Diophantine Approximation and Transcendental Numbers

Connor Goldstick

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1 Introduction

Suppose we have an irrational number, α, that we want to approximate with a rational number, p/q. This question of approximating an irrational number is the primary concern of Diophantine approximation. In other words, we want $|\alpha - p/q| < \epsilon$. However, this method of trying to approximate α is boring, as it is possible to get an arbitrary amount of precision by making q large. To remedy this problem, it makes sense to vary ϵ with q. The problem we are really trying to solve is finding p and q such that

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2} \quad \text{which is equivalent to} \quad |q\alpha - p| < \frac{1}{q} \tag{1}$$

Solutions to this problem have applications in both number theory and in more applied fields. For example, in signal processing many of the quickest approximation algorithms are given by solutions to Diophantine approximation problems. Diophantine approximations also give many important results like the continued fraction expansion of e. One of the most interesting aspects of Diophantine approximations are its relationship with transcendental numbers (a number that cannot be expressed of the root of a polynomial with rational coefficients). One of the key characteristics of a transcendental number is that it is easy to approximate with rational numbers.

This paper is separated into two categories. The first concerns itself with some basic Diophantine approximation and continued fraction results. This section is almost entirely based upon an introductory textbook to the subject by Serge Lang [1]. Many of the results and proofs in this section utilize only algebra and induction, so they should not be too difficult to follow. The second half of the paper is devoted to transcendental numbers. Many of the results in this section are stated without proof because the proofs are outside the scope of this paper. Finally, the paper concludes with the proof of e's continued fraction representation. Although this result is not required in proving the transcendence of e, it is still a neat result and the proof emphasizes how many arguments in this field need to be specifically constructed on a case-by-case basis.

2 Continued Fractions

Definition 1. *A continued fraction is a way of expressing a rational number as an iterated sum of positive integers plus reciprocals. They are usually expressed at* $[a_0, \ldots, a_n]$ *where*

$$[a_0, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \frac{1}{a_n}}}}$$

One property of continued fractions is that $[a_0, \ldots, a_n] = [a \ldots, a_n - 1, +1]$ so it is possible for the length of a continued fraction expansion of any number whose continued fraction expansion has finitely many terms to be either even or odd.

Let p, q be integers defined inductively with $p_0 = a_0$, $q_0 = 1$, and

$$p_n = a_0 p'_{n-1} + q'_{n-1} \quad \text{and} \quad q_n = p'_{n-1}$$

where

$$\frac{p'_n}{q'_n} = [a_1, \ldots, a_n]$$

Then,

$$\frac{p_n}{q_n} = \frac{a_0 p'_{n-1} + q'_{n-1}}{p'_{n-1}} = a_0 + \frac{1}{[a_1, \ldots, a_n]} = [a_0, \ldots, a_n]$$

Theorem 1. *For* p, q *defined above and* $n \geq 2$*,*

$$p_n = a_n p_{n-1} + p_{n-2} \tag{2a}$$

$$q_n = a_n q_{n-1} + q_{n-2} \tag{2b}$$

Proof. Base Case: For $n = 2$,

$$\frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = \frac{a_2(a_0 a_1 + 1) + a_0}{a_0 a_1 + 1} = \frac{a_2(a_0 p'_0 + q'_0) + p_0}{a_2 p'_0 + q_0}$$

$$\frac{p_2}{q_2} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0}$$

Note that we are using the substitution $p'_0 = a_1$ and $q'_0 = 1$.
Inductive case: Assume $n > 2$ and assume that:

$$p'_{n-1} = a_n p'_{n-2} + p'_{n-3}$$

$$q'_{n-1} = a_n q'_{n-2} + q'_{n-3}$$

Using the previous definitions of p_n and q_n:

$$p_n = a_0 p'_{n-1} + q'_{n-1} = a_0(a_n p'_{n-2} + p'_{n-3}) + a_n q'_{n-2} + q'_{n-3}$$

$$p_n = a_n(a_0 p'_{n-2} + q'_{n-2}) + a_0 p'_{n-3} + q'_{n-3}$$

$$p_n = a_n p_{n-1} + p_{n-2}$$

$$q_n = p'_{n-1} = a_n p'_{n-2} + p'_{n-3}$$

$$q_n = a_n q_{n-1} + q_{n-2}$$

□

If we define $p_{-1} = 1$ and $q_{-1} = 0$ then:

$$\frac{p_1}{q_1} = \frac{a_1 p_0 + 1}{a_1 q_0} = \frac{a_1 a_0 + 1}{a_1} = a_0 + \frac{1}{a_1}$$

This makes the above theorem hold for $n \geq 1$. The immediate consequence of this theorem is that both p_n and q_n are strictly increasing. This is due to a_n, p_n, and q_n being greater than or equal to 1 for all $n \geq 0$.

