REVIEW OF FUZZY SETS

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1. Introduction

L. A. Zadeh's paper *Fuzzy Sets**[1] introduces the concept of a fuzzy set, provides definitions for various fuzzy set operations, and proves several properties regarding these operations, culminating in a theorem analogous to the hyperplane separation theorem for traditional sets. The paper aims to introduce and establish a groundwork for these objects, upon which future work may be built. We cover the material in the paper, condensing or elaborating as appropriate, in an attempt to tailor it to students in the 33X series. Finally, we provide a brief summary of the reception and current usage of fuzzy sets in the scientific and mathematical community.

2. Definitions & Theorems

2.1. **Set Properties & Basic Operators.** The concept of a *fuzzy set* is analogous to that of a typical set for which membership is a spectrum rather than a binary. Zadeh's exact definition is: "A *fuzzy set (class)* A in X is characterized by a *membership (characteristic) function* $f_A(x)$ which associates with each point in X a real number in the interval $[0, 1]$." $f_A(x)$ determines the degree to which x is contained in A (or the "grade of membership"). Element x of X is considered to be contained in A if $f_S(x) > 0$. Fuzzy sets are thus an abstraction of standard sets (subsequently referred to as the crisp sets where the context alone does not make it clear) with continuous rather than binary characteristic functions. Fuzzy set operations are defined to be intuitive extensions of crisp set operations, and in all cases reduce to the equivalent operations when applied to crisp sets.

Fuzzy set equality is defined by:

$$A = B \iff f_A(x) = f_B(x),\ \forall x \in \Omega \tag{1}$$

and is stricter than crisp set equality in the sense that the characteristic functions must agree on a value chosen out of the uncountable set $[0, 1]$ rather than the finite set $\{0, 1\}$. A fuzzy set is empty if its characteristic function is equivalent to zero. Set containment is defined as:

$$A \subseteq B \iff f_A(x) \leq f_B(x),\ \forall x \in \Omega \tag{2}$$

substituting in a strict inequality in the case of a strict subset. Note that a crisp set contains all fuzzy sets which contain the same elements as itself. The complement of a

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fuzzy set S, denoted S', is the fuzzy set defined by the characteristic function:

$$f_{S'} = 1 - f_S \tag{3}$$

As a set is entirely defined by its characteristic function, f_S will sometimes be used to denote set S.

The intersection and union operators on fuzzy sets are defined as follows:

$$\begin{aligned} A \cup B &= \max(f_A, f_B) \\ A \cap B &= \min(f_A, f_B) \end{aligned} \tag{4}$$

From the definitions of minimum and maximum functions, it follows that these operations act as expected on crisp sets, and that they maintain the associativity for all fuzzy sets. The characterization of $A \cup B$ as the smallest set containing both A and B (respectively, $A \cap B$ as the largest contained set) holds for fuzzy sets. As a brief demonstration of this in the union case, let $C = A \cup B$. Thus $f_C = \max(f_A, f_B)$ is greater than or equal to both f_A and f_B and C contains both sets. At the same time, $\forall x,\ f_C = f_A$ or else $f_C = f_B$, so any f_D greater than both will also be greater than f_C and thus C is contained by any set containing A and B.

The standard identities of crisp set logic apply to fuzzy sets, most provable by employing some casework. The equality:

$$1 - \max(f_A, f_B) = \min(1 - f_A, 1 - f_B) \tag{5}$$

can be easily verified by examining the three cases $f_A(x)\ \{>, <, =\}\ f_B(x)$ for a given x, and gives the first of the two forms of DeMorgan's Laws:

$$\begin{aligned} (A \cup B)' &= A' \cap B' \\ (A \cap B)' &= A' \cup B' \end{aligned} \tag{6}$$

The equality:

$$\max(f_C, \min(f_A, f_B)) = \min(\max(f_C, f_A), \max(f_C, f_B)) \tag{7}$$

which can be verified by examining all six possible weak orderings $f_X \le f_Y \le f_Z$ of the characteristic functions of A, B, and C for a given x, give the first of the two distributive identities:

$$\begin{aligned} C \cup (A \cap B) &= (C \cup A) \cap (C \cup B) \\ C \cap (A \cup B) &= (C \cap A) \cup (C \cap B) \end{aligned} \tag{8}$$

Other identities such as the idempotent, domination, and commutative laws follow directly from the definitions of the minimum and maximum functions.

