Conformal if and only if Complex Differentiable

Theorem 1. *Suppose $f = u + iv$ is a complex valued function that is defined in a neighborhood of a point z_0 and real differentiable at z_0. Suppose that*

$$\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} (z_0) \neq 0.$$

Then f is complex differentiable at z_0 if and only if f is conformal at z_0

Proof. The proof is a consequence of the following lemma and the characterization of complex differentiability by the Cauchy-Riemann equations. □

Lemma 1. *The real matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det A \neq 0$. preserves angles and orienation if and only if $b = -c, d = a$.*

Proof. If $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then we can consider A to be a complex number $a + ib$. Matrix multiplication $A \begin{bmatrix} x \\ y \end{bmatrix}$ can be interpreted as $(a + ib)(x + iy)$. If $a + ib = re^{it}$, this is scaling by r and rotating to the left by angle t. This multiplication preserves angles between pairs of vectors and the relative orientation.

Next suppose A preserves angles and orientation. Then $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ are orthogonal $ab + cd = 0$. Also $\begin{bmatrix} a + b \\ c + d \end{bmatrix}$ and $\begin{bmatrix} b - a \\ d - c \end{bmatrix}$ are orthogonal. So $b^2 - a^2 + d^2 - c^2 = 0$. By manipulating these equations we can show that $c = \pm b$ and $d = -\frac{ac}{b} = \mp a$. We have to choose $c = -b, d = a$ to preserve relative orientations (interpreted as complex numbers we want $b + id = i(a + ic)$, rotate to the left, not right). □