This theorem is the best way to define both p_n and q_n and is much easier to work with than their original equations. Seeing as almost every result in Diophantine approximation has to be proven inductively, this result will be constantly utilized. Since this result is the basis for almost every result involving Diophantine approximations, many people start with equations (2a) and (2b) as their definitions for p_n and q_n.

3 Rational Approximations

Let α be a real irrational number. Note that we can we can express α as

$$\alpha = a_0 + \frac{1}{\alpha_1}$$

where a_0 is the largest integer smaller than α. This also means that $\alpha_1 > 1$ which lets us inductively say:

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}}$$

So, the continued fraction representation of α is $[a_0, a_1, \ldots, \alpha_{n+1}]$. Since α is irrational, its continued fraction expansion fraction expansion must be infinite because if it ever terminated that would imply that α could be represented as a fraction. So, we can say that

$$\alpha = [a_0, a_1, \ldots]$$

Definition 2. *A principal convergent of $\alpha = [a_0, a_1, \ldots, a_n, \ldots]$ is defined as the rational number p_n/q_n that satisfies:*

$$\frac{p_n}{q_n} = [a_0, \ldots, a_n]$$

This means that if we cut off the continued fraction of α at n, we get a rational number with can be expressed in terms of the p and q that was described in the continued fractions section.

For any real number $\alpha = [a_0, a_1, \ldots, \alpha_{n+2}]$ we can use (2a) and (2b) to prove the following two equations:

$$q_{n+1}\alpha - p_{n+1} = \frac{(-1)^{n+1}}{\alpha_{n+2}q_{n+1} + q_n} \tag{3}$$

and

$$q_n\alpha - p_n = \frac{(-1)^n \alpha_{n+2}}{\alpha_{n+2}q_{n+1} + q_n} \tag{4}$$

Since $\alpha_{n+1} \geq 1$ The n^{th} principal convergent of α satisfy the following properties:

1. For even n, the n^{th} principal convergents form a strictly increasing sequence converging to α.
2. For odd n, the n^{th} principal convergents form a strictly decreasing sequence converging to α.
3. for all n:

$$|q_n\alpha - p_n| < \frac{1}{q_{n+1}} \tag{5}$$

The first two assertions can be obtained from (2a) and (2b) by letting $\alpha = [a_0, a_1, \ldots, \alpha_n]$ and manipulating them into the form:

$$\frac{p_{n-2}}{q_{n-2}} - \frac{p_n}{q_n} = \frac{(-1)^{n-1}\alpha_n}{q_n q_{n-2}}$$

which immediately yields the result we want because if n is even, then $\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} > 0$. The opposite is true if n is odd. The third statement comes from equation (4). Since $q_{n+1} > q_n$ because q is strictly increasing, we can rewrite the above equation as a solution to equation (1):

$$|q_n\alpha - p_n| < \frac{1}{q_n}$$

Also, for $n \geq 1$, $|q_n\alpha - p_n|$ is closer to zero than any other integer so we can say that

$$|q_n\alpha - p_n| = ||q_n\alpha||$$

Where $||x||$ is the integer closest to x.

Theorem 2. *For $n \geq 2$,*

$$||q_{n-1}\alpha|| = a_n||q_n\alpha|| + ||q_{n+1}\alpha||$$

which implies that

$$||q_n\alpha|| < ||q_{n-1}\alpha||$$

Proof. From equations (1a) and (1b)

$$q_{n+1}\alpha - p_{n+1} = \alpha(a_n q_n + q_{n-1}) - a_n p_n - p_{n-1}$$
$$(q_{n+1}\alpha - p_{n+1}) - a_n(\alpha q_n - p_n) = (\alpha q_{n-1} - p_{n-1})$$
$$||q_{n+1}\alpha|| = -a_n||\alpha q_n|| + ||\alpha q_{n-1}||$$

The trick with $||\cdot||$ comes from the fact that $\alpha q_n - p_n$ is the opposite sign of $\alpha q_{n+1} - p_{n+1}$. This means that $||(q_{n+1}\alpha - p_{n+1}) - a_n(\alpha q_n - p_n)|| = ||q_{n+1}\alpha|| + a_n||\alpha q_n||$. The second result follows quickly from the first result. □

This theorem implies that every subsequent approximation is better than the one before it. Now, we are going to define the principal convergents in another way.