2.2. **Mappings & Algebraic Operators.** We define as well several algebraic operations making use of arithmetic on characteristic functions. The algebraic product of two sets is a set with characteristic function equal to the product of their characteristic functions:

$$AB = f_{AB} = f_A f_B \tag{9}$$

and has the immediate properties:

$$AB \subseteq A \cap B, \text{ for fuzzy sets } A \text{ and } B \tag{10}$$
$$AB = A \cap B, \text{ for crisp sets } A \text{ and } B$$

The algebraic sum is analogous, but with the complication of not being defined on all sets.

$$A + B = f_{A+B} = f_A + f_B, \ \forall A, B \text{ s.t. } f_A + f_B \leq 1 \tag{11}$$

The absolute difference is defined:

$$|A - B| = f_{|A-B|} = |f_A - f_B| \tag{12}$$

and acts as an exclusive or operation on crisp sets.

A relation in set theory is a set of ordered pairs $R \subseteq X \times X$ with elements of the form $a, b \in X, \ (a, b) \in R$. The concept can be extended to an n-ary relation on X^n (elements being ordered tuples of length n). A fuzzy relation is a relation R defined by a fuzzy set. For example, consider the relation in $R = \{(a, b)\} \subset \mathbf{R}^2$, "$b$ the value stored in some digital computer to represent a." We know $f_R(0, 0) = f_R(1, 1) = 1$ and $f_R(3, 3) = 0$ since computers can store the exact values of 0 and 1, but not of 3. However without knowing more about the computer we do not know exactly what approximation of 3 it will store, so for b in some small punctured interval around 3 we could assign $f_R(3, b) = .1$. This example is of course highly subjective, as is the point of fuzzy sets. Someone with extensive knowledge of number representation in computers might limit b down to some finite set of values near 3, and accordingly assign higher grades to each of those relations $f_R(3, b)$.

The composition of two individual ordered pairs (a, b) and (b, c) within a fuzzy relations R_1 and R_2, is the pair $(a, c) \in R_2 \circ R_1$ with characteristic function $f_{R_2 \circ R_1}(a, c) = \min[f_{R_1}(a, b), f_{R_2}(b, c)]$. We extend this to define the composition $R_2 \circ R_1$ as:

$$R_2 \circ R_1 = f_{R_2 \circ R_1}(a, c) = \sup_{b \in \Omega} \min[f_{R_1}(a, b), f_{R_2}(b, c)], \ \forall a, c \in \Omega \tag{13}$$

that is, the maximal characteristic value among all such ordered pairs. The composition is associative, as can be demonstrated using supremum and minimum properties, but the proof is ugly and not of interest to us.

Now we turn to fuzzy sets induced by mappings. Consider $T : X \to Y$ for X and Y spaces (or crisp sets) in Ω. If T is one-to-one and A is a fuzzy set in X then T defines a fuzzy set $B \in Y$:

$$f_B(y) = f_A(x), \ y = T(x) \ \forall x \in X \tag{14}$$

and likewise T^{-1} defines A given B. For T not one-to-one we resolve the ambiguity in the same manner was we did with compositions of relations–by using the maximal value.

$$f_B(y) = \sup_{y \in T^{-1}(y)} f_A(x), \ \forall x \in X \tag{15}$$

A convex combination of vectors u and v has the form $\lambda u + (1 - \lambda)v, \ \lambda \in [0, 1]$. For fuzzy sets, the convex combination is a ternary operator, with the third argument standing in for the scalar in the vector operation:

$$(A, B; \Delta) = f_{(A,B;\Delta)}(x) = \Delta A + \Delta' B = f_\Delta(x) f_A(x) + (1 - f_\Delta(x)) f_B(x),\ \forall x \in \Omega \tag{16}$$

$$A \cap B \subseteq (A, B; \Delta) \subseteq A \cup B,\ \forall A, B, \Delta \tag{17}$$

