Hyperreals and a Brief Introduction to Non-Standard Analysis
Math 336

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Abstract

The hyperreals are a number system extension of the real number system. With this number system comes many advantages in the use of analysis and applications in calculus. Non-standard analysis refers to the use of infinitesimals in doing analysis instead of the usual epsilon and distance functions.

The machinery we will build in this papaer will allow us to prove some elementary analytic results. This paer will go through how to construct number systems via equivalence classes, how the hyperreals are constructed, how the hyperreals function, and finally how to use them to prove some theorems about uniform convergence and Riemann Integration.

Contents

1 Introduction

Historically when Leibniz invented calculus the use of infinitesimals was somewhat careless, not many people questioned the vailidity of them as mathematical objects. Besides some detractors, namely George Berkeley in his scathing article about infinitesimals, no one really questioned the intutive nature of them. For instance Euler proved many theorems infinitesimals without much regard for the foundations of the ojects he was working with. For example if one were to calculate the derivative of $f(x) = x^2$, the calculation would be as follows:

$$\frac{(x+\epsilon)^2 - x^2}{\epsilon} = \frac{2x\epsilon + \epsilon^2}{\epsilon} = 2x + \epsilon$$

however ϵ is an infinitesiamal and thus disregarded, so the answer is $2x$. The quest for absolute rigor lead to the demise of infinitesimals in the 19th century, it was during this time that Wierstrass gave his ϵ, δ definition of limit. It wasnt until advances in model theory for Robinson in the 1960s that the full theory of infinitesimals was rigorously defined.

2 Number System Construction

2.1 Axiomatic Set Theory

To begin understanding the construction of the non-standard reals we must be familiar with how number systems are constructed. We construct the natural numbers from axiomatic set theory using two axioms, union and the existence of an inductive set. Start with the empty set, $\emptyset$, and define the sucessor operation S on a set x such that $S(x) = x \cup \{x\}$, for instance 0 is associated with the empty set, $\emptyset$, and so 1 is associated with $\{\emptyset, \{\emptyset\}\}$. The set with the inductive property, that being of closed under the successor operation, we intentionally call $\mathbb{N}$.

Defintion 2.1. ***Equivalence Relation****: An Equivalence Relation, $\sim$ is a relation on sets such that the three following properties hold:*

1. *Reflexivity: $a \sim a$*
2. *Symmetry: $a \sim b$ implies $b \sim a$*

3. Transitivity: $a \sim b$ and $b \sim c$ implies $a \sim c$

Defintion 2.2. ***Equivalence Class*** *An equivalence class is the set of objects satisfing some equivalence relation.*

The integers, $\mathbb{Z}$, is constructed by equivalence classes from orderd pairs of naturals. The rationals, $\mathbb{Q}$, are constructed from equivalence classes on integers.[1] The interesting part comes in constructing the reals, $\mathbb{R}$, from the rationals, where we no longer talk about ordered pairs but infinite sequences. There are two ways of defining the reals one way is consider Dedekind cuts and the other is to look at equivalence classes of Cauchy Sequences of rational numbers. The Cauchy sequence method is more appropriate for what is being done here, as the hyperreals, $^*\mathbb{R}$, are defined in such a way from real numbers.

2.2 Integers $\mathbb{Z}$

Example 2.1. *Lets constuct the integers from the naturals as simple demonstration of taking an equivalence class. $\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim$, i.e. we identify an integer as an equivalence class or the set of ordered pairs of natural numbers that satisfy the following equivalence relation. Where $\sim$ is the equivalence relation $(a, b) \sim (c, d)$ if $a + d = b + c$. We verify this is an equivalence relation by showing reflexivity, symmetricity, and transitivity:*

$$(a, b) \sim (a, b)$$

$$(a, b) \sim (c, d) \to (c, d) \sim (a, b)$$

$$(a, b) \sim (c, d), (c, d) \sim (e, f) \to (a, b) \sim (e, f)$$

Obviously these hold by the commutativity and identity properties of the natural numbers, the details are left for the reader.