Definition 3. *A best approximation to α is a fraction p/q such that*

$$||q\alpha|| = |q\alpha - p|, \quad \textit{and} \quad ||q\alpha|| < ||q'\alpha||$$

where $1 \leq q' < q$.

The reason that these properties are what define a best approximation is because $\alpha - p/q$ are close to 0 (given by the first equations) and there are no denominators smaller than q that give as good of an approximation. One helpful property of this definition of best approximations is that p and q are relatively prime (they share no common factors). If they were not reduced, then we could write $p = p'r$ and $q = q'r$ with $r \geq 2$. This would mean that

$$r \cdot |q'\alpha - p'| = |q\alpha - p|$$
$$|q'\alpha - p'| < |q\alpha - p|$$

which contradicts the requirement that $||q\alpha|| < ||q'\alpha||$.

One useful property of best approximations is that the best approximations to α are the principal convergents to α. On top of that, the smallest integer, q_m, greater than q_n such that $||q_m\alpha|| < ||q_n\alpha||$ is $q_m = q_n + 1$.

Theorem 3. *The best approximations to α are the principal convergents to α. Additionally, for $n \geq 1$, q_n is the smallest integer $q > q_{n-1}$ such that $||q\alpha|| < ||q_{n-1}\alpha||$.*

Proof. This will only prove one direction of the above statement. If p_n/q_n is a principal convergent to α, then p_n/q_n is a best approximation to α. This does require us to assume the other direction of the statement: any best approximation is a principal convergent.

For $n = 0$, since $q_0 = 1$ there is no q such that $1 \leq q < q_0$. Thus, the definition of a best approximation is satisfied for p_0/q_0.

Assume inductively that p_n/q_n satisfies the conditions required for it to be a best approximation of α. Then, we want to prove that p_{n+1}/q_{n+1} is also a best approximation. Let q be the smallest integer $> q_n$ such that

$$||q\alpha|| < ||q_n\alpha||$$

and let p be the integer that satisfies $||q\alpha|| = |q\alpha - p|$. Since we have inductively assumed that p_n/q_n is a best approximation, p/q is also a best approximation. Since we have already asserted that best approximations are principal convergents, p/q must be a principal convergent. Since q is chosen to be as small as possible, $q = q_{n+1}$ which means that $p = p_{n+1}$. This proves that if p_n/q_n is a principal convergent, then p_n/q_n is a best approximation to α. □

4 Transcendental Numbers

To define what a transcendental number is, we must first define algebraic numbers. A complex number, β, is algebraic if and only if there is a non-zero polynomial, $P(z)$ that has rational coefficients with β as a root. In other words, this means that $P(\beta) = 0$. Note that since every coefficient of P is rational, we can clear the denominators are write

$$a_m\beta^m + a_{m-1}\beta^{m-1} + \cdots + a_1\beta + a_0 = 0$$

Where $a_m \in \mathbb{Z}$. The **degree** of an algebraic number is the minimal degree of the polynomial that satisfies $P(\beta) = 0$.

Definition 4. *A transcendental number is defined as a number that is not algebraic. In other words, a number w is transcendental if and only if there is no polynomial with rational coefficients such that $P(w) = 0$.*

There are many interesting properties that transcendental numbers have. One such property is that almost every number is transcendental. In 1873 Georg Cantor proved that the set of transcendental numbers is uncountable and the set of algebraic numbers is countable. [2]

Proof. Cantor (1873). The algebraic numbers are countable because the set of polynomials with rational coefficients are countable and each polynomial has finitely many zeros. Since the algebraic numbers are the zeros of these polynomials, they must be countable. The complex numbers are uncountable by Cantor's diagonalization argument and the union of transcendental and algebraic numbers form $\mathbb{C}$. Because the algebraic numbers are countable, the transcendental numbers must be uncountable. □

What makes this result so striking is the fact that it is difficult to generate transcendental numbers and check a number for its transcendence. Although transcendental numbers vastly outnumber algebraic numbers, they are still difficult to find.

