. First, I refer back to definition 2.3 in the preliminaries section (integration). From this, if we integrate again, we get

$$\begin{aligned} f^{(-2)}(t) &= \int_a^t d\tau_1 \int_a^{\tau_1} f(\tau)d\tau \\ &= \int_a^t f(\tau)d\tau \int_\tau^t d\tau_1 \\ &= \int_a^t (t-\tau)f(\tau)d\tau \end{aligned}$$

The second equality comes from the fact that when we switch our order of integration, we have to switch the bounds of integration in order to preserve the region being integrated over. Folland gives a brief discussion of this on page 170 of *Advanced Calculus*. It can be similarly shown that

$$f^{(-3)}(t) = \frac{1}{2}\int_a^t (t-\tau)^2 f(\tau)d\tau$$

Proceeding inductively, we arrive at what Podlubny calls the "Cauchy formula"

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau \tag{1}$$

This can be called the integral of order n, for future reference. Suppose that in the above equation, $n \geq 1$, and let $k \in \mathbb{Z}, k \geq 0$. Then if we let D^{-k} be k iterations of integrals, as considered above, then

$$f^{(-k-n)} = \frac{1}{\Gamma(n)} D^{-k} \int_a^t (t-\tau)^{n-1} f(\tau)d\tau$$

Likewise, if $k \geq n$, and D^k is the iterated derivative operator, then

$$f^{(k-n)} = \frac{1}{\Gamma(n)} D^{k} \int_a^t (t-\tau)^{n-1} f(\tau)d\tau \tag{2}$$

Therefore, we can simply refer to (2) as a general case of $f^{(k-n)}(t)$, with D^k being iterated integration for $k \leq 0$ and iterated differentiation for $k > 0$. If $k-n < 0$, then (2) is to be interpreted as iterated integrals of $f(t)$. If $k-n = 0$, then (2) represents $f(t)$, and if $k-n > 0$, then (2) represents successive derivatives of $f(t)$.
We can now define the integral of arbitrary order. In (1), replace n with p and require that $p > 0$. Then we can define

$${}_a\mathbf{D}_t^{-p} f(t) = \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1} f(\tau)d\tau \tag{3}$$

Finally, we will define derivatives of all orders. Let $\alpha \in \mathbb{R}$ be a number such that $k-\alpha > 0$. Then rewriting (2), we obtain

$${}_a\mathbf{D}_t^{k-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\frac{d^k}{dt^k}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau \tag{4}$$

4 The Complex Analytic Method

The Complex analytic method of definiing the Riemann-Liouville Fractional Derivative has a different set-up than the real method, however the result are much of the same.
Recall the Cauchy Integral Formula for the m^{th} derivative of a complex-valued function $f : \mathbb{C} \to \mathbb{C}$ on a bounded domain D (Gamelin, 114):

$$f^{(m)}(z) \;=\; \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}dw$$

This result is stated as a theorem proved by Gamelin, wherein f must extend smoothly to the boundary of D. Consider what would happen if m is interchanged with a non-integer, namely any complex number α. The analogy is that $m!$ would be replaced with $\Gamma(\alpha+1)$, and $(w-z)^{-m-1}$ becomes $(w-z)^{-\alpha-1}$. However, consider the functions

$$g \;=\; \frac{1}{(w-z)^{m+1}}$$

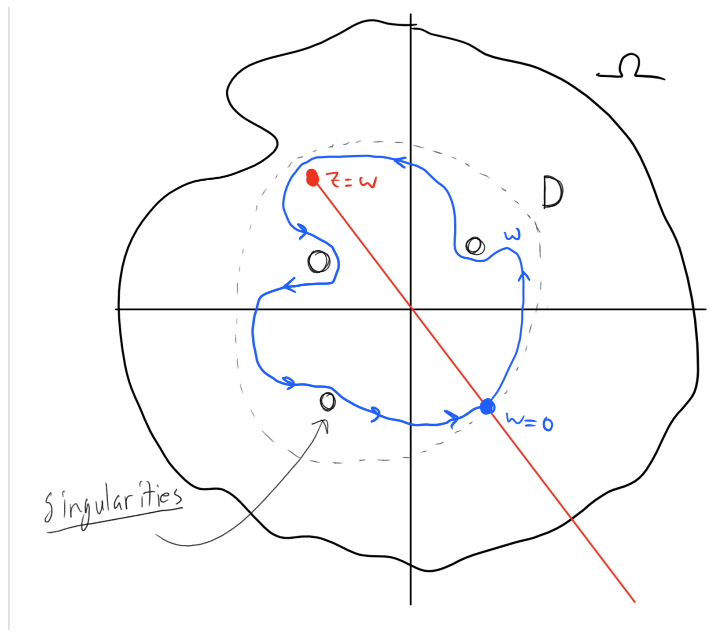
$$h \;=\; \frac{1}{g} \;=\; (w-z)^{m+1}$$