We construct the hyperreals the same way via the set of all real valued sequences indexed by the natural numbers. Now to the construction

[1] See Enderton Chapter 5 for details.[2]

3 Hyperreal Construction

3.1 Preliminaries

The construction of hyperreals requires taking an equivalence classes on $\mathbb{R}^{\mathbb{N}}$, the set of infinite real valued sequences.[2] The relation we have to define for the hyperreals is going to be a special kind of equivalence relation which will need some set theoretic machinery. The first object we will need is what is called an ultrafilter. An ultrafilter, $\mathcal{F}$, on a a set X is a set of subsets of X. The ultrafilter tests for size of the of subsets of X. When we get to the construction we will use size and real number equality as our equivalence relation. As would be guessed, our intuitive defintion of ultrafilter fits with the formal defintion. An ultrafilter acts like a sieve that filters out small sets, when big sets are desired.

Defintion 3.1. ***Power Set*** *:The power set $\mathcal{P}(X)$ of a set X is defined to be $\{x|x \subset X\}$.*

Defintion 3.2. ***Cofinite****: Given set $Y \subset X$, Y is cofinite if $X \setminus Y$ is finite.*

Defintion 3.3. ***Ultrafilter****:An ultrafilter $\mathcal{F}$ on a set X is a collection of subsets of X that satisfy the following properties:*

1. $X \in \mathcal{F}$
2. $\emptyset \notin \mathcal{F}$
3. *If* $A, B \in \mathcal{F}$ *then* $A \cap B \in \mathcal{F}$
4. *If* $A \subset X$ *then either* $A \in \mathcal{F}$ *or* $X \setminus A \in \mathcal{F}$
5. *If* $A \in \mathcal{F}$ *and* $A \subset B$ *then* $B \in \mathcal{F}$.

3.2 The Equivalence Relation $\equiv$

The equivalence relation we want to take is on the set of all infinite real sequences, $\mathbb{R}^{\mathbb{N}}$, will use an ultrafilter to grab sets that are equivalent. We will say that two hyper real numbers are equal if their real number sequences differ by at most a finite number of terms.

[2]This constuction is taken from Goldblatt Chapter 3.[3]

Defintion 3.4. *Define a relation,* $\equiv$*, on* $\mathbb{R}^{\mathbb{N}}$*, by saying that* $(r_n) \equiv (s_n)$ *if and only if* $\{n \in \mathbb{N} | r_n = s_n\} \in \mathcal{F}$*.*

This equivalence relation captures the idea of 'agreeing almost everywhere.' Since the hyperreals are constructed using real numbers we should use real number equality since we know how that works. We say $x = y$ in the hyperreals if the parts of the real number sequences defining x and y differ at only finitly many terms. For example let $x = (1, 2, 3, 4, ...)$ and $y = (1, 2, 2, 4, ...)$, if these sequences progressed in the natural way then we could say that $x = y$ because each sequence only differs by finite terms, namely one term. The less than relation is defined similarly we want the terms for which the $x_j \leqslant y_j$ to be only finite to say that $y_j < x_j$. The cofinite equality of the real number sequences is captured by the ultrafilter.

Defintion 3.5. *Hyperreal Arithmetic is defined componentwise.*

$$[r] + [s] = [(r_n + s_n)]$$
$$[r] \cdot [s] = [(r_n \cdot s_n)]$$

Defintion 3.6. *The set of equivalence classes of an ultrafilter is called an ultraproduct and* $r \in \mathbb{R}^{\mathbb{N}}$ *under* $\equiv$ *denoted by* $[r]$ *will be*

$$[r] = \{s \in \mathbb{R}^{\mathbb{N}} | r \equiv s\}.$$

This leads to the definition of the hyperreals from $\mathbb{R}^{\mathbb{N}} / \equiv$.

$$*\mathbb{R} = \{[r] | r \in \mathbb{R}^{\mathbb{N}}\}$$

3.3 Transfer Principle

Statements about hyperreals require the use of first-order logic. First-order logic discuss quantification over objects, whereas second-order logic allows quantification over predicates or relations.