Property 17 is apparent when written in the characteristic function form: $\min(f_A, f_B) \leq f_\Delta f_A + (1 - f_\Delta) f_B \leq \max(f_A, f_B)$. Note that since we have control over every value $f_\Delta(x)$ individually, given any fuzzy sets A, B, and C with $A \cap B \subseteq C \subseteq A \cup B$ we can find some Δ such that $C = (A, B; \Delta)$. To do so we simply solve the convex combination equation for f_Δ, resulting in the unique expression given by equation 18.

$$f_C = f_\Delta f_A + (1 - f_\Delta) f_B = f_\Delta f_A + f_B - f_\Delta f_B = f_\Delta (f_A - f_B) + f_B$$

$$f_\Delta(x) = \frac{f_C(x) - f_B(x)}{f_A(x) - f_B(x)},\ \forall x \in \Omega \tag{18}$$

2.3. **Convexity & Boundedness.** From here on, we will work with fuzzy sets in Euclidean space E^n. We will also introduce Γ_α as our notation for the crisp set containing only elements belonging with a grade of membership of at least α to the fuzzy set in question.

$$\Gamma_\alpha = \{x \in \Omega |\ f_S(x) \geq \alpha\} \tag{19}$$

Zadeh does not give a name to this type of set, but makes enough use of them to merit one. We will refer to Γ_α as a partition of S for grade α.

A crisp set C in E^n is said to be convex iff all convex combinations of vectors in C are also contained in C, that is, $x, y \in C \implies \lambda x + (1 - \lambda) y \in C,\ \forall \lambda \in [0, 1]$. We define the convexity of a fuzzy set in two equivalent ways. Note they rely more on the vector convex combination than on the fuzzy set convex combination.

Definition 1 (convexity (1))**.** *A fuzzy set S is convex if the crisp sets Γ_α are convex for every $\alpha \in [0, 1]$.*

In words, this means that if we choose an arbitrary grade of membership and redefine S as the crisp set containing only elements that previously belonged to S with that grade of membership or higher, then this new set will be convex in the traditional sense. Alternatively, we define S to be convex iff for any x and y in S, all elements that can be expressed as a convex combination of x and y have at least as high a grade of membership to S as either x or y.

Definition 2 (convexity (2))**.** *A fuzzy set S is convex if $\forall \lambda \in (0, 1),\ f_S(\lambda x + (1 - \lambda) y) \geq \min[f_S(x), f_S(y)]$.*

We prove the equivalence by examining both directions. Consider S convex by the first definition, and choose $\alpha = \min[f_S(x), f_S(y)]$. Then Γ_α convex implies it contains $\lambda x + (1 - \lambda) y$ and for all elements $\gamma \in \Gamma_\alpha$ we have $f_S(\gamma) \geq \alpha = \min[f_S(x), f_S(y)]$. Now consider S convex by the second definition. Then for any α, if x and y are contained in Γ_α we have $f_S(\lambda x + (1 - \lambda) y) \geq \min[f_S(x), f_S(y)] \geq \alpha$, thus $\lambda x + (1 - \lambda) y \in \Gamma_\alpha$ which is the definition of convexity for crisp sets. Thus we have equivalence.