w has a 0 of order $m+1$ at $z=a$, and so h has a pole of order $m+1$ at $z=a$. Notice that before we were able to "wiggle" the contour we were integrating over without much consequence (see Gamelin page 81). Now, however, considering the functions

$$\eta \;=\; \frac{1}{(w-z)^{\alpha+1}}$$

$$\omega \;=\; \frac{1}{\eta} \;=\; (w-z)^{\alpha+1}$$

ω has a branch point at $z=w$, and so η has a branch point at $z=w$. Thus, we will define a branch cut starting at the point $z=w$, passing through the origin, and going out to infinity. Notice now that for z close to the contour, wiggling the contour may cause us big problems. So we will take the contour of our integral to be starting at $w=0$, and enclosing $z=w$ once in the standard positive orientation, avoiding (going around) any singularities that f may have. Note that this contour will not intersect the branch cut at any point except $w=0$. A picture should help clarify:



Finally, since we can write

$$(w-z)^{-\alpha-1} \;=\; e^{(-\alpha-1)(\log{(w-z)})}$$

and log is a multivalued function, we will take the real part of the logarithm when $w - z > 0$. We arrive at the following definition:

Definition 4.1. *Let $f(z) \;=\; z^p g(z)$, where $g(z)$ is analytic on a simply connected domain $D \subset \Omega \subset \mathbb{C} : 0 \in D$, and let $\mathfrak{Re}(p) > -1$. Then we define the Fractional Derivative of order α of f(z) (denoted $D_z^\alpha f(z)$) as*

$$f^{(\alpha)}(z) \;=\; D_z^\alpha f(z) \;=\; \frac{\Gamma(\alpha+1)}{2\pi i}\int_0^{z^+} \frac{f(w)}{(w-z)^{\alpha+1}}dw \tag{5}$$

for $\alpha \neq -1, -2, -3, \ldots$.

5 The Caputo Fractional Derivative

Podlubny writes his book on Fractional Differential equations, and as he describes it, the Riemann-Liouville is not the best definition to take when solving such problems. In application (such as viscoelasticity and hereditary solid mechanics), it is better to use a different definition, such as the Caputo definition (Podlubny, 78). The Caputo approach makes initial conditions for differential equations nicer, while the Riemann-Liouville definition is better from a pure math approach. It is not my aim in this paper to discuss differential equations or the applications of fractional calculus, rather I thought that the reader would find it useful to see a different way of approaching Fractional Calculus. We define the α^{th} Caputo Fractional Derivative of $f(t)$ with respect to t, ${}_a^C D_t^\alpha f(t)$, as

$$ {}_a^C D_t^\alpha f(t) \frac{1}{\Gamma(\alpha - n)} $$

6 Properties and Examples

6.1 The Real Riemann-Liouville

Proposition 6.1. *Suppose $f(t)$ is C^1 for $t \geq 0$. Then*

$$ \lim_{p \to 0} {}_a\boldsymbol{D}_t^{-p} f(t) = f(t) $$

Proof. We will use integration by parts with $u = f(\tau)$ and $dv = (t-\tau)^{p-1}$. Then $du = f'(\tau)d\tau$ and $v = p(t-\tau)^p$. By proposition 0 ($\Gamma(z+1) = z\Gamma(z)$),

$$ \frac{1}{\Gamma(p)} = \frac{p}{\Gamma(p+1)} $$

Thus,

$$ {}_a\mathbf{D}_t^{-p} f(t) \; = \; \frac{(t-a)^p f(a)}{\Gamma(p+1)} + \frac{1}{\Gamma(p+1)} \int_a^t (t-\tau)^p f'(\tau) d\tau $$