Defintion 3.7. *First Order Formula is a formula involving quantification over objects in the domain of discourse.*

Example 3.1. *Let* X *be the set of students in 336, and let* ϕ *be the formula so that* $\phi(x)$ *says that 'x does analysis.' Then we can say that* $\forall x \phi(x)$ *is a first order forumla about students.*

Theorem 3.1. *Transfer Principle:* Any appropriately formulated statement ϕ about $\mathbb{R}$ holds iff $^*\phi$ holds for $^*\mathbb{R}$.

Proof. The rigorous proof uses sophisticated model theory or axiomatic deductions and is beyond the scope of this paper. The idea of why this holds is straightforward. Consider the ultrafilter $\mathcal{U}$ on a cartesian product of sets, M_i:

$$\prod_{i \in \mathbb{N}} M_i / \mathcal{U}$$

Then if a first-order formula, ϕ, holds for each M_i and is captured by the ultrafilter, then ϕ holds in the ultraproduct. ■

Here appropriately formulated means statements in first-order logic. The transfer principle allows statements about the reals to be equivalent statements of the hyperreals and vice versa. For example commutativity is an appropriate statement about the reals and so it is an appropriate statement about the hyperreals. The details of all the properties of the reals that transfer to the hyperreals are too tedious for this paper.

3.4 Ordered Field

Defintion 3.8. *An Ordered Field, $\mathbb{F}$, is an algebraic stucture with a set, F, two operations, $\{+, \cdot\}$, and an ordering relation, $\{\leqslant\}$ that satisfy the following properties:*

1. $\mathbb{F}$ is associative in both operations
2. $\mathbb{F}$ is invertable in both operations
3. $\mathbb{F}$ has an identity element in both operations
4. $\mathbb{F}$ is closed under both operations
5. $\mathbb{F}$ has a well defined notion of order

The Transfer Principle justifies this. For our purpose it is enough to say that since the reals are a ordered field then the hyperreals are an ordered field.

3.5 Infinitesimals and Unlimiteds

The reals are a subfield of the hyperreals, in a similar sense that the rationals are a subfield of the reals. The reals in the hyperreals are identified with infinte sequences of themselves, so in the hyperreals 0 is identified with the infinite sequence of 0's, $(0, 0, 0...)$ and π with $(\pi, \pi, \pi, ...)$. Infinitesimals are defined by seqences of real numbers approaching 0, for instance we may take $\epsilon = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...)$. This number is less than any real number in the hyperreals:

Theorem 3.2. *There exists $\epsilon \in {}^*\mathbb{R}$, called an infinitesimal, such that $\epsilon > 0$ and for all $x \in \mathbb{R}, \epsilon < x$.*

Proof. Let $\epsilon = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...)$, with each term called ϵ_n and $0 = (0, 0, 0, ...)$, with each term called 0_n, then $\epsilon_n > 0_n$ for all n, so $\epsilon > 0$. Take $x = (x, x, x, ...)$ with each term called x_n, then since each x_n is equal there exists n_0 such that for all $n \geqslant n_0$ then $\epsilon_n < x_n$, so there are infinitely many terms such that $\epsilon_n < x_n$, therefore $\epsilon < x$. ■

Theorem 3.3. *There exists $\omega \in {}^*\mathbb{R}$, called unlimited, such that $\omega > x$ for all $x \in \mathbb{R}$.*

Proof. Let $\omega = (1, 2, 3, ...)$ with each term ω_n and take $x = (x, x, x, ...)$ with each term called x_n, then there exists n_0 such that for all $n \geqslant n_0$ then $\omega_n > x_n$. ■

Let's recap what just happened. First we considered the set of infinite sequences of real numbers. Then we defined an ultrafilter that picked out elements of those sequences that were defined to be equivalent if they differed by at most finite number of terms. This defined the hyperreals as an object of study. Next the arithmetic of how the equivlence classes would be worked with in terms of addition and multiplication was defined in terms of infinite sequences. The transfer principal said that any well formulated statement about the reals would hold for the hyperreals, this happened because of model theory. This gave the arithmetic and ordering properties of the hyperreals as objects themselves. Finally we showed that this set had the infinitesimals and unlimited numbers that were desired.