The two most famous transcendental numbers are π and e. However, it is unknown whether $\pi + e$ or $e\pi$ are transcendental although we know at least one of them must be. Due to its difficulty, many of the proofs in transcendental number theory are very complicated and outside the scope of this paper. As such, almost all of the theorems in this section will be stated without proof. The first result in the theory of transcendental numbers was given by Liouville:

Theorem 4 (Liouville's Theorem (1853))**.** *Let β be a real algebraic number with degree $n > 1$. Then, there is a positive constant $c(\beta)$ that depends on β such that for all rational numbers p/q with $\gcd(p, q) = 1$ and $q > 1$ the following equation holds:*

$$\left|\beta - \frac{p}{q}\right| > \frac{c(\beta)}{q^n} \tag{6}$$

The proof of this result is constructive. It starts with the minimal polynomial that satisfies $P(\beta) = 0$ and uses estimates to construct a c that satisfies the desired inequality.

Intuitively, what this result says is that rational numbers approximate transcendental numbers better than algebraic numbers. Liouville used this equation to prove the existence of transcendental numbers by constructing a number that violated (6). Specifically, he proved that the number

$$\sum_{n=0}^{\infty} \frac{1}{10^{n!}}$$

is transcendental. The above number is called Liouville's Constant.

Theorem 5. *Liouville's Constant is transcendental.*

Proof. Let L denote Liouville's Constant for notational convenience. Then, the partial sums of L are:

$$\frac{p_k}{q_k} = \sum_{n=0}^{k} \frac{1}{10^{n!}}$$

This means

$$\left|L - \frac{p_k}{q_k}\right| = \sum_{n=k+1}^{\infty} \frac{1}{10^{n!}} < \frac{1}{10^{(k+1)!}}$$

If φ was algebraic of degree m, then by Liouville's theorem:

$$\left|L - \frac{p_k}{q_k}\right| > \frac{c(L)}{10^{k!m}}$$

for all p_k/q_k. But for large enough k we can say that:

$$\frac{c(L)}{10^{k!m}} > \frac{1}{10^{(k+1)!}}$$

Which contradicts our bounds on $|L - p_k/q_k|$. Thus, L is transcendental. □

5 Irrationality Measure

We know that rational approximations of transcendental numbers converge better than those of algebraic numbers. One question we should ask is how much faster do the approximation of transcendental numbers converge? Sadly, the answer is that there is no general difference in estimates between all transcendental numbers and algebraic numbers. This is due to some transcendental numbers having Diophantine approximations that converge at about the same rate as algebraic number approximations.

One way of classifying how well numbers are approximated is by a slight tweak to equation (1):

$$0 < \left|x - \frac{p}{q}\right| < \frac{1}{q^{\mu}}$$

where x is a real number and there are at most finitely many integers p and q that satisfy the equation. Specifically, we are asking how big we can make the power of q in the denominator before x becomes "badly approximated" by rational numbers.

Definition 5. *Let x be a real number. Then, the irrationality measure, $\mu(x)$, of x is the smallest μ such that the inequality*

$$\left|x - \frac{p}{q}\right| > \frac{1}{q^{\mu+\epsilon}}$$

holds for any $\epsilon > 0$ and large enough p and q.

Both definitions of $\mu(x)$ are equivalent. One could interpret $\mu(x)$ as the number that causes x to become badly approximable. $\mu(x)$'s value directly depends on if x is rational, algebraic, or transcendental:

$$\begin{cases} \mu(x) = 1, & x \text{ is rational} \\ \mu(x) = 2, & x \text{ is algebraic of degree} > 1 \\ \mu(x) \geq 2, & x \text{ is transcendental} \end{cases}$$

Note that there is some overlap between the possibilities for x when $\mu(x) = 2$, x could either be algebraic or transcendental. These results were exceptionally difficult to prove. Showing that $\mu(x) = 2$ for algebraic numbers was the result that Klaus Roth was awarded the Fields Medal for in 1958 [4].

Some of the more interesting irrationality measures are those of Liouville's constant (L) and e. It has been proven that $\mu(L) = \infty$ which is a major reason it is so easy to prove that L is not algebraic. We will prove that $\mu(e) = 2$ later in the paper. Since $\mu(e) = 2$ it is one of the worse transcendental number when it comes to being approximated with its principal convergents.