Taking the limit on either side and passing the limit under the integral (Podlubny does not check for uniform convergence here, 66), we obtain

$$ \lim_{p \to 0} {}_a\mathbf{D}_t^{-p} f(t) \; = \; f(a) + \int_a^t f'(\tau) d\tau \; = \; f(t) $$

■

If we weaken our assumption that $f(t) \in C^1$ for $t \geq 0$ to $f(t) \in C^0$ for $t \geq a$, the result still holds, but an epsilon delta proof is needed.

Proposition 6.2. *If $f(t) \in C^0$ for $t \geq a$, then*

$$ {}_a\boldsymbol{D}_t^{-p} \left({}_a\boldsymbol{D}_t^{-q} f(t) \right) \; = \; {}_a\boldsymbol{D}_t^{-p-q} f(t) \tag{6} $$

The proof of this is given on Podlubny, page 67. I will prove a more general result, but will use this proposition in the proof. Podlubny does not justify swapping limiting operations in his proofs.

Theorem 6.3. *Suppose $p > 0$ and $t > a$. Then*

$$ {}_a\boldsymbol{D}_t^{p} \left({}_a\boldsymbol{D}_t^{-p} f(t) \right) \; = \; f(t) $$

Proof. Consider the case where $p = n \in \mathbb{N}$. Then

$$ {}_a\mathbf{D}_t^n \left({}_a\mathbf{D}_t^{-n} f(t)\right) = \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau $$

Swapping the limiting operations, we obtain

$$ \frac{d}{dt} \int_a^t f(\tau) d\tau = f(t) $$

Now consider the case where $k-1 \leq p < k$. Applying the above proposition,

$$ {}_a\mathbf{D}_t^{-k} f(t) = {}_a\mathbf{D}_t^{-(k-p)} \left({}_a\mathbf{D}_t^{-p} f(t)\right) $$

According to Podlubny (69), this implies that

$$ \begin{aligned} {}_a\mathbf{D}_t^p \left({}_a\mathbf{D}_t^{-p} f(t)\right) &= \frac{d^k}{dt^k} \left[{}_a\mathbf{D}_t^{-(k-p)} \left({}_a\mathbf{D}_t^{-p} f(t)\right)\right] \\ &= \frac{d^k}{dt^k} \left[{}_a\mathbf{D}_t^{-p} f(t)\right] = f(t) \end{aligned} $$

■

Example

Let $\nu \in \mathbb{R}$, $\nu > -1$, and let

$$ f(t) = (t-a)^\nu $$

Suppose $n-1 \leq p < n$. By definition of the Riemann-Liouville Derivative,

$$ {}_a\mathbf{D}_t^p f(t) = \frac{d^n}{dt^n} \left({}_a\mathbf{D}_t^{-(n-p)} f(t)\right) $$

If we let $\alpha = n - p$ and substitute in (3), then we obtain

$$ {}_a\mathbf{D}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} B(\alpha, \nu+1)(t-a)^{\nu+\alpha} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} (t-a)^{\nu+\alpha} $$

Where $B(x,y)$ is the beta function. Thus,

$$ {}_a\mathbf{D}_t^p f(t) = \frac{1}{\Gamma(-p)} B(-p, \nu+1)(t-a)^{\nu-p} = \frac{\Gamma(\nu+1)}{\Gamma(\nu-p+1)} (t-a)^{\nu-p} $$

A similar example will be discussed in the complex section.