4 Working With *$\mathbb{R}$

4.1 Arithmetic in *$\mathbb{R}$

Let $x \in^* \mathbb{R}$, x is called finite if $a < x < b$ for $a, b \in \mathbb{R}$, and x is not infinitesimal. A number ϵ is called infinitesimal if $\epsilon < x$ for all $x \in \mathbb{R}^+$ and $\epsilon > x$ for all $x \in \mathbb{R}^-$. Here ϵ can be positive or negative, but it is not necessary to worry about for the purposes of this paper. One way to think about an infinitesimal is that if the limit of the terms of the real number sequence defining the hyperreal number goes to zero then it will define an infintesimal. Suppose ϵ, δ are infinitesimal, x, y are finite and ω, α are unlimited then the following is true:

1. $\epsilon + \delta$ is infinitesimal
2. $x + \epsilon$ is finite
3. $x + y$ is appreciable possible infinitesimal
4. $\epsilon \cdot \delta$ is infintesimal
5. $\epsilon \cdot x$ is infinitesimal
6. $\omega + x$ is unlimited
7. $\frac{1}{\epsilon}$ is unlimited
8. $\frac{1}{\omega}$ is infinitesimal

Several indeterminate forms arise when working with operations of hyperreal numbers:

1. $\frac{\epsilon}{\delta}$
2. $\frac{\omega}{\alpha}$
3. $\omega \cdot \epsilon$
4. $\omega + \alpha$

4.2 The 'Arbitrarily Close' Equivalence Relation

Defintion 4.1. *Let $\simeq$, be an equivalence relation on $^*\mathbb{R}$ such that $x \simeq y$ means that $x - y$ is infinitesimal or 0.*

Checking this is a well defined equivalence relation is straightforward and left to the reader. It immediately follows that $\epsilon \simeq 0$, this will be important because it captures of idea of "arbitrary close" that lies at the heart of usual ϵ, δ type proofs in analysis.

4.3 Set Enlargement

Defintion 4.2. *Set Enlargement: A set $I \subset \mathbb{R}$ can be extended to a set $^*I \subset {^*\mathbb{R}}$ if for each $r \in \mathbb{R}^{\mathbb{N}}$ then*

$$[r] \in *A \leftrightarrow \{n \in \mathbb{N} | r_n \in A\} \in \mathcal{F}$$

The ultrafilter gives us the concept of almost all, by the cofinite definition of equality. So we can extend an interval to be a hyperreal interval by saying that a hyperreal number is in the interval if the real valued sequence defining it is almost all in the real interval.

4.4 Least Upper Bound Property

The least upper bound (LUB) property for the real number states that every set of real numbers with an upper bound has a least upper bound. This is also called the dedekind completeness property or Cauchy completeness. This has problems in the language of the transfer princple, since here we are not quantifying over objects, but sets of objects. This means that the LUB property is not expressable in first-order language. But the properties of the hyperreals give us an equivalent statement.

Theorem 4.1. *Because Every limited hypereral is infinitely close to a real number implies the completeness of $\mathbb{R}$.*[3]

Proof. Let $s : \mathbb{N} \to \mathbb{R}$ be a Cauchy sequence, so there exists $k \in \mathbb{N}$ so that

$$\forall m \forall n \in \mathbb{N}(m, n \geqslant k \to |s_m - s_n| < 1).$$

[3][3] pg 55

This is a first-order statement(only quantified over numbers, not sets), so the transfer principle applies in a sequence with an unlimited hyperreal N. Take N unlimited so $k, N \geqslant k$ then $|s_k - s_N < 1|$, and so s_N is limited. By assumption that every limited hyperreal is infinitely close to a real number then say that $s_N \simeq L$ for $L \in \mathbb{R}$. Now we show that the the original sequence $s \to L$.