There are two computation formulas for $\mu(x)$ given in Sondow [3] that allow for easier computation of $\mu(x)$ given that we know x's continued fraction:

Theorem 6. *For a real number $x = [a_0, a_1, a_2, ...]$, the irrationality measure of x, $\mu(x)$ is given by:*

$$\mu(x) = 1 + \limsup_{n\to\infty} \frac{\log(q_{n+1})}{\log(q_n)} \tag{7a}$$

$$\mu(x) = 2 + \limsup_{n\to\infty} \frac{\log(a_{n+1})}{\log(q_n)} \tag{7b}$$

The irrationality measure of a number gives us another way of evaluating how fast the principal convergents of an irrational number converge. Also, if a number's principal convergents converge exceptionally quickly, that number must be transcendental. This allows for an important connection between Diophantine approximations and transcendental number theory.

6 The continued fraction for e

The continued fraction expansion for e is given by

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...]$$

This continued fraction expansion was originally given by Euler in 1737. I am presenting the proof given in Lang [1] which is the same that Euler used. The proof does not require any of the machinery that has been developed in this paper other than some basic ideas and definitions of continued fractions. Even so, this result is still important in the theory of Diophantine approximation because it allows us to easily compute partial quotients. Additionally, this does not prove that e is transcendental. The proof of e's transcendence was given by Hermite in 1873 using the fact that $\frac{d}{dx}e^x = e^x$ and doing some clever integration and approximation. [2]

The proof is made up of two parts. In the first, a function is constructed that satisfies a continued fraction relation. This function is then related to e, which in turn allows for the direct proof of e's continued fraction representation. The construction of the function and its relationship to e is given in Appendix A. In it, we prove that

$$\frac{e-1}{e+1} = [0, 2, 6, 10, \dots]$$

With this knowledge, we can prove what e's continued fraction is.

Theorem 7. *The continued fraction of e is*

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...]$$

Proof. Let r_n/s_n be the principal convergents to the number

$$\alpha = \frac{e+1}{e-1} = \left(\frac{e-1}{e+1}\right)^{-1}$$

Rearranging the above equation gives

$$e = \frac{\alpha+1}{\alpha-1} \tag{8}$$

From the continued fraction given in (U), we know that

$$\alpha = \left(\frac{e-1}{e+1}\right)^{-1} = \left(0 + ([2, 6, 10, ...])^{-1}\right)^{-1} = [2, 6, 10, ...]$$

$$\alpha = [2, 6, 10, \dots]$$

This means that the principal convergents of α satisfy

$$r_n = (2+4n)r_{n-1} + r_{n-2} \tag{9a}$$

$$s_n = (2+4n)s_{n-1} + s_{n-2} \tag{9b}$$

Let $\zeta = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...]$ and let p_n/q_n be the principal convergents to ζ. Due to the simple pattern of ζ's continued fraction, we can obtain the following recursive relationship for p and q when $n \geq 2$:

$$p_{3n+1} = (2+4n)p_{3n-2} + p_{3n-5} \tag{10a}$$

$$q_{3n+1} = (2+4n)q_{3n-2} + q_{3n-5} \tag{10b}$$

These relations give us the ability to define p and q in terms of r and s . Specifically:

$$p_{3n+1} = r_n + s_n \quad \text{and} \quad q_{3n+1} = r_n - s_n \tag{11}$$

The argument is inductive with the base case being $n = 0, 1$ and the inductive case being for $n \geq 2$. Here, just p's relationship is derived but the one for q is done is exactly the same manner.

Case 1: $n = 0$. We can verify directly using the definitions of partial quotients (2a) and (2b)

$$r_0 = 2 \quad \text{and} \quad s_0 = 1$$

$$p_{3\cdot 0+1} = p_1 = 1 \cdot p_0 + p_{-1} = 2 + 1$$

$$p_1 = r_0 + s_0$$

Case 2: $n = 1$. We can verify directly using the newly derived recursive formula for p_{3n+1}, (10a) that

$$r_1 = 6 \cdot 2 + 1 = 13 \quad \text{and} \quad s_1 = 6 \cdot 1 + 0 = 6$$

$$p_{3\cdot 1+1} = (2 + 4) \cdot 3 + 1$$

$$p_4 = r_1 + s_1$$

Inductive case: Assume that $p_{3n+1} = r_n + s_n$ and $p_{3n-2} = r_{n-1} + s_{n-1}$. Using (10a) we get

$$p_{3(n+1)+1} = (2 + 4(n + 1))(r_n + s_n) + r_{n-1} + s_{n-1}$$

$$p_{3(n+1)+1} = (2 + 4(n + 1))r_n + r_{n-1} + (2 + 4(n + 1))s_n + s_{n-1}$$

Using the recursive formula form of r and s, (9a) and (9b), gives us

$$p_{3(n+1)+1} = r_{n+1} + s_{n+1}$$

These equations relating p with r and s allows us to relate ζ to α:

$$\frac{p_{3n+1}}{q_{3n+1}} = \frac{r_n + s_n}{r_n - s_n} = \frac{\frac{r_n}{s_n} + 1}{\frac{r_n}{s_n} - 1}$$