6.1.1 Composition of Fractional Derivatives

With Integer-Order Derivatives

Proposition 6.4.

$$ \frac{d^n}{dt^n} \left({}_a\boldsymbol{D}_t^p f(t)\right) = {}_a\boldsymbol{D}_t^{n+p} f(t) $$

The motivation for this property, rather than a detailed proof, will be shown. The full discussion is on Podlubny 73, and uses results that were not discussed in this paper. Using (4), we have that

$$\frac{d^n}{dt^n}\left({}_a\mathbf{D}_t^{k-\alpha}f(t)\right) \;=\; \frac{1}{\Gamma(\alpha)}\frac{d^{n+k}}{dt^{n+k}}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau \;=\; {}_a\mathbf{D}_t^{(n+k)-\alpha}f(t)$$

Wiritng $p = k - \alpha$, we obtain

$$\frac{d^n}{dt^n}\left({}_a\mathbf{D}_t^{p}f(t)\right) \;=\; {}_a\mathbf{D}_t^{(n+p)-\alpha}f(t)$$

The other direction requires a little bit more work, and again is avoided due to the extent of results used to prove it.

With Fractional Derivatives The following proposition will be stated without proof, as Podlubny again using results from earlier in this book, results which I am not covering. The property is quite handy, however.

Proposition 6.5. *Suppose $f(t)$ is k times differentiable, where $k = \max\{m, n\}$, and $m-1 \leq p < m$ and $n-1 \leq q < n$. If $f^{(j)}(a) = 0$ for $j = 1, \ldots, k$, then the following is true:*

$${}_a\boldsymbol{D}_t^{p}\left({}_a\boldsymbol{D}_t^{q}f(t)\right) \;=\; {}_a\boldsymbol{D}_t^{p}\left({}_a\boldsymbol{D}_t^{q}f(t)\right) \;=\; {}_a\boldsymbol{D}_t^{p+q}f(t)$$

In general, these two operators do not commute, and this proposition will be discussed more when discussing the Caputo Derivative's properties.

6.2 The Complex Riemann-Liouville Definition

For the complex case, I will just show one example of the fractional derivative. It is discussed on page 647 on Osler, although I attempt to give more description than he does. He skips quite a few steps in stating his example, and thus I am interpreting some of his results in-between steps.

Example

Let $f(z) = z^p$ for $\mathfrak{Re}(p) > 0$. Note that in the case where $\alpha = N \in \mathbb{N}$. Then

$$D_z^N f(z) \;=\; \frac{p!}{(N-p)!}z^{p-N}$$

In the case where $N > p$, we must invoke the gamma function. In (5), parametrize w in terms of s: $w = zs$ for $0 \leq s \leq 1$. Then $dw = zds$ and we are tasked with evaluation of

$$\begin{aligned} D_z^\alpha f(z) \;&=\; \frac{\Gamma(\alpha+1)}{2\pi i}\int_0^{1^+} z\frac{(zs)^p}{(sz-z)^{\alpha+1}}ds \\ &=\; \frac{z^{p-\alpha}\Gamma(\alpha+1)}{2\pi i}\int_0^{1^+} \frac{s^p}{(s-1)^{\alpha+1}}ds \end{aligned}$$

Osler then prescribes a contour that runs from 0 to $1-\varepsilon$ along the real axis, traverses the circle $|s-1|=\varepsilon$, and then runs back to the origin on the real axis. I assume on the way back along the real axis there was a phase shift caused by traversing the circle, as Osler arrives at the following expression

$$\frac{z^{p-\alpha}\Gamma(\alpha+1)}{2\pi i}[1-e^{-2\pi i(\alpha+1)}]\int_0^1 \frac{s^p}{(s-1)^{\alpha+1}}ds$$

Osler then uses properties about the Gamma and Beta function (not explicitly), and simplifies the above expression to

$$z^{p-\alpha}\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}$$

Compare this to the result in the real case.

6.3 The Caputo Definition

Here, I will discuss some simple properties of the Caputo Fractional Derivative simply by contrasting it to the Riemann-Liouville Fractional Derivative. They are stated as factual results rather than precise properties.
Firstly, the Caputo definition satisfies the property that the Caputo Derivative of a constant is 0. This is familiar to traditional calculus. However, in the Riemann-Liouville definition, if we take $K \neq 0$ to be a constant, assuming we have a finite lower bound(or terminal) on the integral (assume it is 0),

$${}_0\mathbf{D}_t^{\alpha} K \;=\; \frac{Kt^{-\alpha}}{\Gamma(1-\alpha)}$$

Podlubny notes that it is somewhat common to let $a=-\infty$, as this preserves the property from traditional calculus that the derivative of a constant is 0 (Podlubny, 80).
Secondly, recall proposition 6.5 above. The more general case is