We now prove the following theorem:

Theorem 1. *If both A and B are convex then $A \cap B$ is convex.*

Proof. Let $C = A \cap B$ with A and B convex. Then:

$$f_C(\lambda x + (1-\lambda)y) = \min[f_A(\lambda x + (1-\lambda)y), f_B(\lambda x + (1-\lambda)y)]$$

and

$$f_A(\lambda x + (1-\lambda)y) \geq \min[f_A(x), f_A(y)]$$
$$f_B(\lambda x + (1-\lambda)y) \geq \min[f_B(x), f_B(y)]$$

Making substitutions we get:

$$f_C(\lambda x + (1-\lambda)y) \geq \min[\min[f_A(x), f_A(y)], \min[f_B(x), f_B(y)]]$$

which is equivalent to:

$$f_C(\lambda x + (1-\lambda)y) \geq \min[\min[f_A(x), f_B(x)], \min[f_A(y), f_B(y)]]$$
$$f_C(\lambda x + (1-\lambda)y) \geq \min[f_C(x), f_C(y)]$$

and thus C is convex. □

Next we turn to boundedness. A fuzzy set is *bounded* if Γ_α is bounded in norm $\forall \alpha > 0$. Note this definition does not exclude the set of all elements with nonzero membership grade from being unbounded, nor the set of elements with zero membership grade from being empty. Unions and intersections of bounded sets are bounded as well (simply consider the maximum norm in the partitions of the two sets for a given grade).

Lemma 1. *If S is bounded, then for every $\epsilon > 0$, there exists a hyperplane H such that $f_S(x) \leq \epsilon$ for all x opposite H from the origin.*

To see this we construct a sphere around the origin with radius $R > \max(\Gamma_\epsilon)$, containing all $x \in \Gamma_\epsilon$. Any hyperplane tangent to this sphere has the desired property.

The following definitions will be used copiously throughout the rest of the paper:

Definition 3 (essentially attained)**.** *For fuzzy set S, $c \in [0,1]$ is essentially attained at x_0 if $\forall \epsilon > 0$, every spherical neighborhood of x_0 contains points x_i such that $f_S(x_i) \geq c - \epsilon$*

Definition 4 (maximal grade)**.** *For a fuzzy set S, the maximal grade $M = \sup_{x \in \Omega}(f_S(x))$*

We will prove that a fuzzy set attains or essentially attains its maximal grade at at least one point in Ω. Zadeh's original statement of the lemma (omitting the previously stated definitions) is as follows: "*Let A be a bounded fuzzy set and let $M = \sup_x f_A(x)\ldots$ Then there is at least one point x_0 at which M is essentially attained...*" The lemma as stated is slightly incorrect, since M need not be essentially attained if it is literally attained, however he mentions that possibility later on in the paper.[1] First, we state the Bolzano-Weierstrass theorem, which we take to be a basic result of analysis.

[1] It's also of note that the original statement can be made legitimate if we restrict $f_S(x)$ to be continuous, as $f_S(x)$ must then approach M before taking that value.

Theorem 2 (Bolzano-Weierstrass)**.** *If S is a subset of $\mathbf{R}^n$, S is compact $\iff$ every sequence of points in S has a convergent subsequence whose limit lies in S.*

Theorem 3. *Let S be a bounded fuzzy set and M, the maximal grade in S, nonzero. Then M is either obtained or essentially obtained at at least one point.*

Proof. If there exists a point x_0 such that $f_S(x_0) = M$, then we are done. The case of a crisp set containing a single point demonstrates that M need not be essentially attained in this case. Now suppose that no such point exists and consider the sequence of crisp sets: $\{\Gamma_{\alpha(n)}\}$ where $\alpha(n) = M - \frac{M}{n+1}$. By definition of M, for every finite n there exist a point with membership grade higher than $\alpha(n)$ so every element of the sequence is non-empty. By the boundedness of S, $\Gamma_{\alpha(1)}$ is bounded, and its closure $\overline{\Gamma}_{\alpha(1)}$ is compact. Since $\alpha(n)$ is increasing, every $\Gamma_{\alpha(n)}$ is contained in $\overline{\Gamma}_{\alpha(1)}$. Consider the sequence $\{x_n\}$, with x_i an arbitrary point in $\Gamma_{\alpha(n)}$. By the Bolzano-Weierstrass theorem, some subsequence of $\{x_n\}$ converges to some point x_0 in $\overline{\Gamma}_{\alpha(1)}$. Since no point x_n satisfies $f_S(x_n) = M$, for any fixed x_k we can take N large enough that $x_{n>N} \neq x_k$, and thus every neighborhood of x_0 must contain an infinite number of unique points in $\{x_n\}$. Since we can also take N high enough that every $x_{n>N}$ has a membership grade within ϵ of M, M is essentially obtained at x_0. □