Let $\epsilon > 0$, since s is Cauchy then there exists $j_0 \in \mathbb{N}$ so that for $j \geqslant j_0$ then:

$$\forall m \forall n \in \mathbb{N}(m, n \geqslant j \to |s_m - s_n| < \epsilon).$$

We can show that beyond s_j that all the terms are within ϵ of L. This is because all such terms are within ϵ of s_N, which is itself with in ϵ of L. Let $m \in \mathbb{N}$ be such that $m \geqslant j$ then $m, N \geqslant j$ so by transfer principle then $|s_m - s_N| < \epsilon$. Pushing the inequalities this becomes:

$$|s_m - L| < |s_m - s_N| + |s_N - L| < \epsilon + \delta.$$

Where here δ is another infinitesimal. And since Cauchy sequence convergence implies completeness, this completes the proof. ■

The last result is interesting in that it shows how transfer can be used in ingenous ways to give us properties that we might have lost. In this case we might lose the LUB property since it is a second-order formula, but the first-order formulation of convergent sequences gave use that property. Though the work of showing that Cauchy Completeness is equivalent to LUB and to Dedekind completeness is taken as given here.

5 Non-Standard Calculus

5.1 Continuity

Defintion 5.1. *Let $f(x)$ be a real-valued function on $[a, b]$, then say that $f(x)$ is continuous if when $x \simeq c$ then $f(x) \simeq f(c)$ for all $x \in [a, b]$.* [4]

Example 5.1. *All lines of the form $y = mx + b$ are continuous.*

Suppose $x \simeq c$ then $x = c + \epsilon$ and:

$$f(x) - f(c) = m(c + \epsilon) + b - (mc + b) = m\epsilon \simeq 0.$$

[4]These defintions are taken from Keisler[4]

This simple example shows the power of infinitesimal caluclus. In regular continuity proofs, much inequality pushing was necessary and finding appropriate bounding inequalities could be very difficult. But here we appeal to the intuition that Leibniz and many great mathematicians had, but that intuition now has a very rigorous foundation that could be laid out at any time. No longer is it necessary in many cases to find a δ because that δ exists.

5.2 Derivative

Defintion 5.2. *Let $f(x)$ be real valued, the derivative of $f(x)$ denoted $f'(x)$ is given by*

$$f'(x) = \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

provided that the function is defined at x and $f'(x) \simeq f'(x) + \epsilon$.

5.3 Riemann Sum

Defintion 5.3. *Let $f(x)$ be defined on $I = [a, b]$ and let $\pi = \{x_0, x_1, ..., x_\omega\}$ be a partition of I, and let $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_\omega\}$, where $x_j \leqslant \alpha_j \leqslant x_{j+1}$, then define*

$$S(\pi, \alpha) = \sum_{j=0}^{\omega} f(\alpha_j)(x_{j+1} - x_j) \simeq \int_a^b f(x)\mathrm{d}x.$$ [5]

Since there exists unlimited numbers in the hyperreals we can use those to capture the idea "for sufficiently large" in a rigourous sense.

6 Non-Standard Analysis

We know present some theorems from introductory analysis from a non-standard point of view. The goal here is to see simplications that come from not having to choose ϵ or 'sufficiently large n.'