$$\begin{gathered}{}_a\mathbf{D}_t^{\alpha}\left({}_a\mathbf{D}_t^{m} f(t)\right) \;=\; {}_a\mathbf{D}_t^{m}\left({}_a\mathbf{D}_t^{\alpha} f(t)\right) \;=\; {}_a\mathbf{D}_t^{\alpha+m} f(t)\\ m\in\mathbb{N}, n-1<\alpha<n\end{gathered}$$

which is only satisfied if $f^{(s)}(0)=0$ for $s=0,\ldots,m$. However, the same condition with the Caputo definition,

$$\begin{gathered}{}_a^C D_t^{\alpha}\left({}_a^C D_t^{m} f(t)\right) \;=\; {}_a^C D_t^{m}\left({}_a^C D_t^{\alpha} f(t)\right) \;=\; {}_a^C D_t^{\alpha+m} f(t)\\ m\in\mathbb{N}, n-1<\alpha<n\end{gathered}$$

is satisfied if $f^{(s)}(0)=0$ for $s=n,n+1,\ldots,m$ Hence the Caputo integral can be nicer in applications and formulas.

7 Concluding Remarks

Many of the papers and books considered in writing this paper were either too big or too small. By this I mean the author either completely ignored measure theory, and did not discuss how fractional calculus related to Hausdorff measure and other such topics, or the author did cover these things, but such papers and journals were intended for readers well versed in Lebesgue integration, measure theory, and occasionally more advanced topics. Thus, I will give an extremely brief discussion of Lebesgue measure, adopted from Folland pg. 207-208 [3]. Then I will briefly explain Hausdorff measure and it's loose-applications to fractional calculus (as I don't have time to go into further detail).

Definition 7.1. *Suppose T is a tiled set such that it is composed of a finite number of rectangles R_k with disjoint interiors. That is, $T = \bigcup_{k=1}^{K} R_k$. Then the Lebesgue measure $m(T)$ is the sum of the area's of the R_k's.*

The Lebesgue measure of a compact set K is

$$m(K) \;=\; \sup\{m(T) : \text{T is a tiled set and} K \subset T\}$$

While the Lebesgue measure of an open set U is given by

$$m(U) \;=\; \inf\{m(T) : \text{T is a tiled set and } T \subset U\}$$

A subset S of $\mathbb{R}^2$ is called Lebesgue measurable if for compact $K \subset S$ and open $S \subset U$,

$$\sup\{m(K)\} \;=\; \inf\{m(U)\}$$

in which case we denote the Lebesgue measure of S by $m(S)$, which is equal to both of these values. Hausdorff measure is slightly more difficult to define, but it is a more general extension of Lebesgue measure, and can be defined by taking the inf defined by of a sum of diameters of small coverings of a set, where the diameter of a set is the supermom of of the distance between any two points in the set [7]. While this may seem abstract, Hausdorff measure can be used to define such spaces as $\mathbb{R}^\alpha$ for $0 < \alpha \leq 1$. Hausdorff measure also allows one who is interested in fractal geometry to make more precise statements[5]. Once spaces such as $\mathbb{R}^\alpha$ are set up, one can talk about mapping functions into $\mathbb{R}^\alpha$, and develop a more advanced theory of fractional calculus that may better resemble undergrad analysis. This kind of rigorous extension can lead to inequalities that some might consider to be "hard analysis", such as integral inequalities [6]. As stated at the start of the paper, the idea of fractional calculus at least in concept dates back to the late 1600's, so it is no surprise that has many extensions and can be widely used.