Since p_n/q_n are partial quotients of ζ and r_n/s_n are partial quotients of α, we know that $p_{3n+1}/q_{3n+1} \to \zeta$ and $r_n/s_n \to \alpha$ as $n \to \infty$. Equation (8) gives us the relationship that we want:

$$\zeta = \frac{\alpha + 1}{\alpha - 1} = e$$

So, e has the continued fraction expansion of

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$$

□

6.1 e's Irrationality measure

Using the continued fraction of e, it becomes easy to determine what its irrationality measure is.

Theorem 8. *The irrationality measure of e is equal to 2.*

Proof. Since $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$ we can say that for $a_0 = 2$ and $m \geq 1$,

$$a_{3m} = a_{3m-2} = 1 \quad \text{and} \quad a_{3m-1} = 2m$$

From Theorem 6 we know that

$$\mu(e) = 2 + \limsup_{n\to\infty} \frac{\log(a_{n+1})}{\log(q_n)}$$

Since $\log(1) = 0$ and $a_{3m} = a_{3m-2} = 1$ we know that $\mu(e) \geq 2$. With those two cases out of the way, the only case we need to worry about is when $n = 3m - 2$. For this case we need an estimate of q_{3m-2}.

I assert that $q_{3m+1} \geq (m+1)^m$ for all $m \geq 0$. In the case where $m = 0$ this is easy to verify as $q_1 = a_1 = 1$. Now, assume inductively that for some $m \geq 1$ both $q_{3m-2} \geq m^{m-1}$ and $q_{3m-5} \geq (m-1)^{m-2}$. By equation (10b):

$$q_{3m+1} = (4m+2)q_{3m-2} + q_{3m-5} \geq 4m^m + 2m^{m-1} + (m-1)^{m-2}$$

In other words:

$$q_{3m+1} \geq 4m^m + \text{positive} \geq 4m^m$$

factoring out a $(m+1)^m$ gives us:

$$q_{3m+1} \geq (m+1)^m \cdot 4\Big(\frac{m}{m+1}\Big)^m = (m+1)^m \cdot 4\Big(1 + \frac{1}{m}\Big)^{-m}$$

Since $(1 + \frac{1}{x})^{-x}$ is a decreasing function that tends to e^{-1} as $x \to \infty$, we can say that $(1 + \frac{1}{m})^{-m} \geq e^{-1}$ for $m \geq 1$ which means:

$$q_{3m+1} \geq \frac{4}{e}(m+1)^m \geq (m+1)^m$$

This in turn implies that $q_{3m-2} \geq m^{m-1}$ for all $m \geq 1$. Thus:

$$\mu(e) = 2 + \limsup_{m\to\infty} \frac{\log(2m)}{\log(q_{3m-2})} \leq 2 + \lim_{m\to\infty} \frac{\log(2m)}{(m-1)\log(m)} = 2$$

Since $2 \leq \mu(e) \leq 2$, $\mu(e) = 2$. □

7 Conclusion

Diophantine approximations give us many useful results that can be applied to a plethora of problems. This paper only touched the surface of what Diophantine approximation is used for. Some further applications of Diophantine approximation include approximating roots of functions and analysis of approximation formulas. The relationship between transcendental numbers and Diophantine approximation is exceptionally important. The future advances in the theory of transcendental numbers will most likely be highly reliant on results from Diophantine approximation.

A The Lambert Continued Fraction

Let f be the function described as:

$$f(c,x) = 1 + \frac{1}{c}x + \frac{1}{c(c+1)}\frac{x^2}{2!} + \frac{1}{c(c+1)(c+2)}\frac{x^3}{3!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{1}{c(c+1)\cdots(c+k-1)}\frac{x^k}{k!}$$

Where c is taken to be any real number that is not an integer because otherwise the series would not be defined. The domain of f is $\mathbb{R}$ because for large enough k the terms in f's series expansion are smaller in absolute value than the corresponding terms in the series expansion for e.