[5] [5] Pg 71-72

6.1 Uniform Convergence

Defintion 6.1. ***Uniform Convergence of a Sequence of Functions***

A sequence of functions f_n is said to converge uniformly to f on I if $f_n \simeq f$ for n unlimited.[6]

6.2 Adapted Analysis Proofs

Theorem 6.1. ***Cauchy Criterion for Uniform Convergence***

Let f_n be a sequence of bounded funtions on $I \subset \mathbb{R}$. Then this sequence converges uniformly on I to a bounded function f if and only if for unlimited n and m then $f_n - f_m \simeq 0$.[7]

Proof. Suppose f_n converges uniformly to f on I, then for unlimited n and m, $f_n \simeq f$ and $f_m \simeq f$. So by transitivity $f_m - f_n \simeq 0$. Now suppose that $f_m - f_n \simeq 0$, so for each $x \in I$ then $f_n(x) - f_m(x) \simeq 0$. Because $f_n(x)$ adn $f_m(x)$ are cauchy they both converge to $f(x)$. So $f_n(x)$ converges to $f(x)$ for each $x \in I$ then f_n converges uniformly to f on I. ■

Theorem 6.2. *Suppose f_n is a sequence of continuous functions on and $f_n \to f$ uniformly on I, then f is continous I.*[8]

Proof. Take $x \simeq c$ and n unlimited then $f_n(x) \simeq f_n(c)$ for all n from the assumption of continuity of each f_n. Because of uniform convergence $f_n(x) \simeq f(x)$ and $f_n(c) \simeq f(c)$. So then by transitivity $f(x) \simeq f(c)$. ■

We know that for

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

to hold then f_n must converge uniformly to f on $[a, b]$, we prove that now using non-standard analysis. Where $n \to \infty$ is replaced with the unlimited hyperreal number ω.

Theorem 6.3. *Let f_n be a sequence of functions in $\mathcal{R}[a, b]$ and suppose f_n converges uniformly on $[a, b]$ to f. Then $f \in \mathcal{R}[a, b]$.*[9]

[6] [5] Pg 130-131
[7] [1] Pg 246
[8] [6] Pg 443-444
[9] [1] Pg250-251

Proof. Since Riemann integrable functions are bounded then by the Cauchy Criterion for uniform convergence we know that for $x \in [a, b]$ that $f_m - f_n \simeq 0$ for $n, m \geqslant \omega$, so then f_m converges and this implies that

$$\int_a^b f_m \simeq I.$$

Now take $m \geqslant \omega$ so that $f_m(x) - f(x) \simeq 0$ for all $x \in [a, b]$. And consider a significant partition, π, of $[a, b]$ then:

$$S(\pi, f_m) - S(\pi, f) \simeq \sum_{i=1}^{\omega} (f_m(x_i) - f(x_i))(x_i - x_{i-1}) \simeq 0(b - a) \simeq 0.$$

Considering the fact that for unlimited n that:

$$\int_a^b f_m - I \simeq 0.$$

And we can say that because each f_m is Riemann integrable

$$\int_a^b f_m - S(\pi, f_m) \simeq 0.$$

We finally get:

$$S(\pi, f) - I \simeq S(\pi, f) - S(\pi, f_m) + S(\pi, f_m) + \int_a^b f_m - \int_a^b f_m - I \simeq 0.$$

■

7 Conclusions

The importance in non-standard analsis comes from its simplification of proofs. No triangle inequality, no adding and subtracting terms, and no ingenuity when trying to find inequalties. You could say that inequalities are in the equivalence relation, $\simeq$, but its function is almost that of equality.

Non-Standard Analysis works great for heuristic arguments when it comes to analysis. If we want to show something, it could suffice to show that it does work for the hyperreals and use the transfer principle to claim that it workds for the reals too, though this can be tricky. This paper left alot out because of the nature of the paper, and the deep topic that Non-Standard Analysis is. Goldblatt's book is a very are lecture ntoes for a course that he taught on this subject.

In 1976, Kielser, who was involved in the development of the some of the machinery that helped developed for the hyperreals, wrote a book called "Elementary Calculus." This book, is a textbook style calculus text, that uses the hyperreal number system for its proofs. Though it never caught on, this could be a great teaching method for those people learning elementary calculus, since it relies more on the intuitions and less on the seemily criptic analytical definitions of calculus topics.

8 References

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[6]C.L. Thompson. Nonstandard Continuity and Uniform Convergence. The American Mathematical Monthly, Vol. 96, No. 5(May 1989), pg 443-444.