Note, Zadeh assumed $\Gamma_{\alpha(1)}$ itself was compact and applied the Bolzano-Weierstrass theorem directly to that set. As equation 20 shows, we can easily define a set such that $\Gamma_{\alpha(1)}$ is open and does not contain the limit of $\{x_n\}$ (0, in this case). Fortunately, the original proof does not rely on $\Gamma_{\alpha(1)}$ actually containing x_0.

$$x \in \mathbf{R}^1,\ f_S(x) = \begin{cases} \frac{1}{1+x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{20}$$

A crisp set is said to be strictly convex if the midpoint of any two distinct points within it lies in its interior. As with other crisp set properties, we extend it to fuzzy sets applying it separately to partitions.

Definition 5 (strict convexity)**.** *A fuzzy set S is strictly convex if for any $\alpha \in (0, 1]$, $\forall x, y \in \Gamma_\alpha,\ .5x + .5y \in \Gamma_\alpha^{int}$*

We introduce as well the concept of *strong convexity.*

Definition 6 (strong convexity)**.** *A fuzzy set S is strongly convex if for any $\lambda \in (0, 1)$ and for any distinct x and y, $f_S(\lambda x + (1-\lambda)y) > \min(f_S(x), f_S(y))$*

Here Zadeh offhandedly mentions some basic properties, which we will elaborate on slightly. First, strict convexity does not imply strong convexity, nor the other way around. The crisp set containing the unit disk in $\mathbf{R}^2$ provides an example of a strictly but not strongly convex fuzzy set. A counter-example in the other direction is much more difficult to construct. Strong convexity does in fact imply strict convexity in $\mathbf{R}^1$, since if x and y are contained in Γ_α then by strong convexity the entire interval between them is contained in it as well, and they form the boundary of that interval. We will leave off finding a counterexample in this direction, since the result is not especially useful.

Second, intersections maintain both strict and strong convexity. We will walk through the reasoning.

Proof. Let A and B be fuzzy sets in $\mathbf{R}^n$. Let $C = A \cap B$.

Suppose A and B are strongly convex, let x and y be two arbitrary, distinct points, and let z be an arbitrary convex combination of x and y. Suppose $f_C(z) \le \min(f_A(x), f_A(y))$. Then by the strong convexity of B and by the definition of intersection, either $f_B(x) < f_C(z)$ and $f_C(x) = f_B(x)$ or $f_B(y) < f_C(z)$ and $f_C(y) = f_B(y)$. The same logic applies if $f_C(z) \le \min(f_B(x), f_B(y))$. Thus x, y, and z still satisfy strong convexity properties for C.

Now suppose A and B are strictly convex, x and y arbitrary points belonging to partition of grade α for both A and B, and z is their midpoint $.5x + .5y$. Then by strict convexity of both sets, for every point u near z, $f_A(u) \ge \alpha$ and $f_B(u) \ge \alpha$, and thus $f_C(u) = \min(f_A(u), f_B(u)) \ge \alpha$. Thus $u \in \Gamma_\alpha$ the partition by grade α of C, and since this is true for every u near z, $z \in \Gamma_\alpha^{int}$. □

We note a few interesting properties of strongly convex sets that Zadeh skipped over. These are: the characteristic function of a strongly convex set has no zeroes, attains its supremum at no more than one point, and has at most one maximum on any line. To show the first we assume z is a zero of f_S and take x and y to be two points such that z is a convex combination of them. These three points determine a line, with z lying between x and y. If $f_S(x) \neq 0 \neq f_S(y)$ then we have violated strong convexity. Otherwise, if x has nonzero grade then select a new point on the line, on the same side of z as x. Repeat until $f_S(x)$ is nonzero, and follow the same procedure with y. If for either x or y, no such point can be found, then f_S is uniformly zero on that ray so strong convexity is violated. For the second statement, assume f_S assumes its maximum at two points and note that strong convexity is violated for any convex combination of the two points. For the final statement assume that the maximum value of f_S on the line occurs twice, and again strong convexity is violated for convex combinations of the two points.