Theorem 9. *The function described above can be represented using continued fractions as:*

$$\frac{z}{c}\frac{f(c+1,z^2)}{f(c,z^2)} = \left[0, \frac{c}{z}, \frac{c+1}{z}, \frac{c+2}{z} \ldots\right] \tag{12}$$

where $(c+n)/z$ is an integer that's greater than or equal to 1 for all $n \geq 0$.

Proof. Some messy series manipulation yields the relationship:

$$f(c,x) = f(c+1,x) + \frac{x}{c(c+1)} f(c+2,x) \tag{13}$$

By inverting both sides then multiplying by $f(c+1,x)$ we get:

$$\frac{f(c+1,x)}{f(c,x)} = \frac{1}{1 + \frac{x}{c(c+1)} \frac{f(c+2,x)}{f(c+1,x)}}$$

This expression of f looks similar to one of a continued fraction. However, there is a small problem stopping us from writing the function as a continued fraction: we don't know how to express a_n. The substitution $z^2 = x$ solves this issue:

$$\frac{z}{c}\frac{f(c+1,z^2)}{f(c,z^2)} = 0 + \frac{1}{\frac{c}{z} + \frac{z}{(c+1)} \frac{f(c+2,z^2)}{f(c+1,z^2)}}$$

Which is generalized as:

$$\frac{z}{(c+n)}\frac{f(c+n+1,z^2)}{f(c+n,z^2)} = \frac{1}{\frac{c+n}{z} + \frac{z}{(c+n+1)} \frac{f(c+n+2,z^2)}{f(c+n+1,z^2)}}$$

Which finally gives us the repeated fraction expression of f.

$$\frac{z}{c}\frac{f(c+1,z^2)}{f(c,z^2)} = \left[0, \frac{c}{z}, \frac{c+1}{z}, \frac{c+2}{z} \ldots\right]$$

The restriction on c and z are given in the theorem are due to the restrictions that were imposed upon continued fractions when we defined them. □

The first thing we should note about this continued fraction is that it works for c and z given by $c = 1/2$ and $z = 1/(2y)$ where y is a positive integer greater than or equal to 1. The expansion is given as:

$$\frac{1}{y}\frac{f(3/2, 1/4y^2)}{f(1/2, 1/4y^2)} = [0, y, 3y, 5y, ...]$$

For these specific values of c and z, we get what is called the **Lambert continued fraction**. Additionally, the $f(c,z)$ for the values above has a special relationship to e.

Theorem 10. *For all $w \in \mathbb{R}$,*

$$e^w - e^{-w} = 2wf\left(\frac{3}{2}, \frac{w^2}{4}\right) \tag{14a}$$

$$e^w + e^{-w} = 2f\left(\frac{1}{2}, \frac{w^2}{4}\right) \tag{14b}$$

The proof of this theorem is given by looking at the k^{th} terms in the series expansions of f and $e^w \pm e^{-w}$ and doing some algebraic manipulations. This eventually leads to the conclusion that every term in both series expansions are the same and the equations above hold.

Since it has already been confirmed that f's continued fraction is legitimate for $c = 1/2$ and $z = 1/y$, if we let $w = 1/y$ we obtain:

$$\frac{e^{1/y} - e^{-1/y}}{e^{1/y} + e^{-1/y}} = \frac{1}{y}\frac{f(3/2, 1/4y^2)}{f(1/2, 1/4y^2)} = \left[0, y, 3y, 5y, ...\right]$$

In the special case where $y = 2$ we get:

$$\frac{e^{1/2} - e^{-1/2}}{e^{1/2} + e^{-1/2}} = \frac{e-1}{e+1} = \left[0, 2, 6, 10, ...\right] \tag{15}$$

This actually proves that e is irrational. This is because the sum and product of rational numbers are rational. Since $(e+1)/(e-1)$ is irrational and a composition of rational numbers and e, this implies that e must be irrational. Another useful consequence of this result is that we can use it to approximate hyperbolic tangent because of the relationship:

$$\tanh\left(\frac{1}{y}\right) = \frac{\sinh(1/y)}{\cosh(1/y)} = \frac{e^{1/y} - e^{-1/y}}{e^{1/y} + e^{-1/y}} = \left[0, y, 3y, 5y, ...\right]$$

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