8 List of Definitions

Definition. *Let $\gamma(t)$ for $a \leq t \leq b$ be a closed path in a domain D. Let z_1 be the constant path at some point in D. We say that $\gamma)$ is <u>deformable to a point</u>*

if $\forall 0 \leq s \leq 1$*, there exist closed paths* $\gamma_s(t)$ *for* $a \leq t \leq b$ *such that* $\gamma_s(t)$ *depend continuously on* s *and* t*,* $\gamma_0(t) = \gamma(t)$*, and* $\gamma_1(t) \equiv z_1$*. We say that a domain* D *is simply connected if every closed path in* D *is deformable to a point.*

Definition. *The Gamma Function, denoted* $\Gamma(z)$*, is given by*

$$\Gamma(z) \;=\; \int_0^\infty \frac{t^{z-1}}{e^t} dt$$

Proposition. $\Gamma(z+1) \;=\; z\Gamma(z)$

Definition. *Let* $f(\tau)$ *be a continuous and integrable function. Then define the integral of* f *by*

$$f^{(-1)}(t) \;=\; \int_a^t f(\tau) d\tau$$

Definition (Riemann-Liouville Fractional Derivative)**.** *Let* $f(t)$ *be a* $m{+}1$ *times differentiable function. The let* ${}_aD_t^p f(t)$ *be the* p^{th} *fractional derivative with respect to* t *(with lower bound* a*). fractional derivative of of*

$${}_aD_t^p f(t) \;=\; \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau) d\tau \quad (m \leq p < m+1)$$

$$f^{(-n)}(t) \;=\; \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau \tag{1}$$

$$f^{(k-n)} \;=\; \frac{1}{\Gamma(n)} D^k \int_a^t (t-\tau)^{n-1} f(\tau) d\tau \tag{2}$$

$${}_a\mathbf{D}_t^{-p} f(t) \;=\; \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau \tag{3}$$

$${}_a\mathbf{D}_t^{k-\alpha} f(t) \;=\; \frac{1}{\Gamma(\alpha)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{4}$$

Definition. *Let* $f(z) \;=\; z^p g(z)$*, where* $g(z)$ *is analytic on a simply connected domain* $D \subset \Omega \subset \mathbb{C} : 0 \in D$*, and let* $\mathfrak{Re}(p) > -1$*. Then we define the Fractional Derivative of order* α *of f(z) (denoted* $D_z^\alpha f(z)$*) as*

$$f^{(\alpha)}(z) \;=\; D_z^\alpha f(z) \;=\; \frac{\Gamma(\alpha+1)}{2\pi i} \int_0^{z^+} \frac{f(w)}{(w-z)^{\alpha+1}} dw \tag{5}$$

for $\alpha \neq -1, -2, -3, \ldots$*.*

Proposition. *Suppose* $f(t)$ *is* C^1 *for* $t \geq 0$*. Then*

$$\lim_{p \to 0} {}_a\boldsymbol{D}_t^{-p} f(t) = f(t)$$

Proposition. *If $f(t) \in C^0$ for $t \geq a$, then*

$$ {}_a\boldsymbol{D}_t^{-p} \left({}_a\boldsymbol{D}_t^{-q} f(t) \right) \;=\; {}_a\boldsymbol{D}_t^{-p-q} f(t) \tag{6} $$

Theorem. *Suppose $p > 0$ and $t > a$. Then*

$$ {}_a\boldsymbol{D}_t^{p} \left({}_a\boldsymbol{D}_t^{-p} f(t) \right) \;=\; f(t) $$

Proposition.

$$ \frac{d^n}{dt^n} \left({}_a\boldsymbol{D}_t^{p} f(t) \right) \;=\; {}_a\boldsymbol{D}_t^{n+p} f(t) $$

Proposition. *6.5 Suppose $f(t)$ is k times differentiable, where $k = \max\{m, n\}$, and $m - 1 \leq p < m$ and $n - 1 \leq q < n$. If $f^{(j)}(a) = 0$ for $j = 1, \ldots, k$, then the following is true:*

$$ {}_a\boldsymbol{D}_t^{p} \left({}_a\boldsymbol{D}_t^{q} f(t) \right) \;=\; {}_a\boldsymbol{D}_t^{p} \left({}_a\boldsymbol{D}_t^{q} f(t) \right) \;=\; {}_a\boldsymbol{D}_t^{p+q} f(t) $$

Definition. *Suppose T is a tiled set such that it is composed of a finite number of rectangles R_k with disjoint interiors. That is, $T = \bigcup_{k=1}^{K} R_k$. Then the Lebesgue measure $m(T)$ is the sum of the area's of the R_k's.*

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