Definition 7 (core of a fuzzy set)**.** *If S is a fuzzy set with maximum grade M, its core, denoted $C(S)$ is the crisp set of all points at which M is essentially attained.*

Theorem 4. *If S is a convex fuzzy set, $C(S)$ is a convex set.*

Proof. Let x and y be any two points in $C(S)$, and let l be the line segment connecting them (the set of all convex combinations of the two points). The for any given $\epsilon > 0$ and any $\delta > 0$ we can choose x_0 and y_0 such that $|x_0 - x|,\ |y_0 - y| < \delta$ and $f_S(x_0),\ f_S(y_0) > M - \epsilon$. By the convexity of S, all points on the line segment l_0 connecting x_0 to y_0 are also contained in $\Gamma_{M-\epsilon}$. If we let P be the cylinder of radius δ around l (extended slightly to contain the balls around x and y at either end), then P contains x_0 and y_0, and by the convexity of cylinders, all points on l_0 are contained within P. Thus for any convex combination of x and y we can choose any ball B_δ and any ϵ and a point with the desired membership grade lying on the resultant segment l_0 will be contained in B_δ. □

Corollary 1. *A strongly convex fuzzy set $S \subset \mathbf{R}^1$ attains or essentially attains its maximal grade at exactly one point.*

Proof. By lemma 3 S attains or essentially attains M at a point. As previously stated, a strongly convex set cannot attain M at two points. Suppose it attains M at one point and essentially attains M at another point x. Then choose any point between them, and let its membership grade be $\alpha < M$. We can choose a point close to x with membership grade arbitrarily close to M, so we may choose a point with grade higher than *alpha* too violate strong convexity. Finally suppose M is essentially attained at two points x and y. Then by theorem 4, M is essentially attained on the interval $[x, y]$. Then again we can choose any point on the interval with grade *alpha* and find points on either side in the interval with grades greater than α. □

We now define the *shadow* of a fuzzy set, that is, its projection of a fuzzy set in $\mathbf{R}^n$ onto a hyperplane H (dimension $n-1$) in $\mathbf{R}^n$. This can be viewed as a map from every point in $\mathbf{R}^n$ to the nearest point on H (mapping each line normal to the plane onto the point at which it intersects the plane). In accordance with equation 15, the shadow takes on the supremum of all values mapped to it. Zadeh provides the following definition for the shadow on a hyperplane aligned with the axis:

Definition 8 (shadow of a fuzzy set (axis version))**.** *For fuzzy set $A \subset \mathbf{R}^n$ and hyperplane H normal to the i^{th} basis vector, the shadow of A on H is defined by:*

$$f_{S_H(A)}(\hat{x}) = f_{S_H(A)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \sup_{x_i \in \mathbf{R}^1} [f_A(x_1, \ldots . x_n)] \tag{21}$$

However, using dot product notation for planes, this concept can easily be stated for an arbitrary hyperplane:

Definition 9 (shadow of a fuzzy set)**.** *For fuzzy set $A \subset \mathbf{R}^n$, $u, v \in \mathbf{R}^n$, and hyperplane H defined as $\{x \in \mathbf{R}^n | (x-u) \cdot v = 0\}$, the shadow of A on H is defined by:*

$$f_{S_H(A)}(\hat{x}) = f_{S_H(A)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \sup_{x_i \in \mathbf{R}^1} [f_A(x_1, \ldots . x_n)] \tag{22}$$

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Lemma 2. *The shadow $f_{S_H(A)}(x)$ of convex fuzzy set A on hyperplane H is convex.*

To demonstrate this, let x and y be any two points on H. Assume f_A has a maximum on the normals lines intersecting H at x and y, and let x_0 and y_0 be points on the lines at which f_A takes on those values. Then by the convexity of A, f_A is at least equal to $\min(f_A(x_0), f_A(y_0))$ on the segment connecting those two points. The projection of that segment onto H will be the line segment connecting x and y, and thus convex combinations of x and y have the necessary minimum grade. Suppose instead either one or both of the lines have no maximum. Then we can instead take x_0 or y_0 so that their grades are arbitrarily close to the supremums to show that the connecting line segments have grade at least equal to the supremum.

