Automorphisms of the Unit Disk

Let \( \mathbb{D} = \{ z : |z| < 1 \} \). We want to describe all conformal maps from \( \mathbb{D} \) onto \( \mathbb{D} \). We will postpone doing this and instead describe all linear fractional transformations \( T \) from \( \partial \mathbb{D} \) onto \( \partial \mathbb{D} \) that take \( \mathbb{D} \) into \( \mathbb{D} \). A linear fractional transformation takes circles to circles, so \( T \) must take all points in \( \mathbb{D} \) to points in \( \mathbb{D} \) and all points \( z \) with \( |z| > 1 \) to points \( z \) with \( |z| > 1 \); or all points in \( \mathbb{D} \) to \( |z| > 1 \) and points with \( |z| > 1 \) to points in \( \mathbb{D} \). This is proved using the intermediate value theorem applied to \( |T(z)| \) and the fact that \( |T(z)| = 1 \) exactly when \( |z| = 1 \).

**Theorem 1.** The linear fractional transformations that map \( |z| = 1 \) to \( |z| = 1 \) and \( \mathbb{D} \) to \( \mathbb{D} \) can be described by

\[
\lambda \frac{z - a}{1 - \bar{a}z}, \quad |a| < 1, \quad |\lambda| = 1;
\]

and also by

\[
\frac{az + \bar{b}}{bz + \bar{a}}, \quad |a|^2 - |b|^2 = 1.
\]

**Proof.** We’ll organize the proof in steps. Assume (new meaning of the letters \( a, b, c, d \)).

First we prove \( d \neq 0 \). If \( d = 0 \), the condition \( ad - bc \neq 0 \) implies \( bc \neq 0 \). Hence \( T \) can be written as \( \frac{a}{z} + \frac{b}{cz} \). This implies that \( T(0) = \infty \), which can’t happen. So \( d \neq 0 \).

Now we know \( d \neq 0 \). Next we consider \( c = 0 \). Then since \( ad - bc \neq 0 \), \( a \neq 0 \). We have

\[
Tz = \frac{az + b}{cz + d}.
\]

The image of \( |z| = 1 \) by this \( T \) is a circle with center \( \frac{b}{d} \) and radius \( |\frac{a}{d}| \). This is supposed to be the circle \( |z| = 1 \), so \( b = 0 \) and \( |a| = |d| \). This implies that \( Tz = \lambda z \), with \( |\lambda| = 1 \). That is one of our cases.

Next we consider \( d \neq 0, c \neq 0 \), and prove that in this case \( a \neq 0 \). If \( a = 0 \), we have

\[
Tz = \frac{b}{cz + d}.
\]

This implies that \( T(\infty) = 0 \) and that is not possible.

Finally, we prove that \( b \neq 0 \) when \( adc \neq 0 \). If \( b = 0 \), then

\[
Tz = \frac{az}{cz + d}.
\]

Then

\[
|az|^2 = |cz|^2 + |d|^2 + 2Re(c\bar{d}z),
\]

\[
|a|^2 = |c|^2 + |d|^2 + 2Re(c\bar{d}z).
\]
Let $c \bar{d} = re^{it}$ and $z = e^{i\theta}$. Then $2Re(c \bar{d}z) = re^{i(t+\theta)}$ and this varies with $\theta$ unless $r = 0$. This implies $cd = 0$, which is contrary to our assumption. So $b \neq 0$.

Now introduce new letters and write $T$ as

$$Tz = \lambda \frac{z - a}{1 - \bar{d}z}.$$ 

Since $Ta = 0$, $|a| < 1$. Also $T(0) = -\lambda a$ so $|\lambda a| < 1$. 

The following relations, when $|z| = 1$,

$$|\lambda z|^2 + |a\lambda|^2 - 2Re(\bar{a}|\lambda|^2z) = 1 + |dz|^2 + 2Re(dz),$$

$$|\lambda|^2 + |a|^2||\lambda|^2 = 1 + |d|^2 + 2Re((\bar{a}|\lambda|^2 - d)z),$$

imply

$$d = |\lambda|^2 \bar{a},$$

$$|\lambda|^2 + |a|^2||\lambda|^2 = 1 + |d|^2,$$

by an argument similar to a previous argument. Substituting, we get a quadratic equation for $|\lambda|^2$,

$$|\lambda|^4 - (1 + |a|^2)|\lambda|^2 + 1,$$

with solutions $|\lambda|^2 = 1, \frac{1}{|a|^2}$. Since $|a\lambda| < 1$, the second solution is ruled out. So $|\lambda| = 1, d = \bar{2}$, and these are the only possible linear fractional transformations that map $\mathbb{D}$ onto $\mathbb{D}$. It’s easy to verify that they do map $\mathbb{D}$ onto $\mathbb{D}$.

By rescaling by, we can produce the second form.