Theorem 5. *Let A and B be convex fuzzy sets in $\mathbf{R}^n$. If $S_H(A) = S_H(B) \forall H$, then $A = B$.*

Before we prove this, a note on convexity. Zadeh's initial definition of convexity for fuzzy sets was stated int terms of partitions using weak inequalities ($\{x|f_S(x) \geq \alpha\}$). In the latter half of the paper he begins making use of strict partitions ($\{x|f_S(x) > \alpha\}$) without acknowledging the switch. It is easy, but still worthwhile, to verify that convexity for weak partitions implies convexity for strict partitions, and thus that this usage is valid. We prove this by contradiction. Assume there exists some convex fuzzy set S and some value α such that $\{x|f_S(x) > \alpha\}$ is not convex. Then for some point x with $f_S(x) \leq \alpha$, there exist two points x_1 and x_2, for which $x = \lambda x_1 + (1-\lambda)x_2$ for some $\lambda \in (0,1)$ and $f_S(x_1), f_S(x_2) > \alpha$. We define $\beta = \min[f_S(x_1), f_S(x_2)]$ and consider the weak partition Γ_β. Then clearly $x_1, x_2 \in \Gamma_\beta$ and $x \notin \Gamma_\beta$, so we have a contradiction.

We have thus show that for a fuzzy set S to be convex implies that both its weak and its strict partitions are convex. We now prove theorem 5

Proof. We show that the existence of a point x_0 with $f_A(x_0) \neq f_B(x_0)$ implies the existence of a hyperplane H such that $f_{S_H(A)}(x_0*) \neq f_{S_H(B)}(x_0*)$ where x_0* is the projection of x_0 onto H. Suppose $\exists x_0$ with $f_A(x_0) = \alpha > \beta = f_B(x_0)$. Define the partition of B, $\Gamma_\beta = \{x \in \mathbf{R}^n | f_B(x) > \beta\}$. By convexity of Γ_β, and since x_0 is not contained in the set, we can find a hyperplane F such that Γ_β lies entirely on one side of it (and not on F itself). [2] Then if we let H be a hyperplane orthogonal to F and take the shadow of B on H, $f_{S_H(B)}(x^*)$, then $f_{S_H(B)}(x_0^*) = \beta$ since a line normal to H at x_0 must lie in F, and $\max_{x \in H} f_B(x) = \beta$. On the other hand $f_A(x_0) = \alpha > \beta$ so $f_{S_H(A)}(x_0^*) \geq \alpha > \beta$, and the two shadows are unequal. □

We now derive a fuzzy set analog to the convex set separation theorem, stating that two disjoint convex sets can be separated by a hyperplane in the ambient space. We start by defining the *degree of separation* of two fuzzy sets.

Definition 10 (degree of separation of fuzzy sets)**.** *Let A and B be convex fuzzy sets in $\mathbf{R}^n$, and let H be a hypersurface in $\mathbf{R}^n$. Suppose $\exists K_H \in \mathbf{R}$ such that $f_A(x) \leq K_H$ on one side of H, and $f_B(x) \leq K_H$ on the other, and moreover let it be the infimum of all such values. Then the degree of separation of A and B by H is $D_H = 1 - K_H$.*

The problem of minimizing the degree of separation, as stated above, of two fuzzy sets is beyond the scope of Zadeh's paper. We consider a special case of this situation, where H is a hyperplane (rather than a hypersurface), and denote the minimum degree of separation across all hyperplanes $D = 1 - \bar{M}$, $\bar{M} = \inf_H(K_H)$. From here on, the *degree of separation* will refer to the case of H limited to hyperplanes, and we now state the separation theorem's analog:

Theorem 6. *Let A and B be bounded convex fuzzy sets in $\mathbf{R}^n$ with maximal grades M_A and M_B. Let $M = \sup_{x \in \mathbf{R}^n}[\min(f_A(x), f_B(x))]$ be the maximal grade of their intersection. Then the degree of separation of A and B is $D = 1 - M$.*

[2] Here Zadeh states that we can find "supporting hyperplane," implying that x_0 must lie on the boundary of Γ_β. This is not quite trivial since it is a result of A's convexity and the relationship $\alpha > \beta$, but for the proof, the weaker statement featured above is sufficient.

Proof. We divide the problem into two cases, $M = \min(M_A, M_B)$ and $M < \min(M_A, M_B)$.

Case 1: Assume $M = M_A < M_B$. Then by the boundedness of B there exists a hyperplane H such that $f_B(x) \leq M$ for all x on one side of H. On the other side, $f_A(x) \leq M$ by our assumptions on M. Now suppose there exists some other hyperplane H' and constant $M' < M$ such that $f_A(x) \leq M'$ on one side and $f_B(x) \leq M'$ on the other. We denote the set of points on the first side of the plane H^+ and the set of points on the second side of the plane H^- (both sets including the boundary H). Then on H^+, $f_A(x) \leq M' < M_A$ and thus the core of A must lie entirely on the second side. By theorem 3 A has a core, or else attains its maximal grade, so $f_A > M'$ on H^-, and thus if $f_B(x) > M'$ for any x, it occurs on H^+. We now have $\sup_{x \in H^+}(\min[f_A(x), f_B(x)])$ limited by $f_A(x)$ and $\sup_{x \in H^-}(\min[f_A(x), f_B(x)])$ limited by $f_B(x)$, and thus $\sup_{x \in H^+ \cup H^-}(\min[f_A(x), f_B(x)]) = M \leq M'$, which contradicts our assumptions on M'.

Case 2: Now assume $M < \min(M_A, M_B)$. Let $\Gamma_M^A = \{x \in \mathbf{R}^n | f_A(x) > M\}$, and respectively for Γ_M^B. For either partition to be empty would violate our assumption for case 2, and for the partitions to not be disjoint would violate our definition of M. Since Γ_M^A and Γ_M^B are convex disjoint crisp sets, we simply apply the regular separation theorem to guarantee the existence of H. The same argument by contradiction used in case 1 guarantees that no higher degree of separation is possible. $\square$

3. Conclusion & Applications

Zadeh introduces fuzzy sets as a method of modeling subjective situations, but that characterization is somewhat vague. In the most abstract sense, as we stated in the first section, a fuzzy set is just a map with some nice properties of a space to the real numbers. The concept, under the name "fuzzy sets" has rarely been cited since 1965, however Zadeh's work under the name of "fuzzy logic" is an often referenced topic. Fuzzy logic is essentially a direct extension of fuzzy sets to boolean logic, in the same manner that boolean logic is derived from crisp sets. Though infinite-valued logic existed earlier than 1965, Zadeh's paper is the reason it is known today as fuzzy logic.

Fuzzy logic appears to be of less interest to the pure math community than to engineers and applied mathematicians. Fuzzy logic is cited especially often in the field of robotics, controllers, and language synthesis, as autonomous machines operating in the real world often need to simulate an animal's more adaptable thought processes. This is, of course, precisely Zadeh's stated reason for introducing fuzzy set theory. Though rarer, additional academic explorations into fuzzy logic do occur. The relationship between fuzzy sets and probability (readers may have noted the similarity between a the characteristic function of a set and the probability density function of a probability distribution) was explored by Bart Kosko in his paper *Fuzziness vs. Probability*[2].

References

[1] L. A. Zadeh. *Fuzzy Sets**, Information and Control, Vol. 8 (1965), 338-353.
[2] Bart Kosko. *Fuzziness vs. Probability*, Int. J. General Systems, Vol. 17 (1990), 211-240.